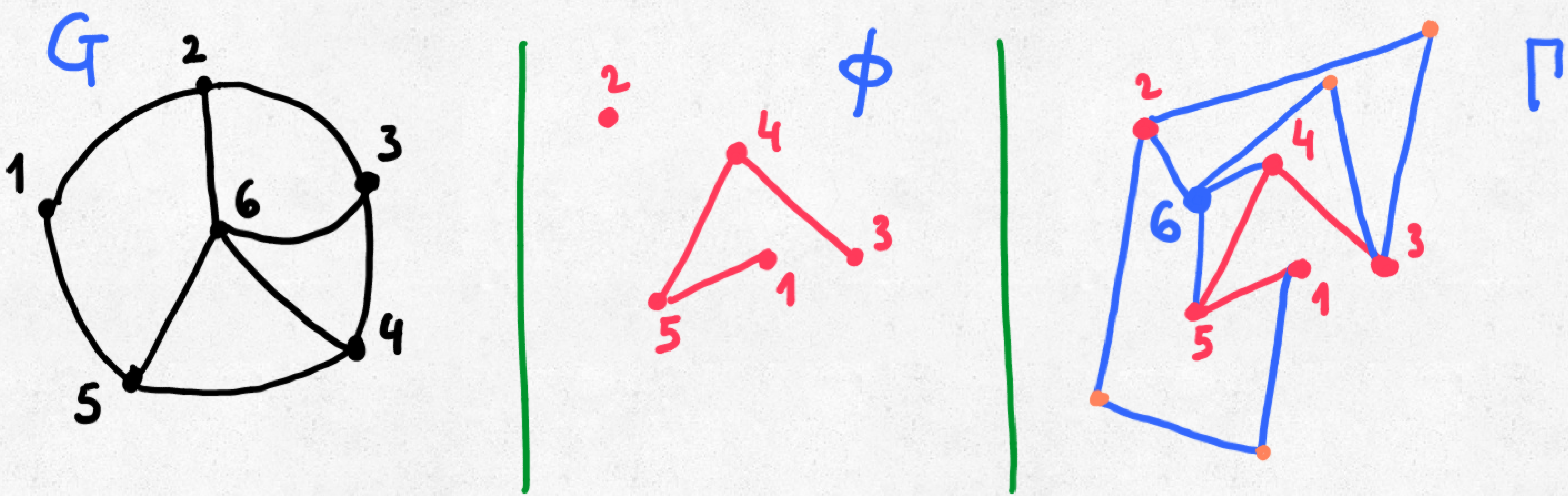
DRAWING PARTIALLY EMBEDDED AND SIMULTANEOUSLY PLANAR GRAPHS

TIMOTHY M. CHAN[1], FABRIZIO FRATI[2], CARSTEN GUTWENGER[3], ANNA LUBIW[1], PETRA MUTZEL[3], MARCUS SCHAEFER[4]

1: UNIVERSITY OF WATERLOO, CANADA
2: THE UNIVERSITY OF SYDNEY, AUSTRALIA
3: TECHNISCE UNIVERSITÄT DORTMUND, GERMANY
4: DE PAUL UNIVERSITY, CHICAGO, USA

THEOREM 1 LET G BE A PLANE GRAPH
LET H BE A SUBGRAPH OF G WITH N VERTICES
LET ϕ BE A PLANAR STRAIGHT-LINE DRAWING OF H

THERE EXISTS A PLANAR STRAIGHT-LINE DRAWING Γ OF G
SUCH THAT Γ EXTENDS ϕ
SUCH THAT EACH EDGE HAS $O(N)$ BENDS



MOTIVATIONS

THE PROBLEM FITS IN THE FOLLOWING COOL TOPICS

1. DRAWING A PLANAR GRAPH WHILE SATISFYING GEOMETRIC OR TOPOLOGICAL CONSTRAINTS (UPWARD PLANARITY [GARG-TAMASSIA 2001], CLUSTERED PLANARITY [FENG-COHEN-EADES 1995], LEVEL PLANARITY [JÜNGER-LEIPERT-MUTZEL 1998], ...)

2. EXTENDING PARTIAL REPRESENTATIONS OF A GRAPH TO COMPLETE ONES (PLANAR GRAPHS [ANGELINI et al. 2010], INTERVAL GRAPHS [KLAVÍK-KRATOCHVÍL-VYSKOCIL 2011], CIRCLE GRAPHS [CHAPLICK-FULEK-KLAVÍK 2013], ...)

WHAT IS KNOWN (1)

THERE EXIST INSTANCES (G, H, ϕ) SUCH THAT
EVERY PLANAR DRAWING OF G THAT EXTENDS ϕ
HAS $\Omega(N)$ EDGES EACH WITH $\Omega(N)$ BENDS

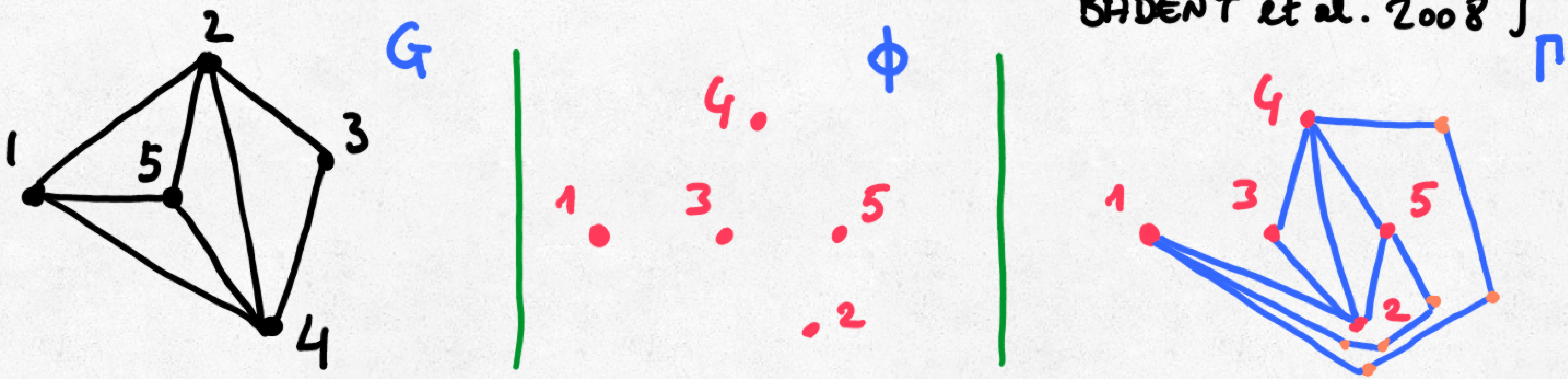
$|G| = |H| = N$

[PACH-WENGER 2001
BADENT et al. 2008]

EVERY PLANE GRAPH G CAN BE DRAWN WITH
ITS VERTICES AT FIXED LOCATIONS
S.T. EACH EDGE HAS $O(N)$ BENDS

$|G| = |H| = N$

[PACH-WENGER 2001
BADENT et al. 2008]



WHAT IS KNOWN (2)

DECIDING WHETHER A STRAIGHT-LINE PLANAR EXTENSION EXISTS
IS NP-HARD [PATRIGNANI 2006]
(EVEN IF G HAS FIXED EMBEDDING)

DECIDING WHETHER A PLANAR EXTENSION EXISTS
IS AN $O(|G|)$-TIME SOLVABLE PROBLEM
[ANGELINI et al. 2010]

WHAT IS KNOWN (3)

CLAIM [FOWLER et al. 2011]

LET G BE A PLANE GRAPH WITH N VERTICES

LET H BE A SPANNING SUBGRAPH OF G

LET ϕ BE A PLANAR STRAIGHT-LINE DRAWING OF H

LET σ BE AN ORDER OF THE EDGES IN $G-H$ SUCH THAT EDGES IN DISTINCT COMPONENTS COME FIRST

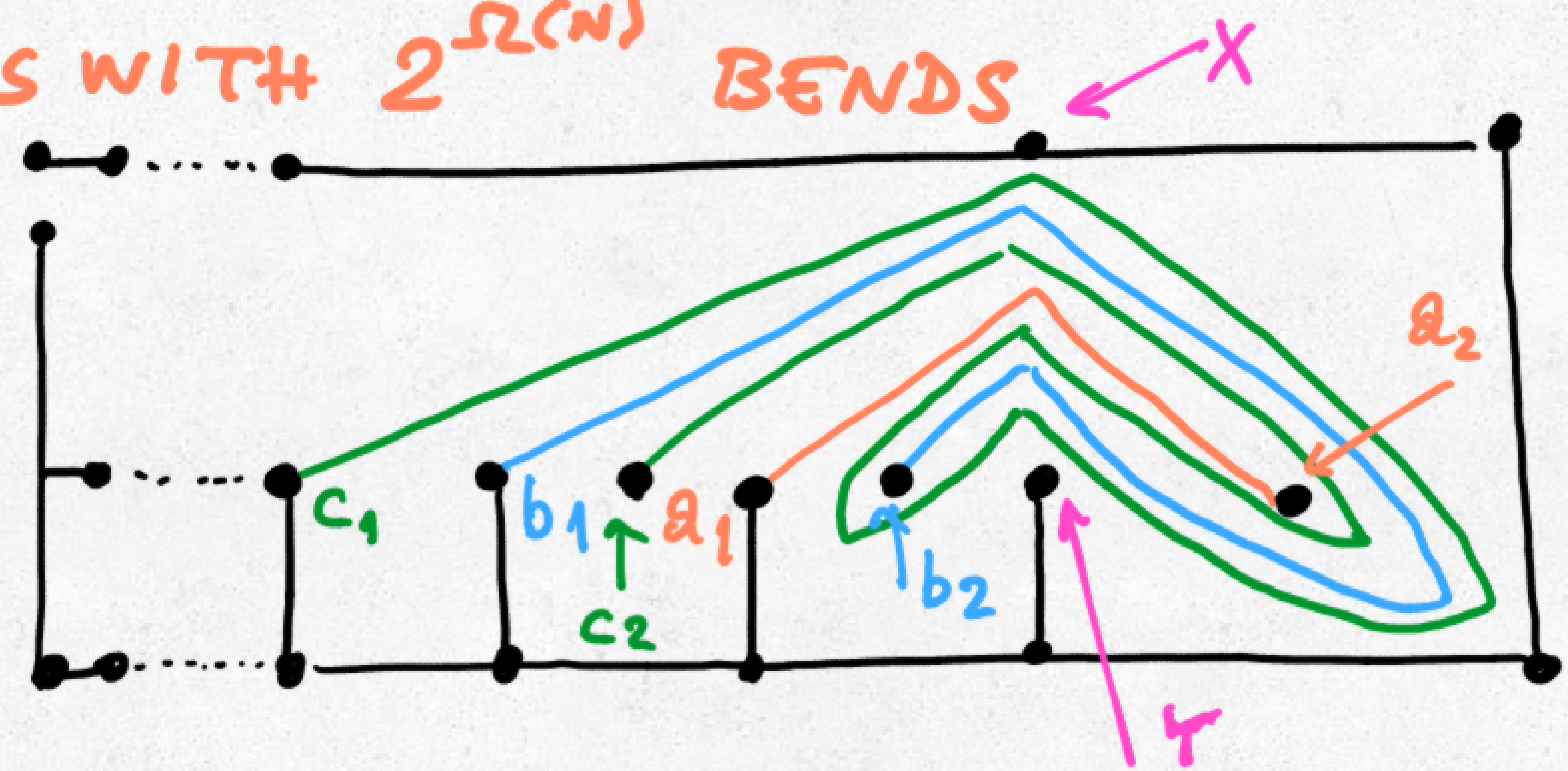
THE PLANAR STRAIGHT-LINE DRAWING Γ OF G EXTENDING ϕ OBTAINED BY DRAWING THE EDGES IN $G-H$ ONE BY ONE IN THE ORDERED GIVEN BY σ AS PATHS WITH THE MINIMUM NUMBER OF BENDS IS SUCH THAT EACH EDGE HAS $O(N)$ BENDS.

THEOREM 2 THERE EXISTS A TREE G WITH N VERTICES A DRAWING Φ OF ITS VERTEX SET H AND AN ORDER σ OF THE EDGES IN G SUCH THAT

THE PLANAR STRAIGHT-LINE DRAWING Γ OF G EXTENDING Φ OBTAINED BY DRAWING THE EDGES IN G ONE BY ONE IN THE ORDERED GIVEN BY σ AS PATHS WITH THE MINIMUM NUMBER OF BENDS IS SUCH THAT

THERE EXIST EDGES WITH $2^{\Omega(N)}$ BENDS

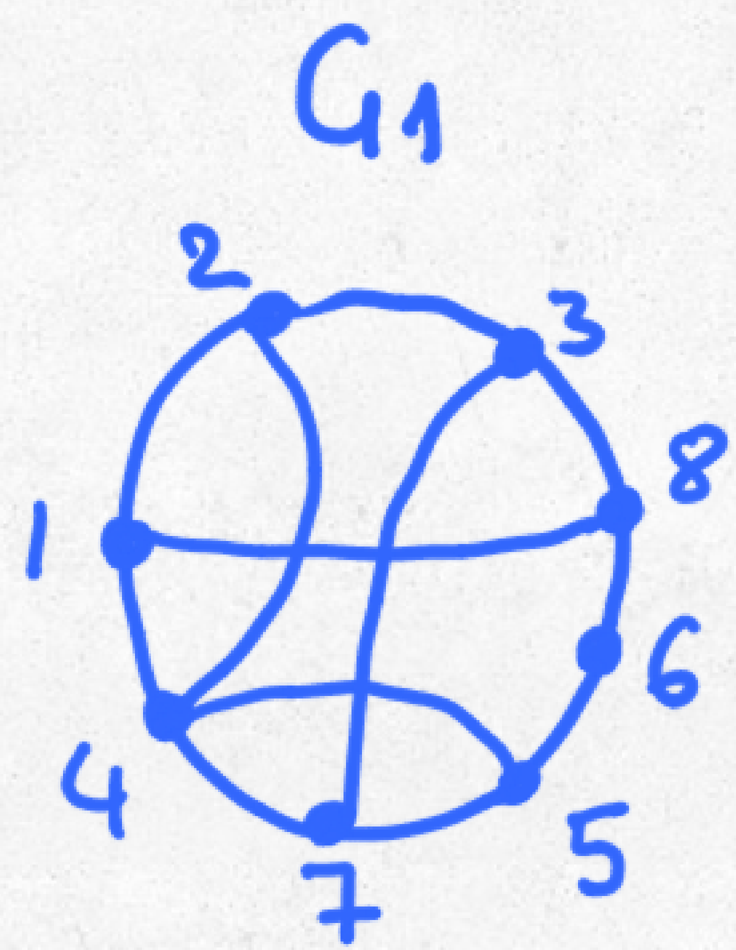
SIMILAR TO [KRATOCHVIL-MATOUSEK 1991]

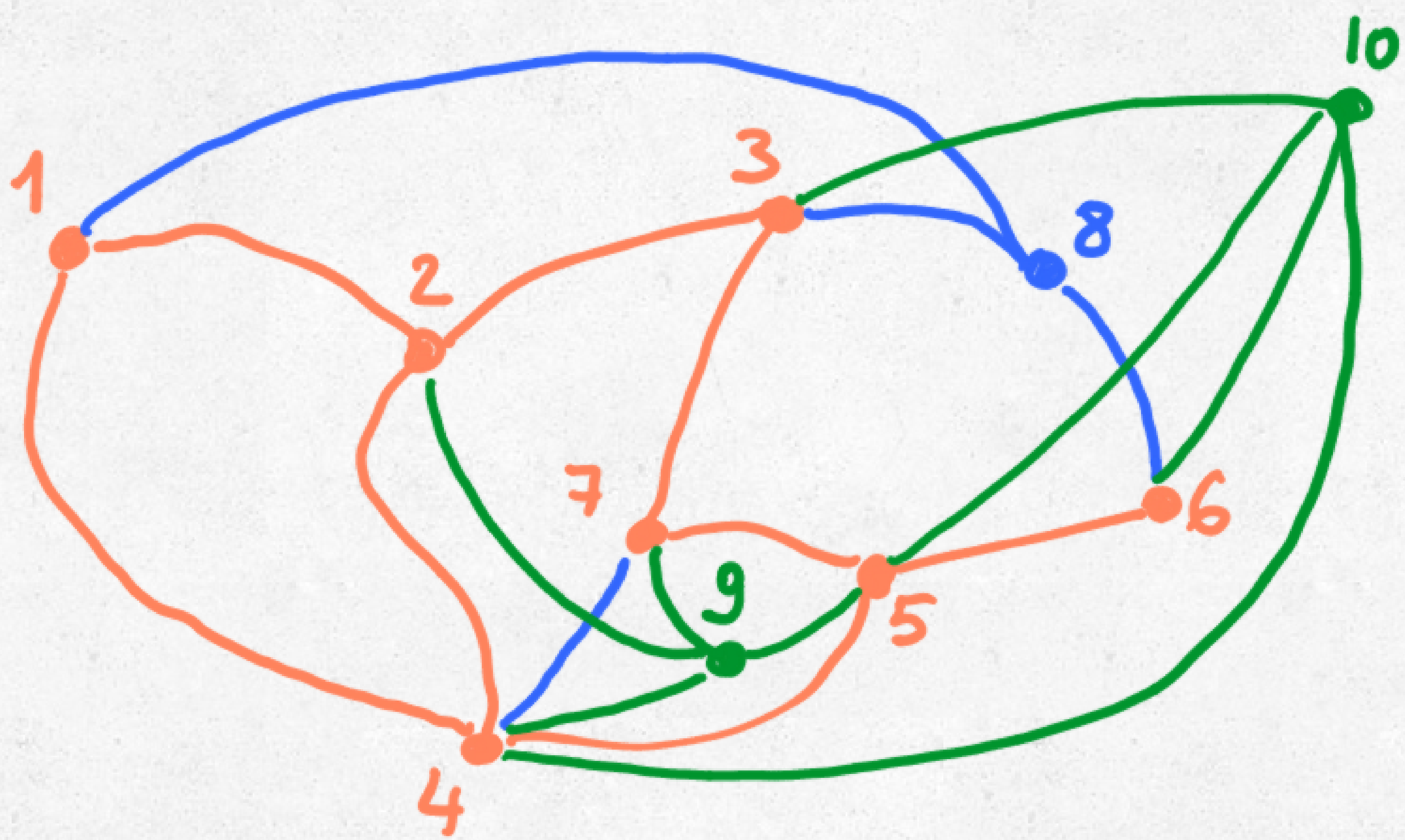


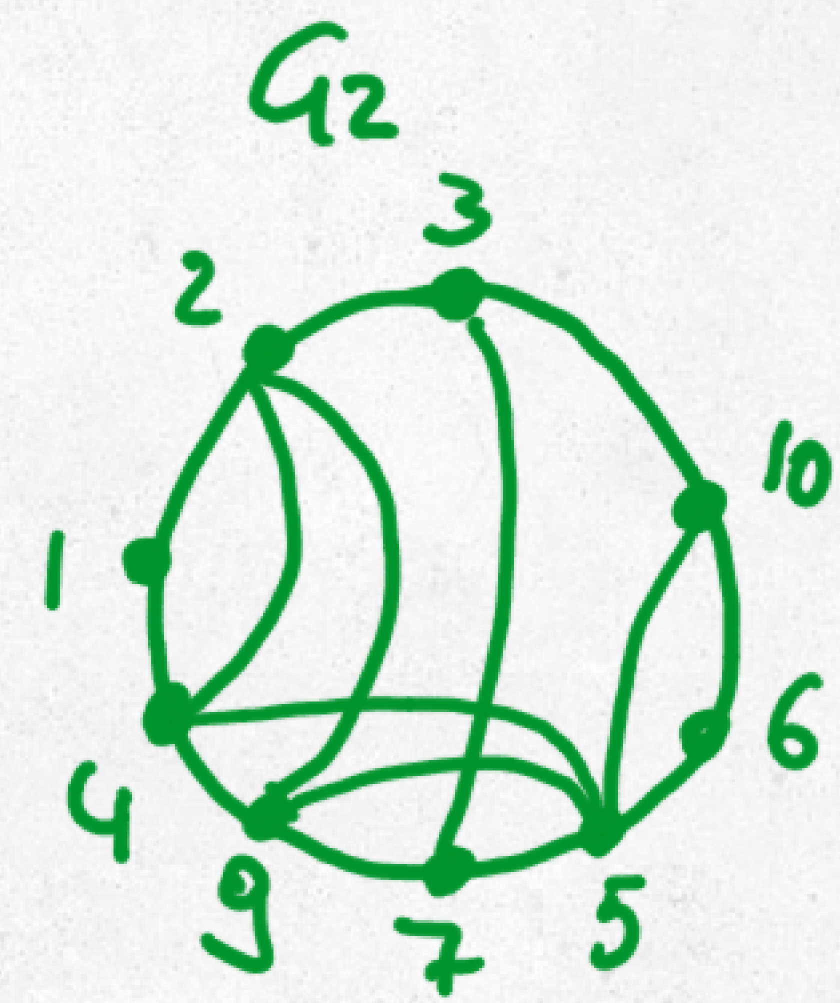
SIMULTANEOUS PLANARITY (SEFE)

LET G_1 AND G_2 BE TWO GRAPHS THAT SHARE A COMMON SUBGRAPH G.

A SIMULTANEOUS PLANAR DRAWING OF G_1 AND G_2 CONSISTS OF A PLANAR DRAWING Γ_1 OF G_1 AND A PLANAR DRAWING Γ_2 OF G_2 THAT COINCIDE ON G.







THEOREM 3 LET G_1 AND G_2 BE TWO SIMULTANEOUSLY PLANAR GRAPHS WITH A TOTAL OF N VERTICES THAT SHARE A COMMON SUBGRAPH G.

THERE EXISTS A SIMULTANEOUS PLANAR DRAWING OF G_1 AND G_2 WHERE ANY EDGE OF $G_1 - G$ AND ANY EDGE OF $G_2 - G$ INTERSECT AT MOST 24 TIMES AND ARBITRARILY SATISFYING ONE OF THE FOLLOWING:

1. EACH EDGE OF G_1 IS STRAIGHT AND EACH EDGE OF G_2 HAS $O(N)$ BENDS.

2. EACH EDGE OF G IS STRAIGHT, EACH EDGE OF $(G_1 - G) \cup (G_2 - G)$ HAS $O(N)$ BENDS, AND VERTICES, BENDS, AND CROSSINGS LIE IN A $O(N^2) \times O(N^2)$ GRID.

WHAT IS KNOWN (1)

IF THE COMMON SUBGRAPH G HAS NO EDGES

3 BENDS PER EDGE ON AN $O(n^3) \times O(n^3)$ GRID [ERTEN-KOBOUROV 2005]

2 BENDS PER EDGE (WITH AN EXPONENTIAL INCREASE IN THE AREA)
[ERTEN-KOBOUROV 2005] PLUS [KAUFMANN-WIESE 2002]

WHAT IS KNOWN (2)

IF THE COMMON SUBGRAPH G IS CONNECTED

$O(n)$ BENDS PER EDGE, AT MOST 1 CROSSING PER PAIR OF EDGES, AND VERTICES, BENDS, AND CROSSINGS LIE ON AN $O(n^2) \times O(n^2)$ GRID [HAUPLER et al. 2013]

WHAT IS KNOWN (3)

AT MOST 9 BENDS PER EDGE, EACH EDGE OF G IS STRAIGHT
[GRILLI et al. NEXT TALK !!]

WHAT IS NOT KNOWN

DETERMINING THE TIME COMPLEXITY OF TESTING SIMULTANEOUS PLANARITY IS A LONG-STANDING OPEN PROBLEM

[E.G. SURVEY BY BLÄSIUS-KOBOUROV-RUTTER IN THE HANDBOOK OF GRAPH DRAWING AND VISUALIZATION. CRC PRESS 2013]

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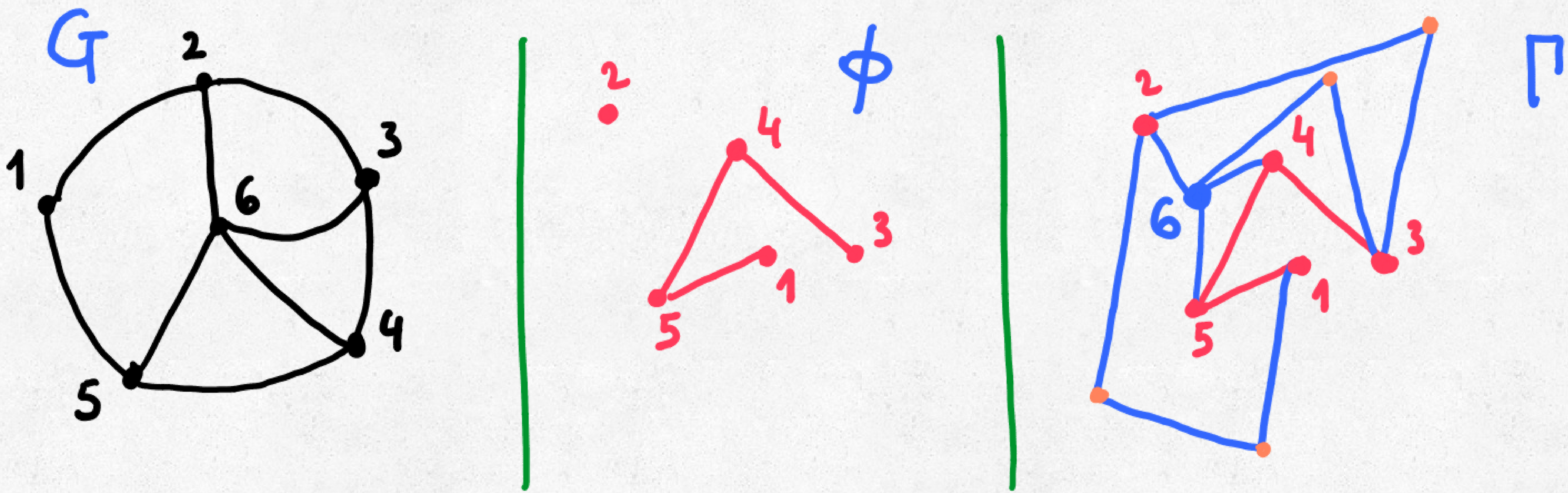
THERE EXISTS A PLANAR STRAIGHT-LINE DRAWING Γ OF G

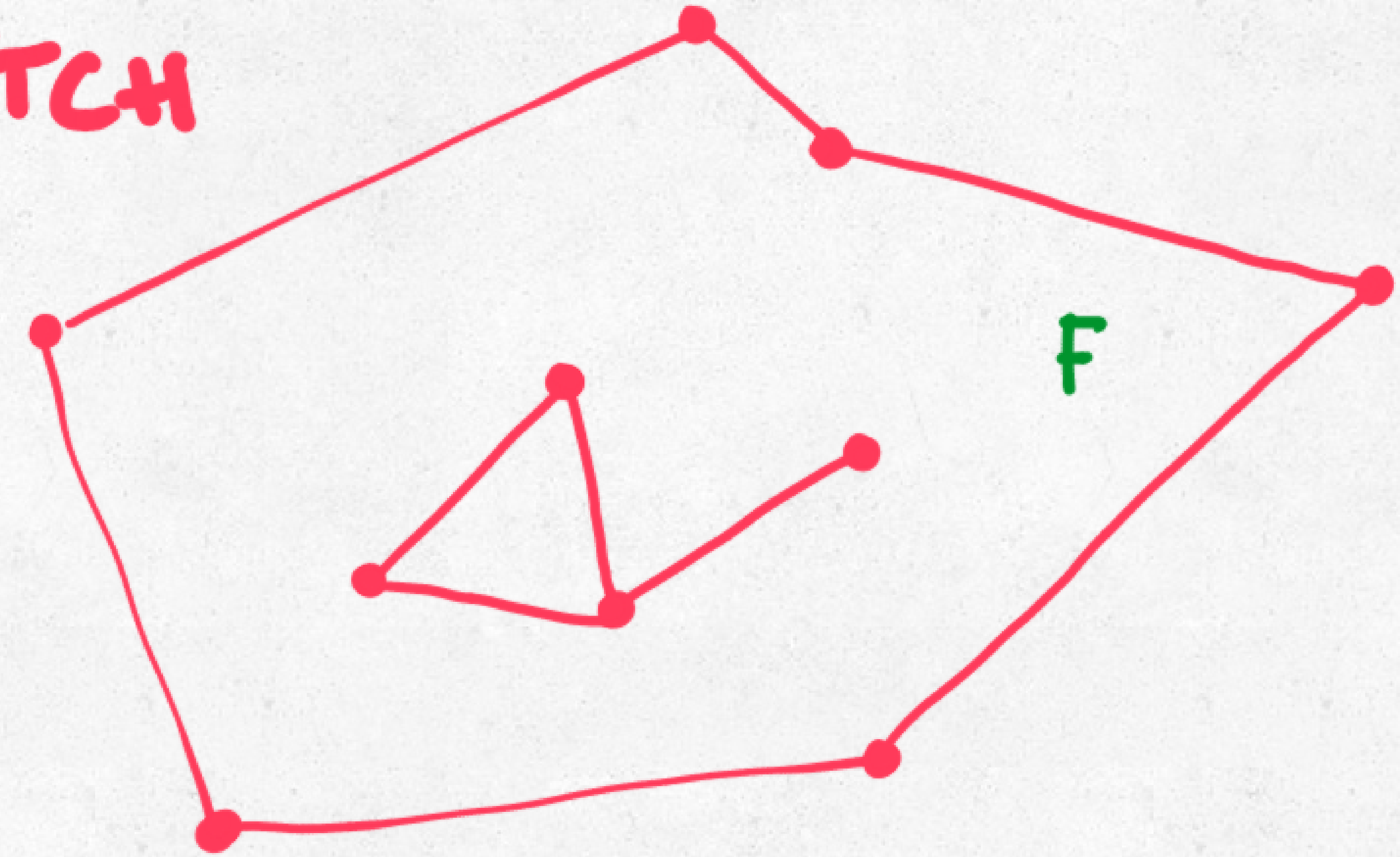
SUCH THAT Γ EXTENDS ϕ

SUCH THAT EACH EDGE HAS $O(N)$ BENDS

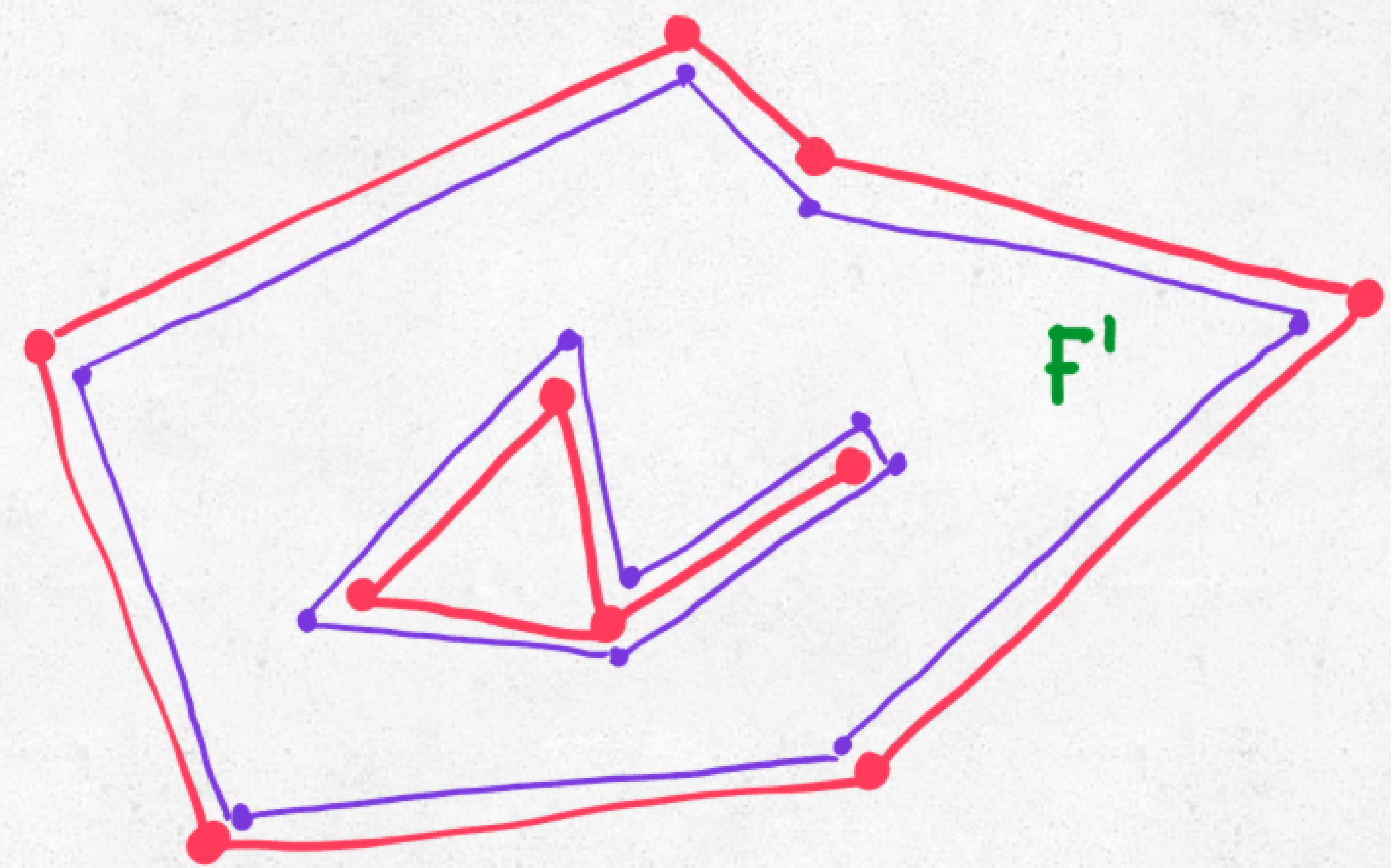


PROOF SKETCH

ONE FACE F OF ϕ AT A TIME



COMPUTE AN INNER ε-APPROXIMATION F' OF F



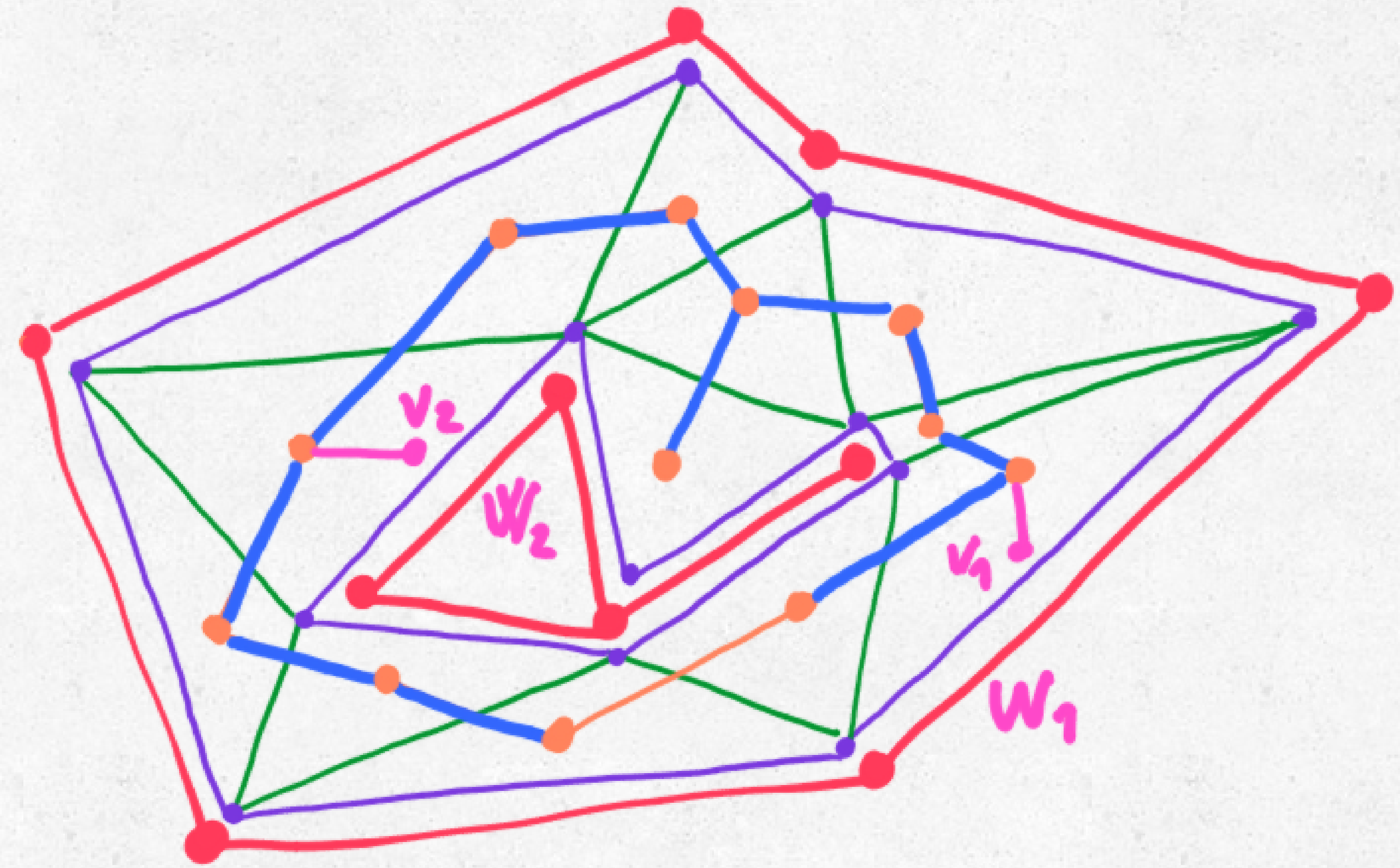
$\forall p \in F \; , \exists \; p' \in F' \text{ s.t. } d(p,p') < \varepsilon$

AND

$\exists \delta : \forall p' \in F' , \forall p \text{ s.t. } d(p,p') < \delta$

IT HOLDS $p \in F$

TRIANGULATE F' INTO A GRAPH $K_{F'}$



CONSTRUCT A GEOMETRIC DUAL $D_{F'}$ OF $K_{F'}$ [BERN-GILBERT 1992]

COMPUTE A SPANNING TREE T OF $D_{F'}$

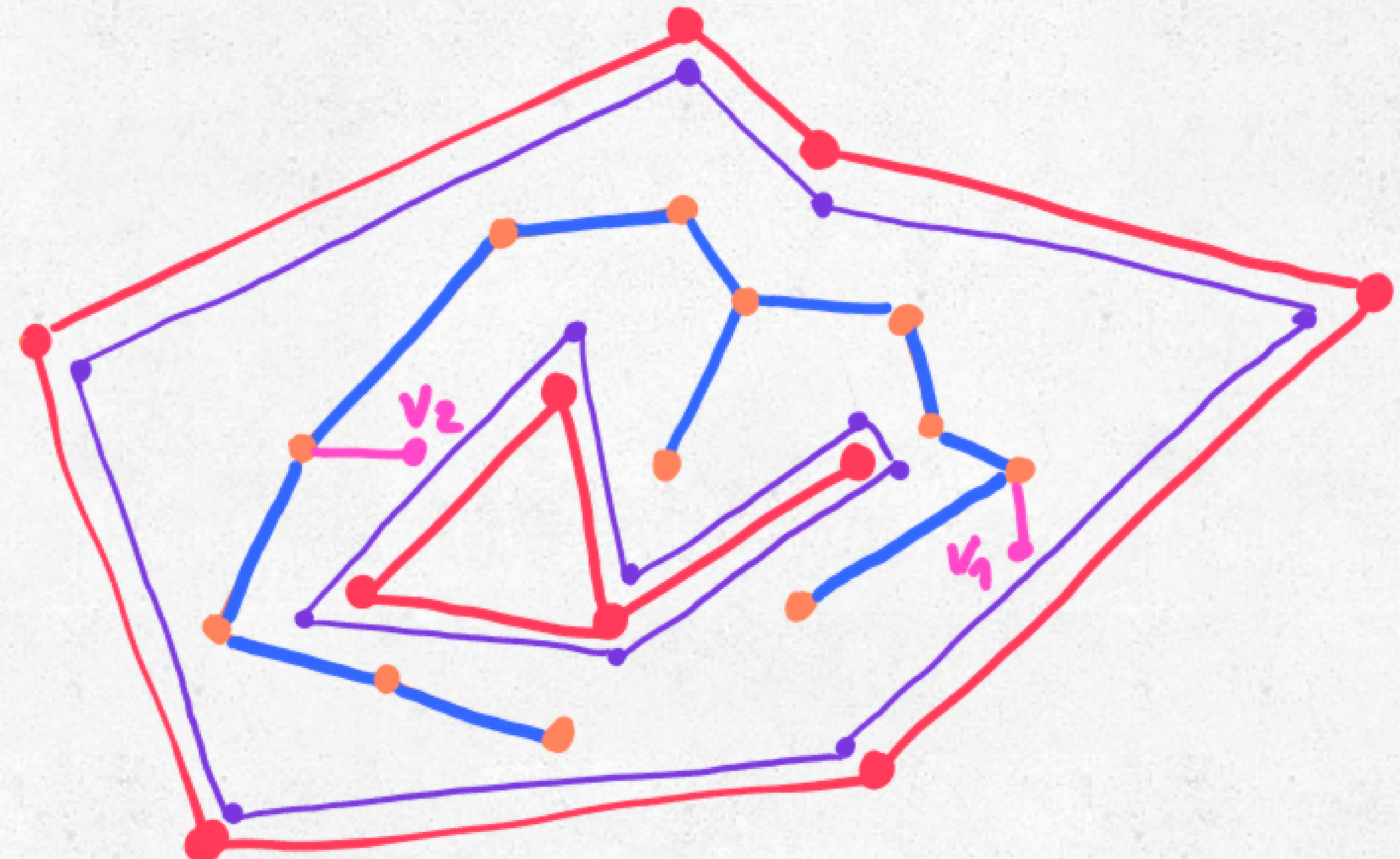
OBSERVATION:
$|T| \in O(n)$

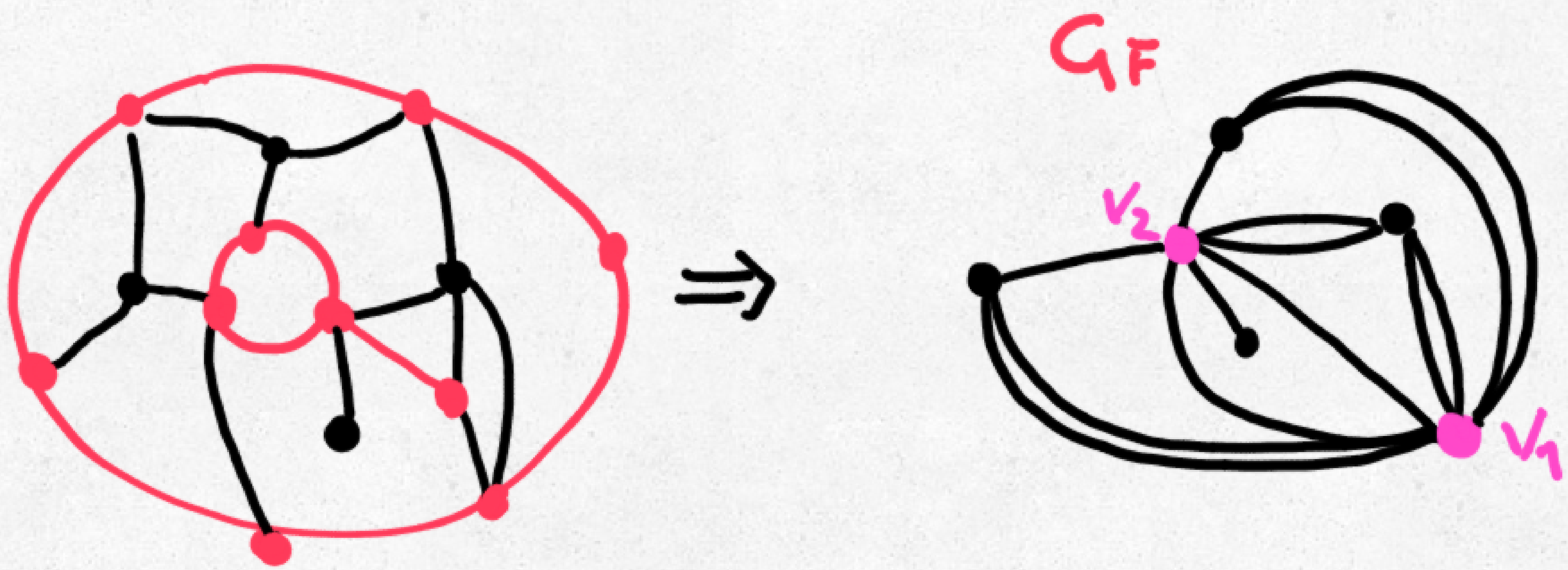
FOR EACH FACIAL WALK W_i OF F, WE AUGMENT T WITH A LEAF V_i CLOSE TO W_i IN F'

DENOTE BY G_F THE EMBEDDED MULTIGRAPH OBTAINED BY

1) RESTRICTING G TO THE VERTICES AND EDGES INSIDE F OR INCIDENT TO F

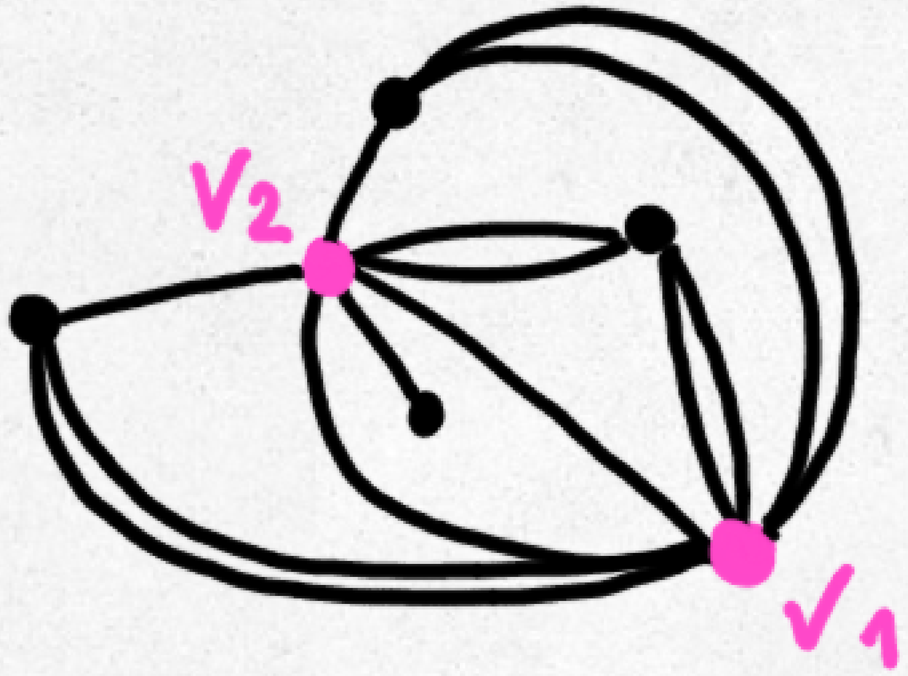
2) CONTRACTING EACH FACIAL WALK W_i OF F TO A SINGLE VERTEX V_i



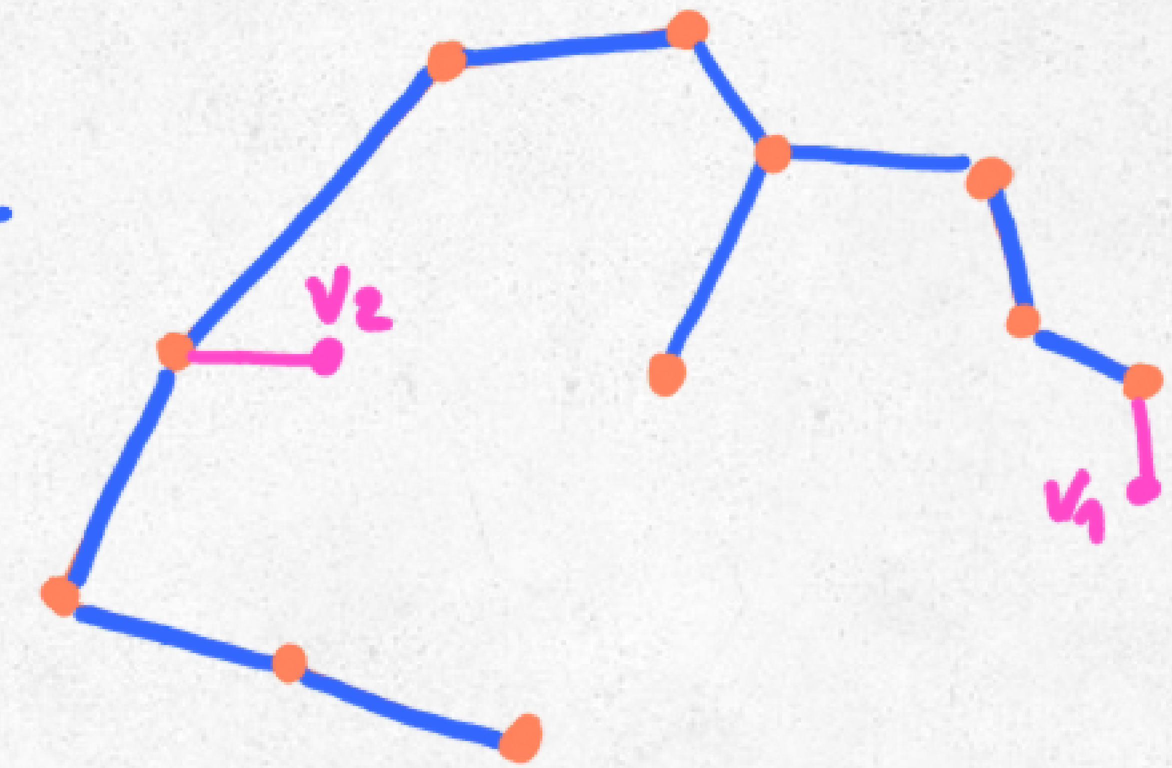


DRAW G_F AROUND T

WE DRAW G_F



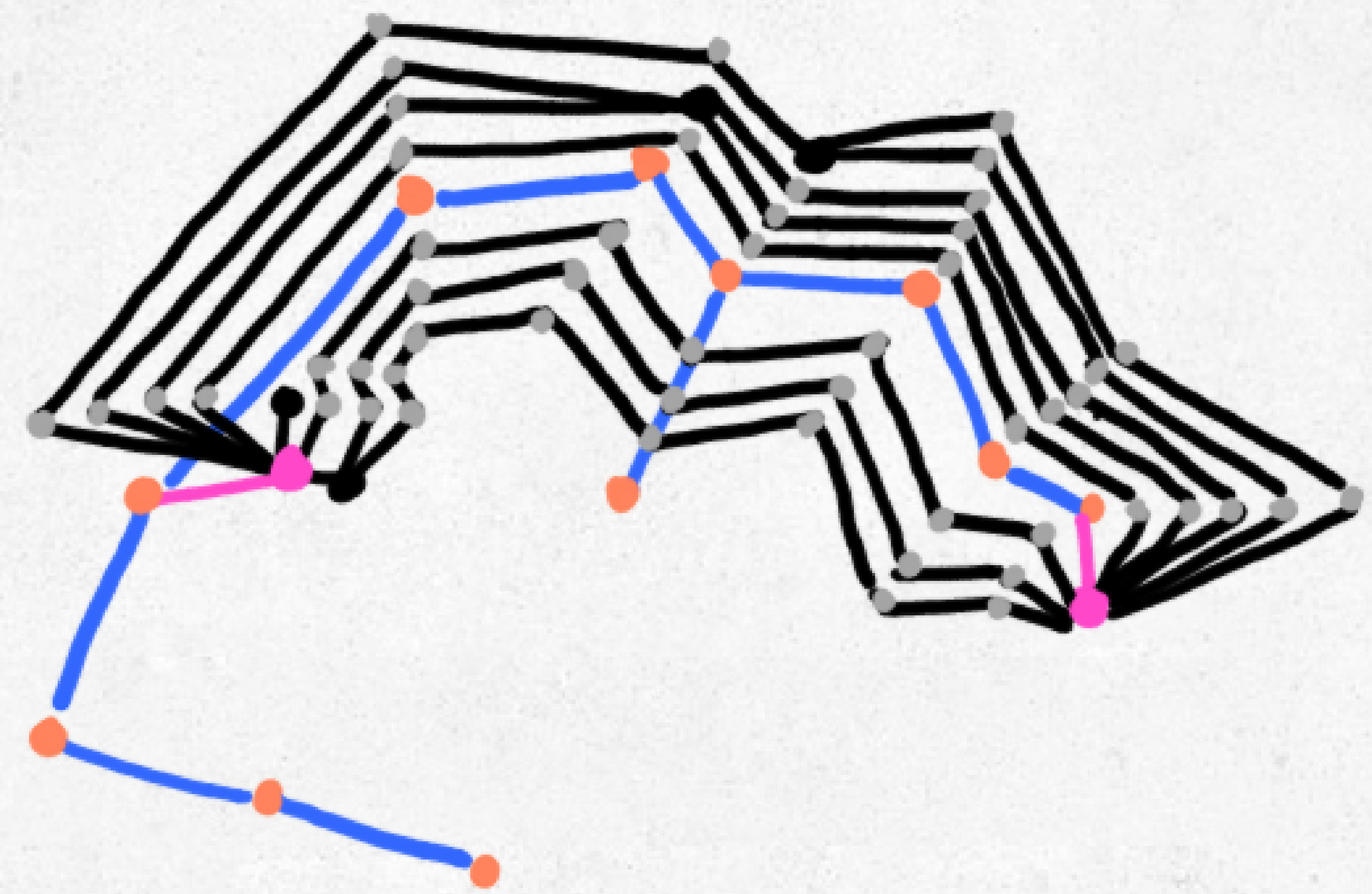
"ε-CLOSE" TO T



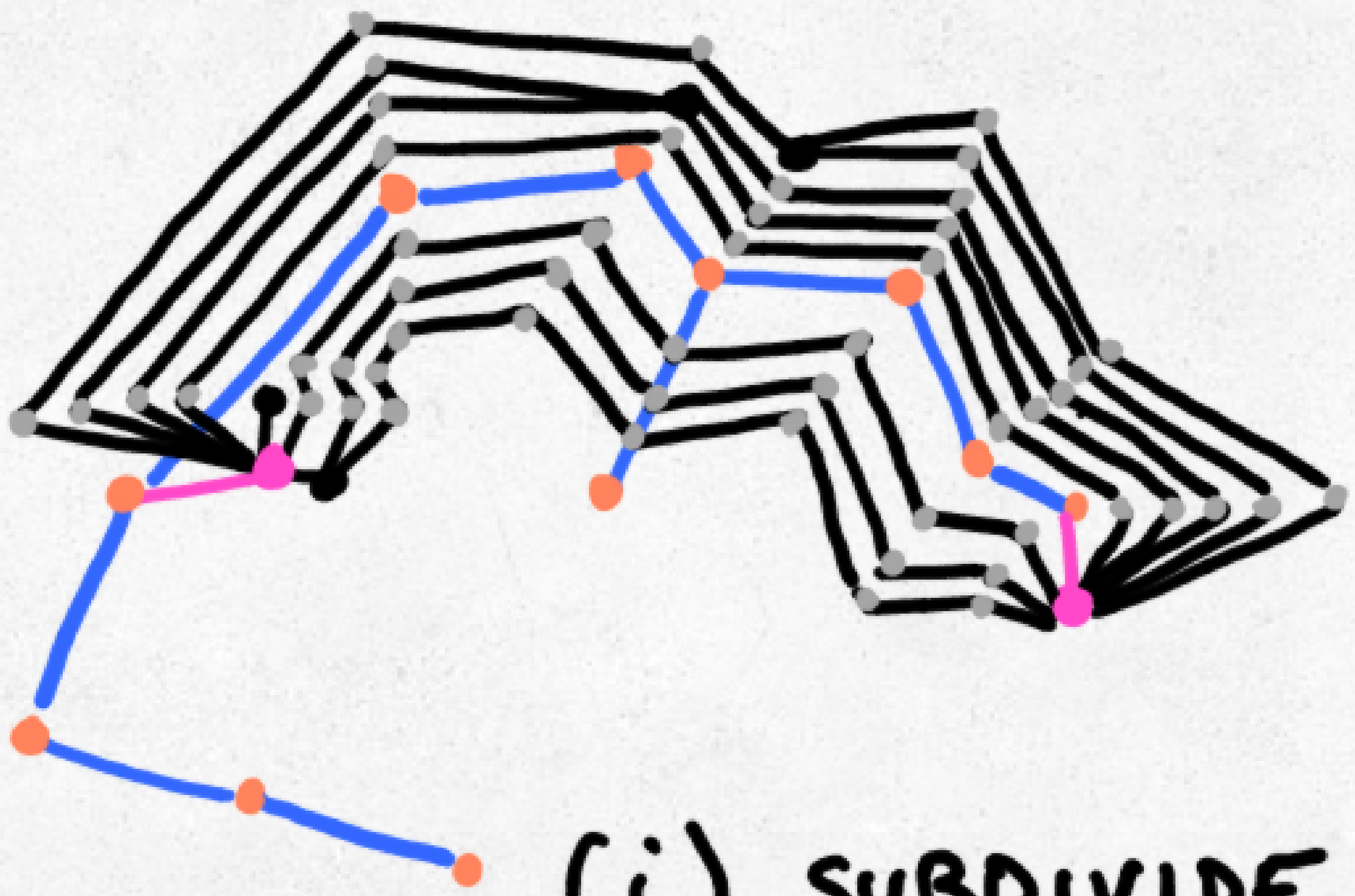
SO THAT $v_1, v_2, \dots, v_k$ ARE DRAWN AT THE POINTS THEY ARE DRAWN IN T,

SO THAT G_F MAINTAINS ITS COMBINATORIAL EMBEDDING,

SO THAT EACH EDGE HAS $O(|T|)$ BENDS, AND



SO THAT EACH EDGE "COMES CLOSE" TO EACH VERTEX v_i $O(1)$ TIMES.



ONLY VERTICES $V_1, V_2, \dots, V_K$ (AND NO EDGES) OF G_F ARE ALREADY DRAWN IN T

$\Downarrow$

SETTING SIMILAR TO [PACH-WENGER 2001]

(i) SUBDIVIDE AND AUGMENT G_F SO THAT IT IS HAMILTONIAN

(ii) DRAW THE HAMILTONIAN CYCLE

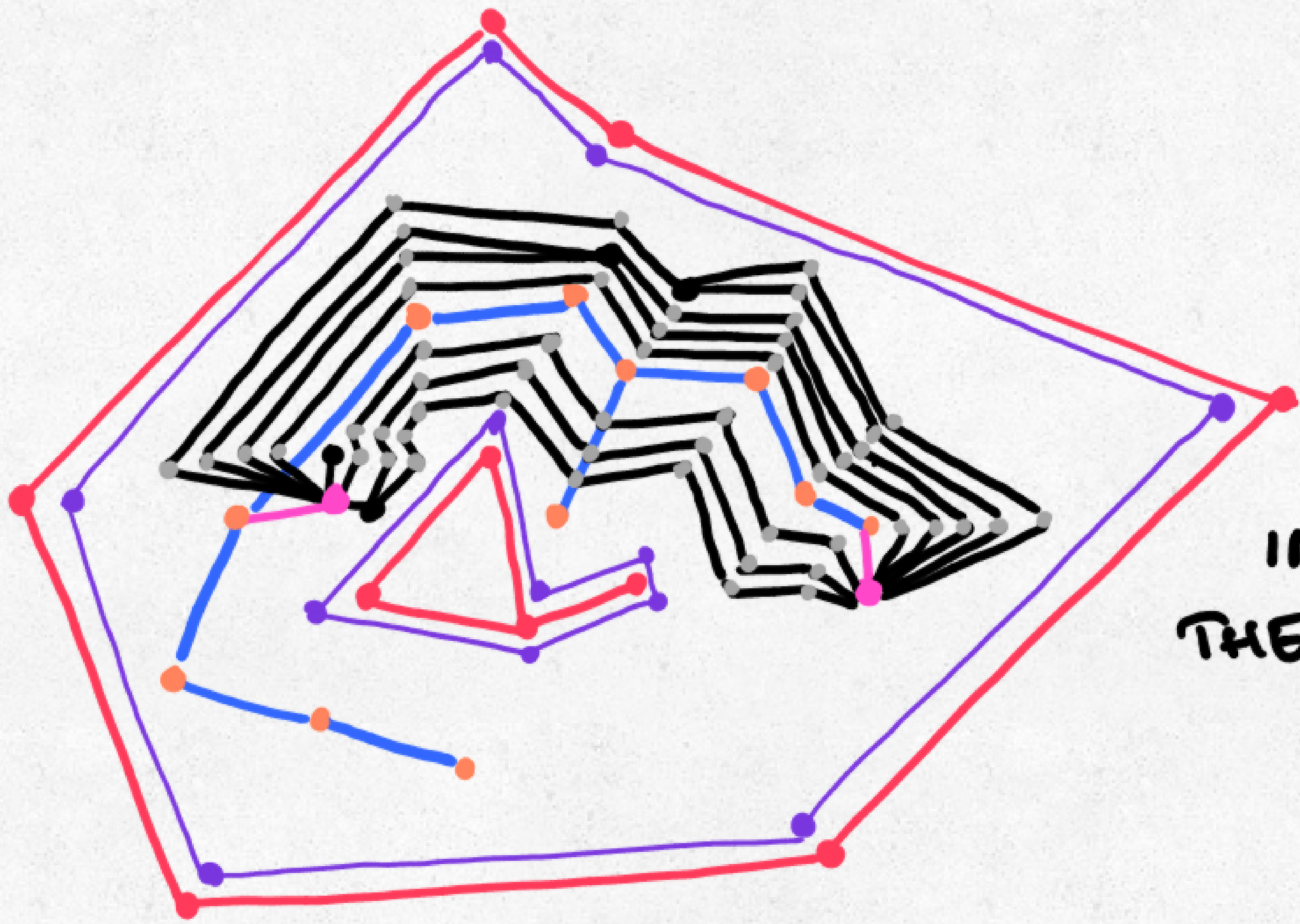
EACH EDGE (V_J, V_{J+1}) FOLLOWS THE BOUNDARY OF THE SUBTREE OF T WHOSE LEAVES ARE $V_1, V_2, \dots, V_{J+1}$

(iii) DRAW THE REMAINING EDGES

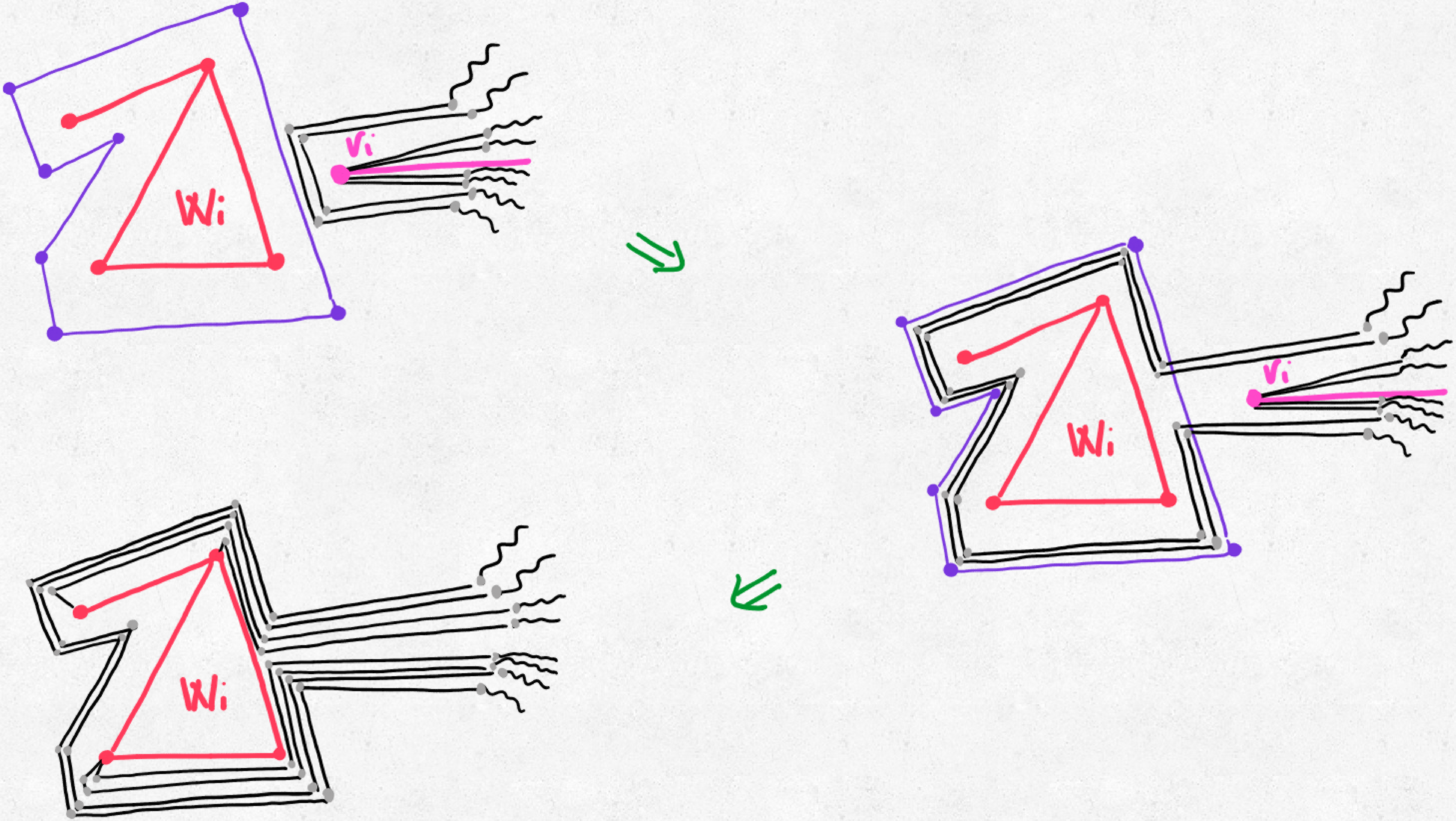
EACH EDGE (V_i, V_J) IS COMPOSED OF TWO PARTS, ONE CONNECTING V_i WITH A POINT P_{iJ} CLOSE TO V_1, ONE CONNECTING V_J WITH P_{iJ}.

T IS NOT A STAR!

NOT ALL THE VERTICES OF G_F ARE ALREADY DRAWN AS LEAVES OF T!



NEED TO REROUTE THE EDGES INCIDENT TO V_i OR PASSING CLOSE TO V_i SO THAT THEY ARE INCIDENT TO A VERTEX OF W_i OR THEY PASS CLOSE TO W_i


Wi
vi
Wi
vi
Wi

HOW MANY BENDS?

AFTER DRAWING GRAPH G_F AROUND T, EACH EDGE HAS $O(|T|) \in O(N)$ BENDS.

SINCE EACH EDGE OF G_F COMES CLOSE TO EACH V_i $O(1)$ TIMES, REROUTING EDGES AROUND FACIAL WALKS W_i ADDS $O(\Sigma |W_i|) \in O(N)$ BENDS TO EACH EDGE.

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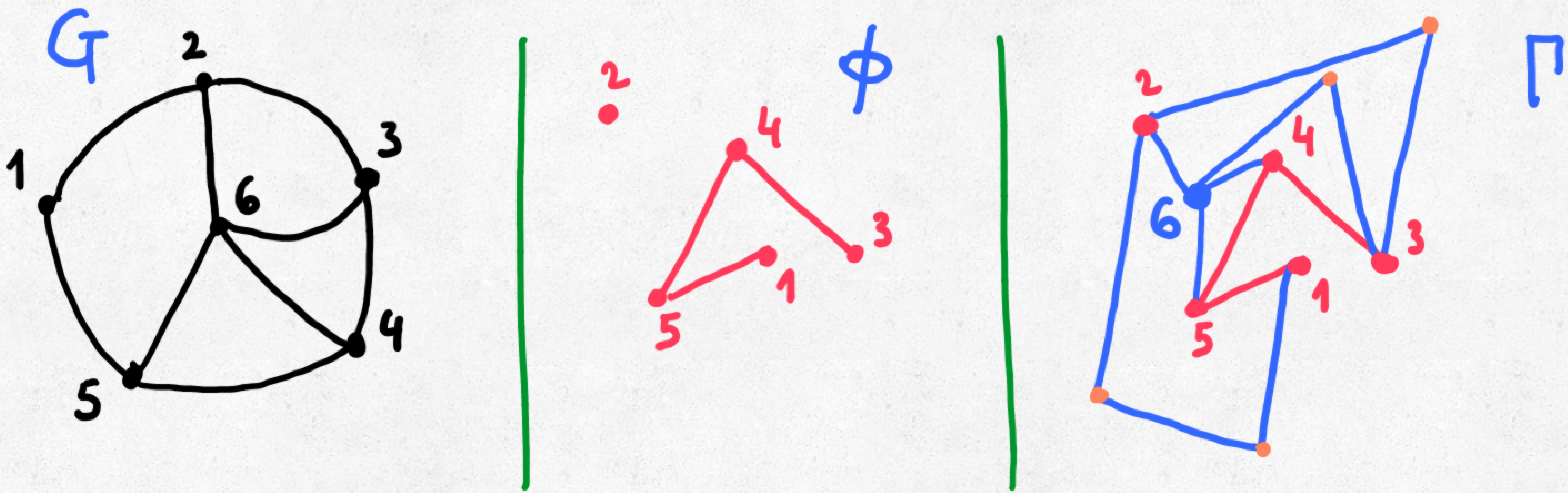

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PROOF SKETCH

CONSTRUCT ANY STRAIGHT-LINE PLANAR DRAWING OF G_1. WITH A SUITABLE EMBEDDING

THIS INDUCES A STRAIGHT-LINE PLANAR DRAWING Φ OF G.

WE CONSTRUCT A STRAIGHT-LINE PLANAR DRAWING OF G_2 WITH A SUITABLE EMBEDDING
BY APPLYING THE PARTIAL DRAWING EXTENSION ALGORITHM (THM 1).

FOR THIS SAKE, WE USE ALL THE VERTICES AND EDGES OF G_1
IN ORDER TO TRIANGULATE THE INTERIOR OF EVERY FACE.

OPEN PROBLEMS

REDUCE THE CONSTANTS! (IN OUR PARTIAL DRAWING EXTENSION RESULT, EACH EDGE HAS AT MOST 102N BENDS)

DO SIMULTANEOUS PLANAR DRAWINGS ALWAYS EXIST S.T. ANY TWO EDGES INTERSECT AT MOST TWICE? (ONCE MIGHT NOT BE ENOUGH)

↳ BEST KNOWN UPPER BOUND IS 24.

THANKS!