Planar Induced Subgraphs of Sparse Graphs

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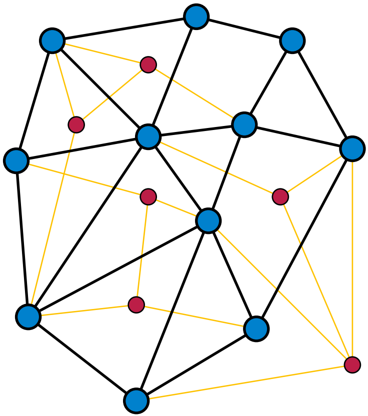
The planarization problem

Goal: find big planar subgraphs in nonplanar graphs

Equivalently: delete as little as possible so the rest is planar

In the version we study, the planar subgraphs are *induced* so we're deleting as few vertices as possible to get a planar graph



What type of result should we look for?

Optimal planarization is known to be NP-hard

Fixed-parameter tractable algorithms are known where the parameter is the number of deleted vertices [Kawarabayashi 2009]

Our results: worst-case bounds on the number of deleted vertices as a function of the number of edges

(and planarization algorithms that achieve those bounds)

Previous results

All previous results restrict the input graph in some way, e.g.:

Triangle-free $\Rightarrow$ delete $m/4$ vertices to get a forest [Alon et al. 2001]

Max degree $\Delta \Rightarrow$ has a planar induced subgraph with $\dfrac{3n}{\Delta+1}$ vertices [Edwards and Farr 2002]

$m \geq 2n \Rightarrow$ same formula replacing Δ by average degree [Edwards and Farr 2008]

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Good news and bad news

Our results:

Every graph can be planarized by deleting $\frac{m}{5.2174}$ vertices

For some graphs, deleting $\frac{m}{6} - o(m)$ vertices is not enough

The same $m/6$ barrier exists for all minor-closed graph properties

Ary Scheffer, *The Temptation of Christ*, 1854

A simple planarization algorithm

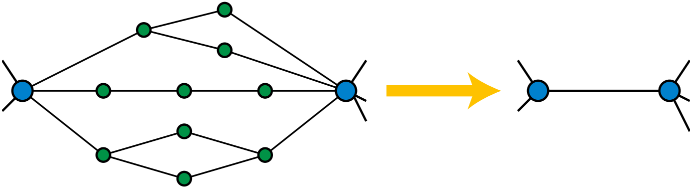
While the remaining graph has a nonplanar component:

- If some edge e has an endpoint of degree at most two:
  1. Contract e (forming a graph without the low-degree endpoint)
  2. Mark the endpoint as being part of the planar output graph
  3. Simplify any self-loops and multiple adjacencies formed by the contraction
- Else, within any nonplanar component:
  1. If max degree ≥ 5, let v be a vertex of maximum degree; otherwise, let v have degree four with a degree-three neighbor (if such a vertex exists); otherwise, let v be any vertex.
  2. Delete v and mark it as not part of the output

Correctness of the algorithm

Contracting and later un-contracting an edge with a degree-one endpoint, or removing and re-adding isolated vertices, cannot change planarity of the result

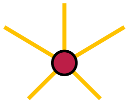
At intermediate steps of the algorithm, degree-two contraction and simplification replaces *series-parallel subgraphs* by single edges.



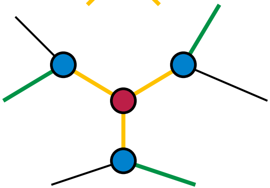
Eventually, either both endpoints of such an edge are kept (and the whole series-parallel subgraph can be re-expanded) or one endpoint is deleted (and the rest of the graph is safe to re-add)

Proof that algorithm deletes $\leq m/5$ vertices (I)

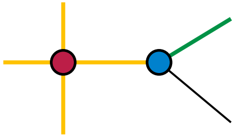
Deleting a vertex of degree ≥ 5 removes at least five edges



Deletion in a 3-regular graph removes three edges and causes at least three more to be contracted



Deletion in an irregular graph eliminates at least five edges



But what about 4-regular graphs?

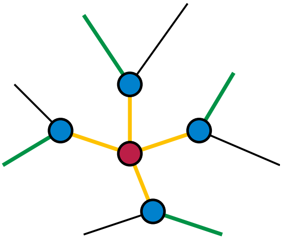
Proof that algorithm deletes $\leq m/5$ vertices (II)

When we delete a vertex from a 4-regular graph, only four edges are deleted and there are no immediate edge contractions

but...

If the remaining graph is 3-regular, the next step eliminates six edges, one more than it needs

If the remaining graph is irregular, then the last degree-four vertex to be deleted within it eliminates at least eight edges, three more than it needs



Every vertex deletion leads to ≥ 5 eliminated edges, QED

Better analysis of the same algorithm

Allow degree-3 and -4 vertices to carry "debts" up to credit limits c_3 or c_4

Also allow graphs that have at least one degree-three vertex to carry one more debt, limit τ

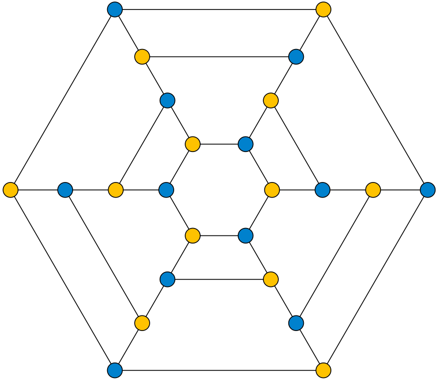
When an operation creates a low-degree vertex, credit its debt to #edges eliminated, but require all debts to be cleared by a later operation that pays for the extra edges

Use linear programming to find optimal choices for c_3, c_4, and τ

$\Rightarrow$ same algorithm deletes at most $\dfrac{23m}{120}$ vertices

Ramanujan graphs

An infinite family of 3-regular graphs
with shortest cycle length $\Omega(\log n)$ [Lubotzky et al. 1988]

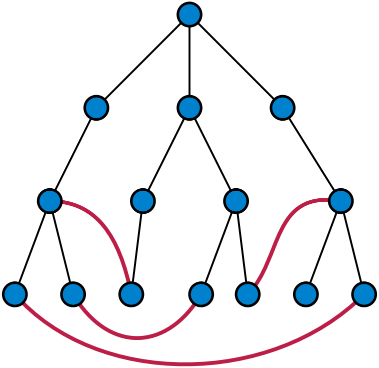


$X^{2,3}$ from [Chiu 1992] = truncated octahedron

These turn out to be difficult to planarize (for large n)

Deleting too few vertices

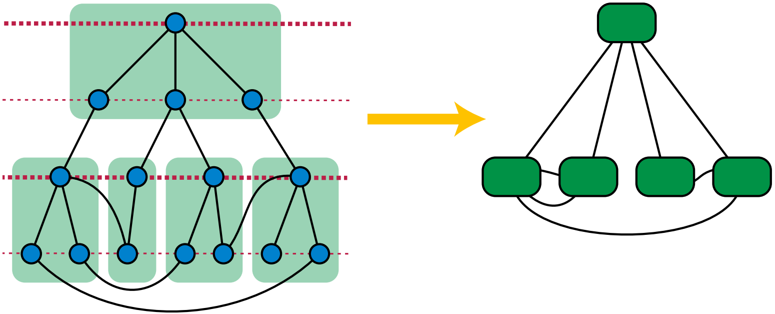
In a 3-regular graph, each vertex deletion removes ≤ 3 edges



If we delete $\frac{m}{6} - k$ vertices, *cyclomatic number* (extra edges beyond a spanning tree) remains $\Omega(k)$, with no short cycles

Densification

Graphs with no short cycles can be made more dense by contracting BFS tree to ancestors on evenly-spaced subset of levels

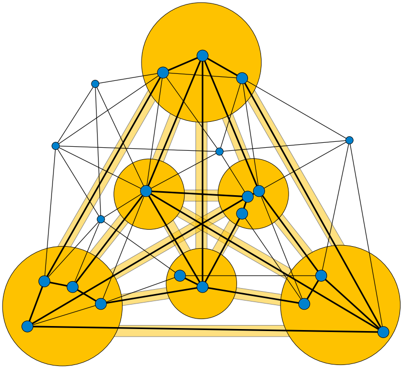


No short cycles ⇒ no self-loops or multiple adjacencies
⇒ cyclomatic number remains unchanged

But #vertices is much smaller (divided by level spacing)

Lower bound

Delete too few vertices ⇒ high cyclomatic # ⇒ dense contraction ⇒ has large clique minors [Thomason 2001] ⇒ nonplanar



To make a planar subgraph, we must reduce the cyclomatic number to $O(n/\log n)$, by deleting $\dfrac{m}{6} - O\left(\dfrac{m}{\log n}\right)$ vertices

Conclusions

Our upper bounds and lower bounds for induced planarization are near each other but with different divisors (5.2174 vs 6).

Can we close this gap?

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