Planar and Quasi Planar Simultaneous Geometric Embedding

Emilio Di Giacomo[1], Walter Didimo[1], Giuseppe Liotta[1], Henk Meijer[2], Stephen Wismath[3]

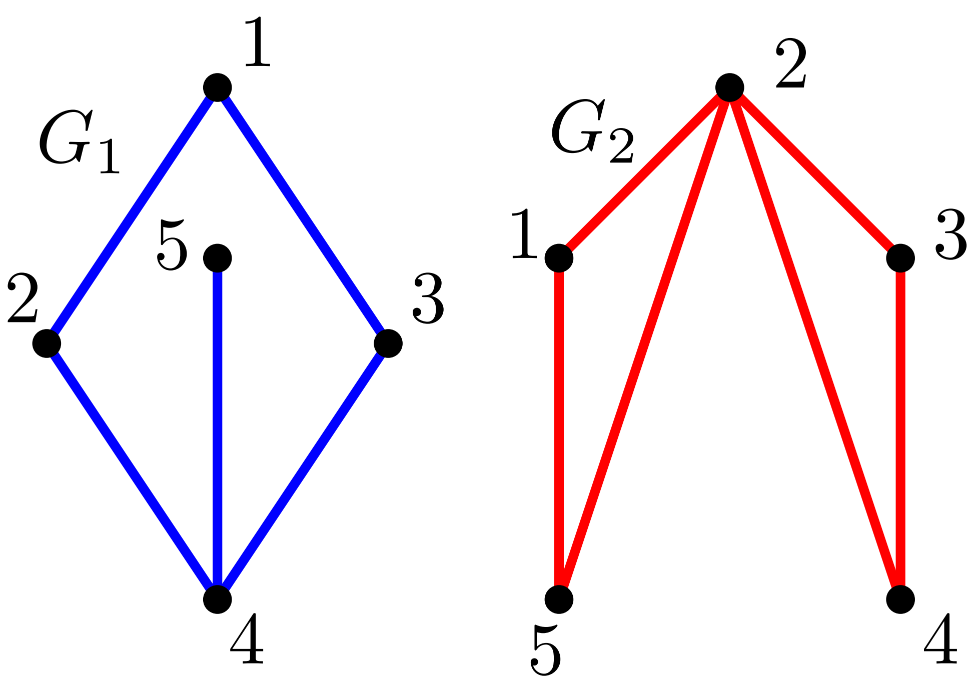
[1]Università degli Studi di Perugia, Italy
[2]University College Roosevelt, the Netherlands
[3]University of Lethbridge, Canada

Simultaneous Geometric Embedding (SGE)

- Let $\langle G_1 = (V, E_1), G_2 = (V, E_2) \rangle$ be a pair of planar graphs with the same vertex set.
- A *simultaneous geometric embedding* (*SGE*) of $\langle G_1, G_2 \rangle$ is a pair of drawings $\langle \Gamma_1, \Gamma_2 \rangle$ such that:
  - Γ_i is a planar straight-line drawing of G_i for $i = 1, 2$;
  - each vertex $v \in V$ is represented by the same point in Γ_1 and Γ_2.

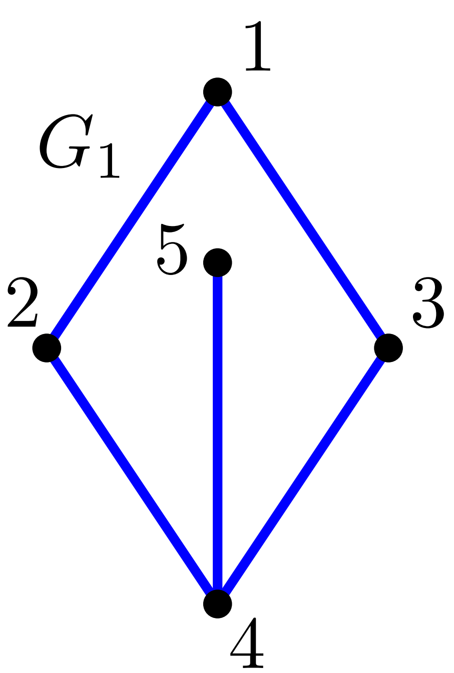
Simultaneous Geometric Embedding (SGE)

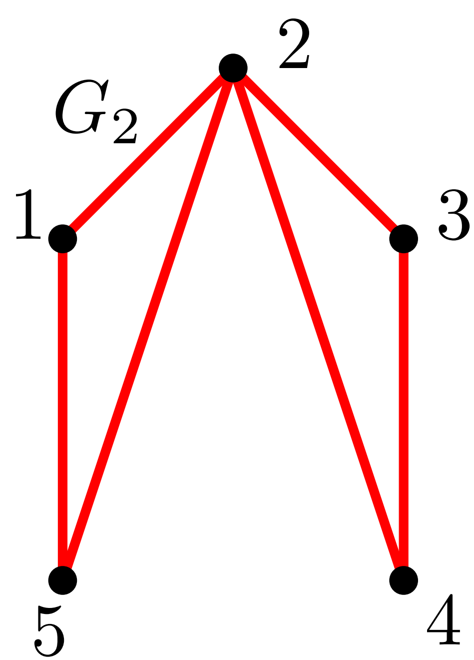
- Let $\langle G_1 = (V, E_1), G_2 = (V, E_2)\rangle$ be a pair of planar graphs with the same vertex set.
- A *simultaneous geometric embedding* (*SGE*) of $\langle G_1, G_2\rangle$ is a pair of drawings $\langle \Gamma_1, \Gamma_2\rangle$ such that:
  - Γ_i is a planar straight-line drawing of G_i for $i = 1, 2$;
  - each vertex $v \in V$ is represented by the same point in Γ_1 and Γ_2.

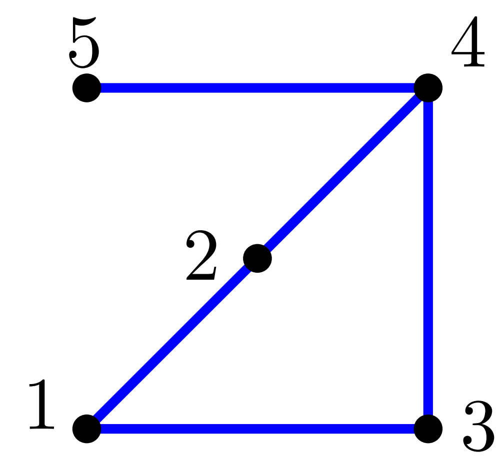


Simultaneous Geometric Embedding (SGE)

- Let $\langle G_1 = (V, E_1), G_2 = (V, E_2) \rangle$ be a pair of planar graphs with the same vertex set.
- A *simultaneous geometric embedding* (*SGE*) of $\langle G_1, G_2 \rangle$ is a pair of drawings $\langle \Gamma_1, \Gamma_2 \rangle$ such that:
  - Γ_i is a planar straight-line drawing of G_i for $i = 1, 2$;
  - each vertex $v \in V$ is represented by the same point in Γ_1 and Γ_2.

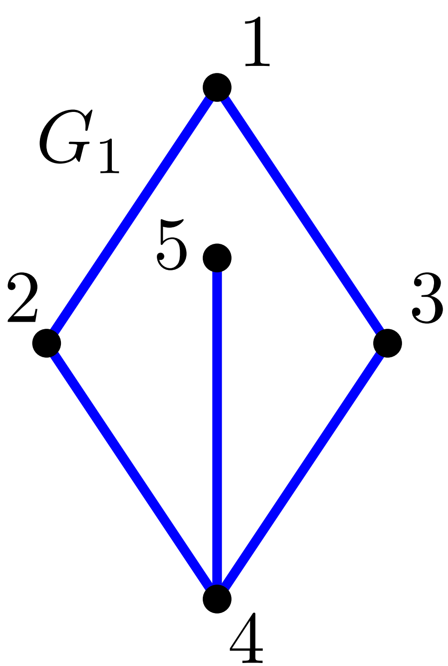


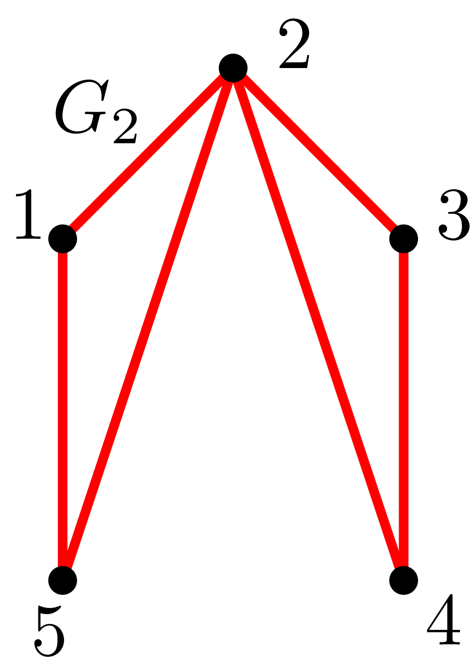


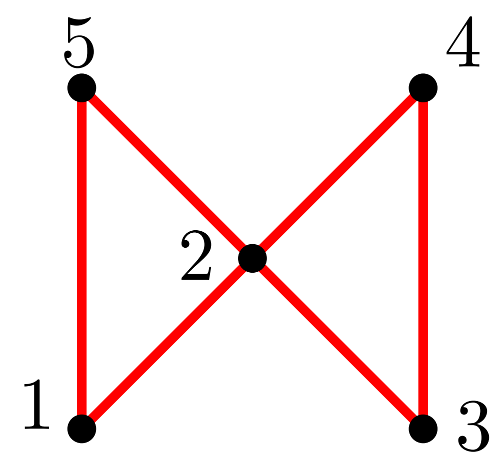


Simultaneous Geometric Embedding (SGE)

- Let $\langle G_1 = (V, E_1), G_2 = (V, E_2)\rangle$ be a pair of planar graphs with the same vertex set.
- A *simultaneous geometric embedding* (*SGE*) of $\langle G_1, G_2\rangle$ is a pair of drawings $\langle \Gamma_1, \Gamma_2\rangle$ such that:
  - Γ_i is a planar straight-line drawing of G_i for $i = 1, 2$;
  - each vertex $v \in V$ is represented by the same point in Γ_1 and Γ_2.

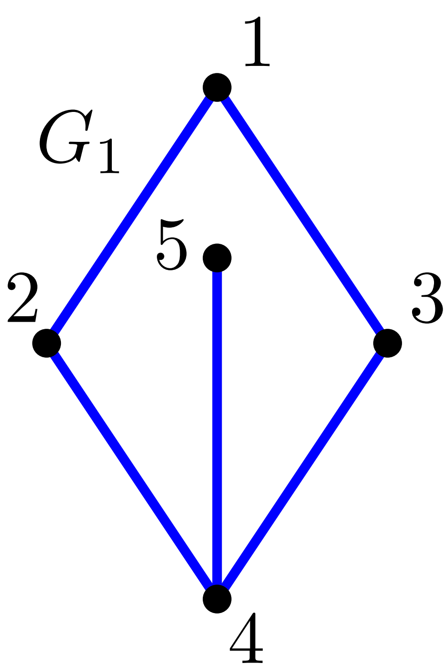


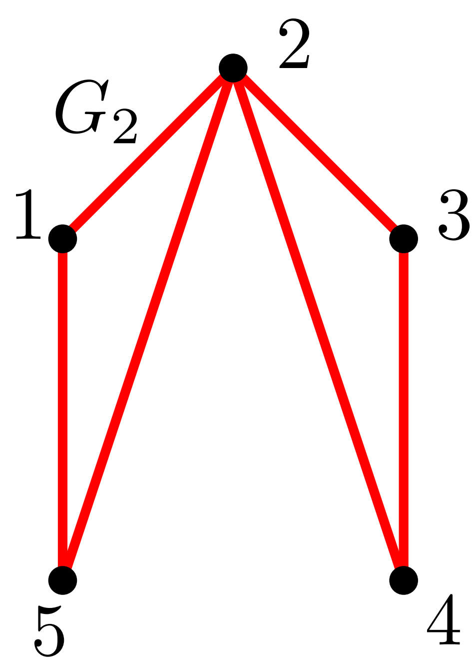


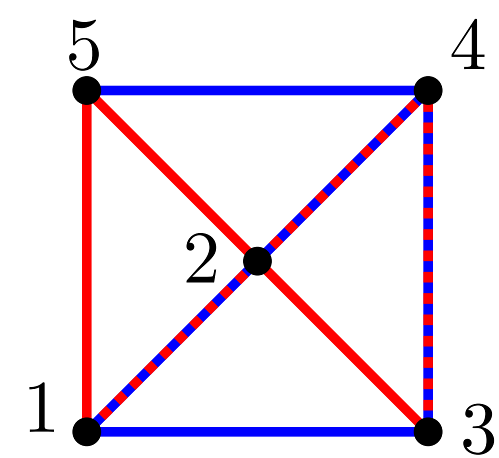


Simultaneous Geometric Embedding (SGE)

- Let $\langle G_1 = (V, E_1), G_2 = (V, E_2)\rangle$ be a pair of planar graphs with the same vertex set.
- A *simultaneous geometric embedding* (*SGE*) of $\langle G_1, G_2\rangle$ is a pair of drawings $\langle \Gamma_1, \Gamma_2\rangle$ such that:
  - Γ_i is a planar straight-line drawing of G_i for $i = 1, 2$;
  - each vertex $v \in V$ is represented by the same point in Γ_1 and Γ_2.





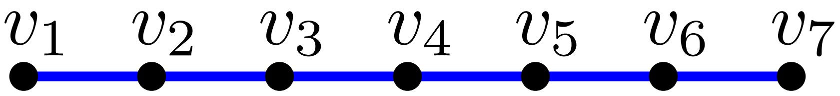


SGE: Some results

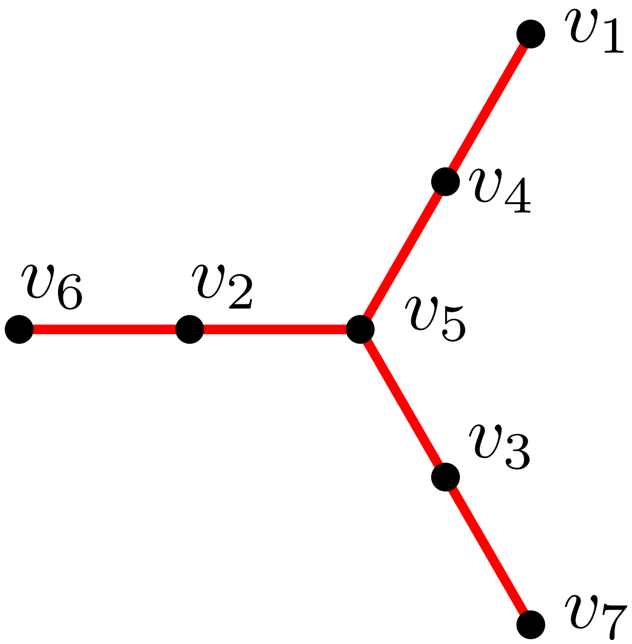
G_1	G_2		
Path	Path	✓	Braß et al., 2007
Caterpillar	Caterpillar	✓	Braß et al., 2007
Cycle	Cycle	✓	Braß et al., 2007
Tree	Matching	✓	Cabello et al., 2011
Outerpath	Matching	✓	Cabello et al., 2011
Planar	Planar	✗	Braß et al., 2007
Outerplanar	Outerplanar	✗	Braß et al., 2007
Tree	Tree	✗	Geyer et al., 2009
Planar	Path	✗	Braß et al., 2007
Planar	Matching	✗	Cabello et al., 2011
Tree	Path	✗	Angelini et al., 2012

An example to start

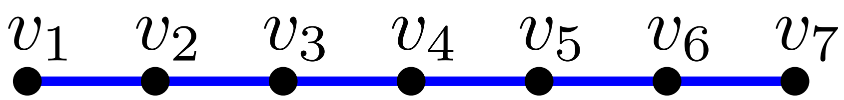
G_1: path



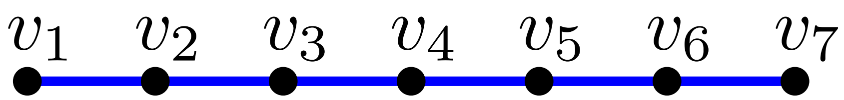
G_2: radius-2 star

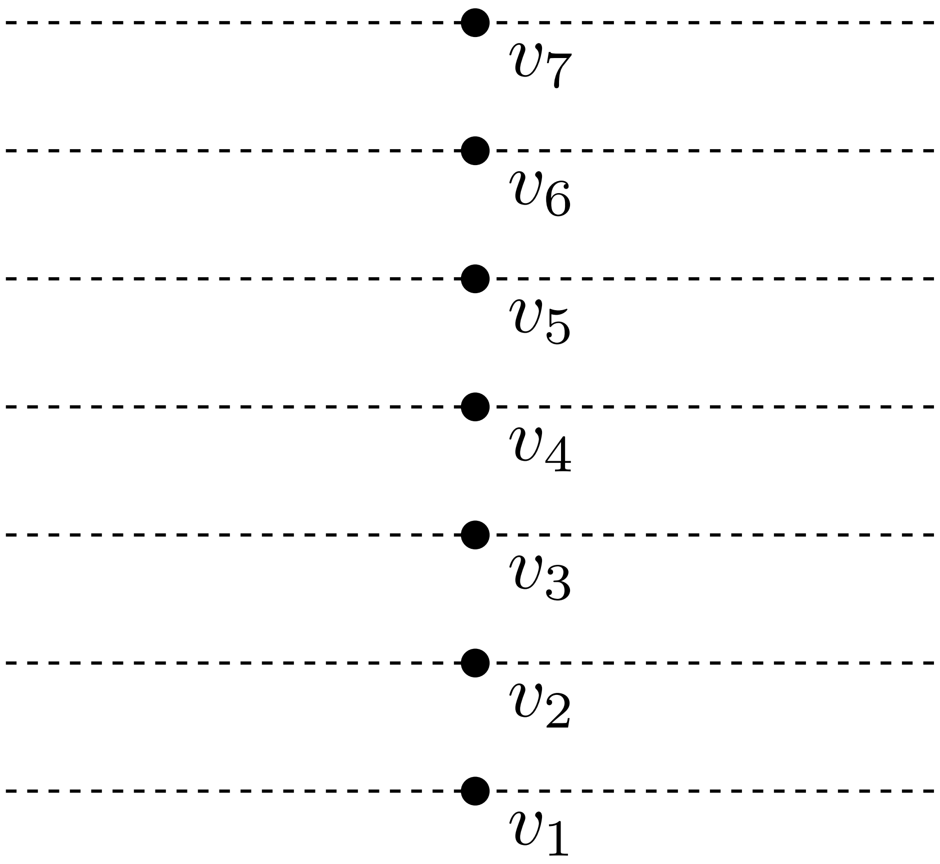


An useful property of paths

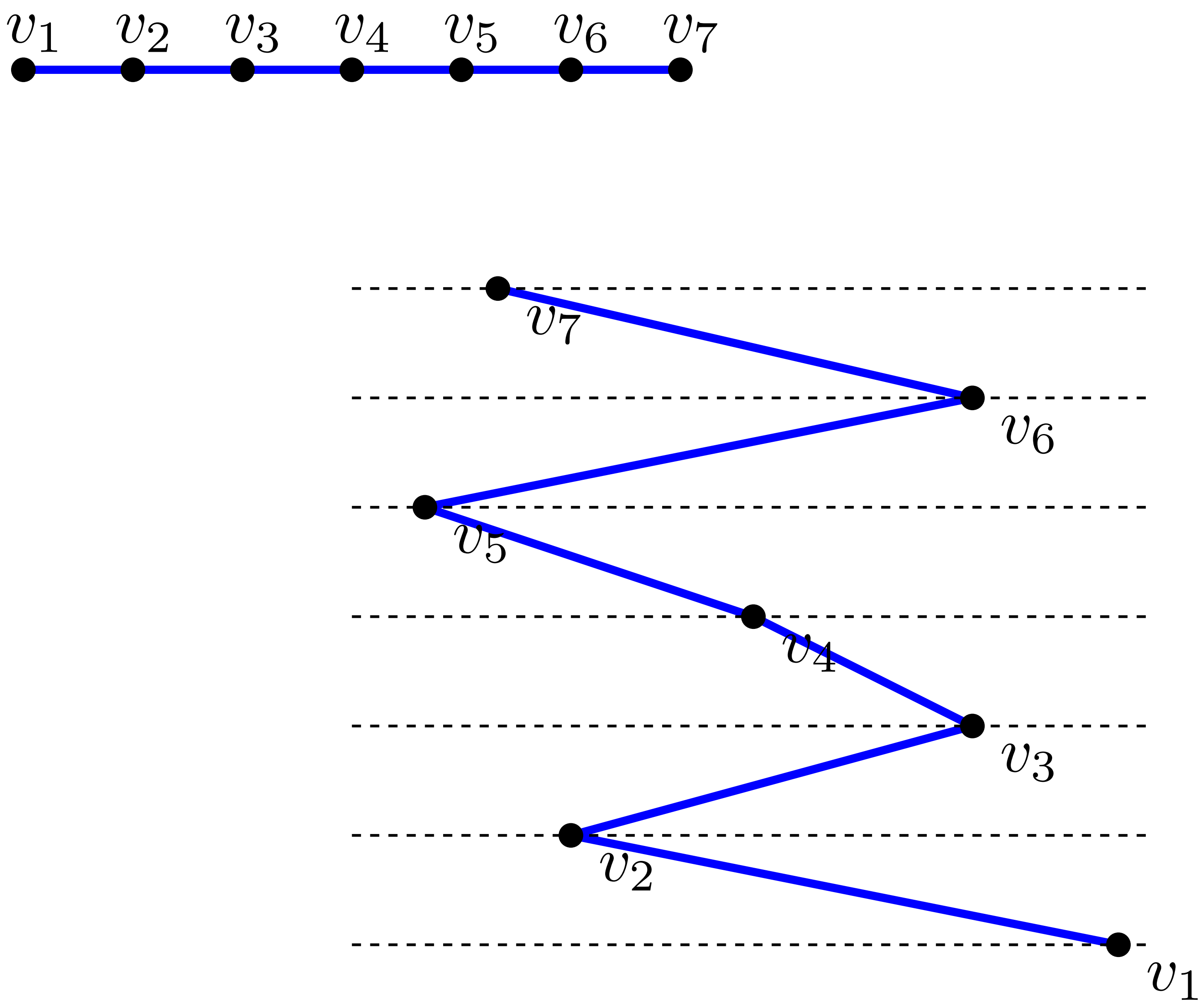


An useful property of paths

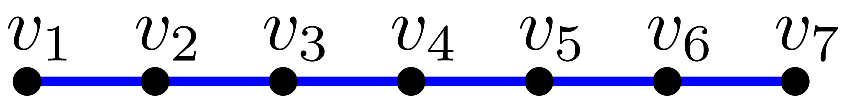


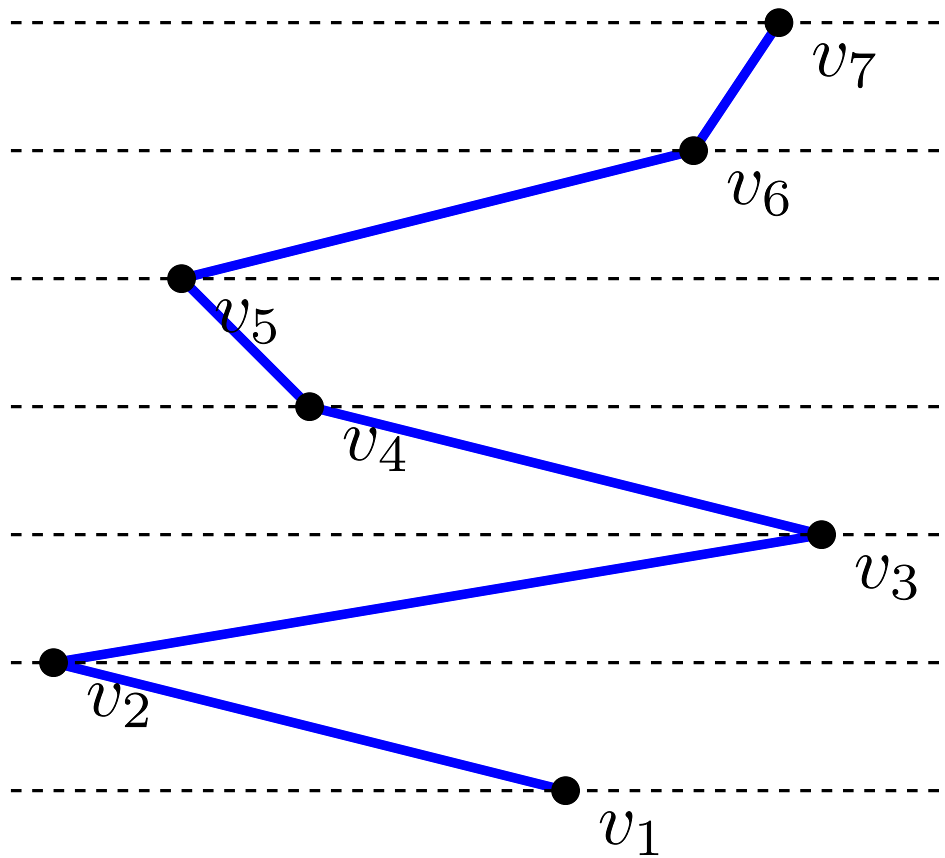


An useful property of paths

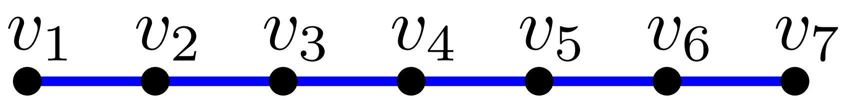


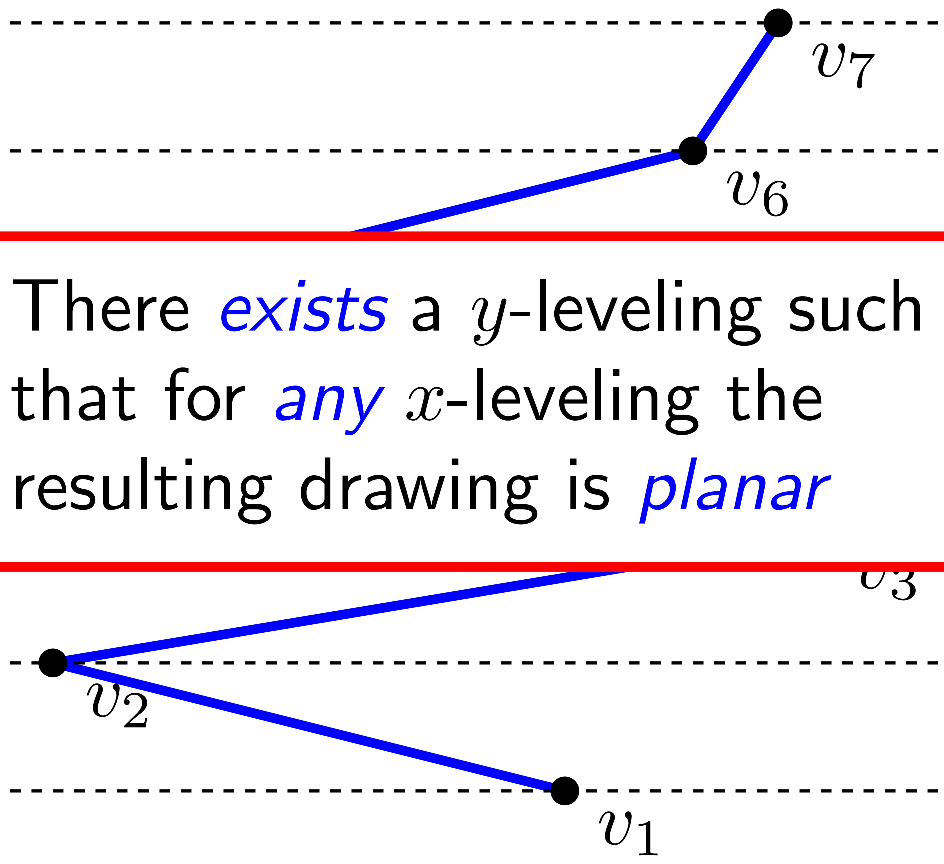
An useful property of paths





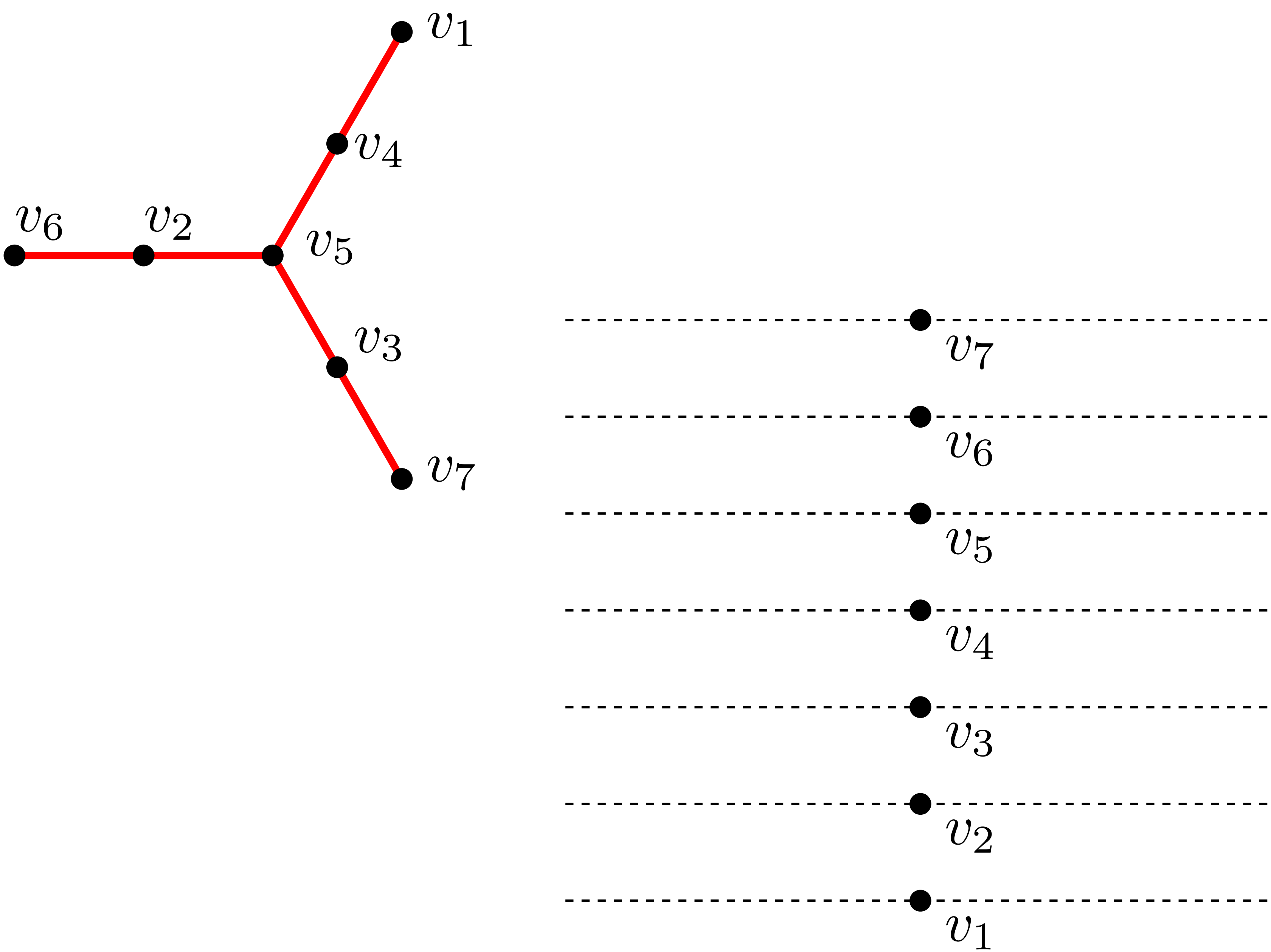
An useful property of paths



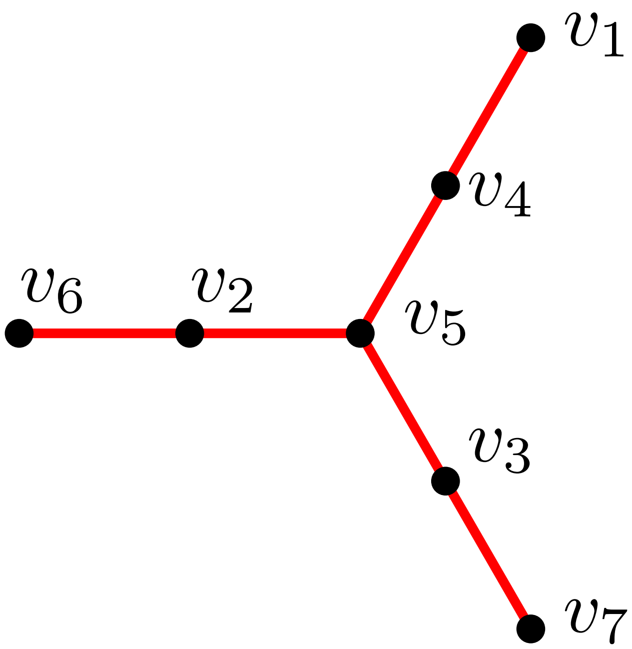


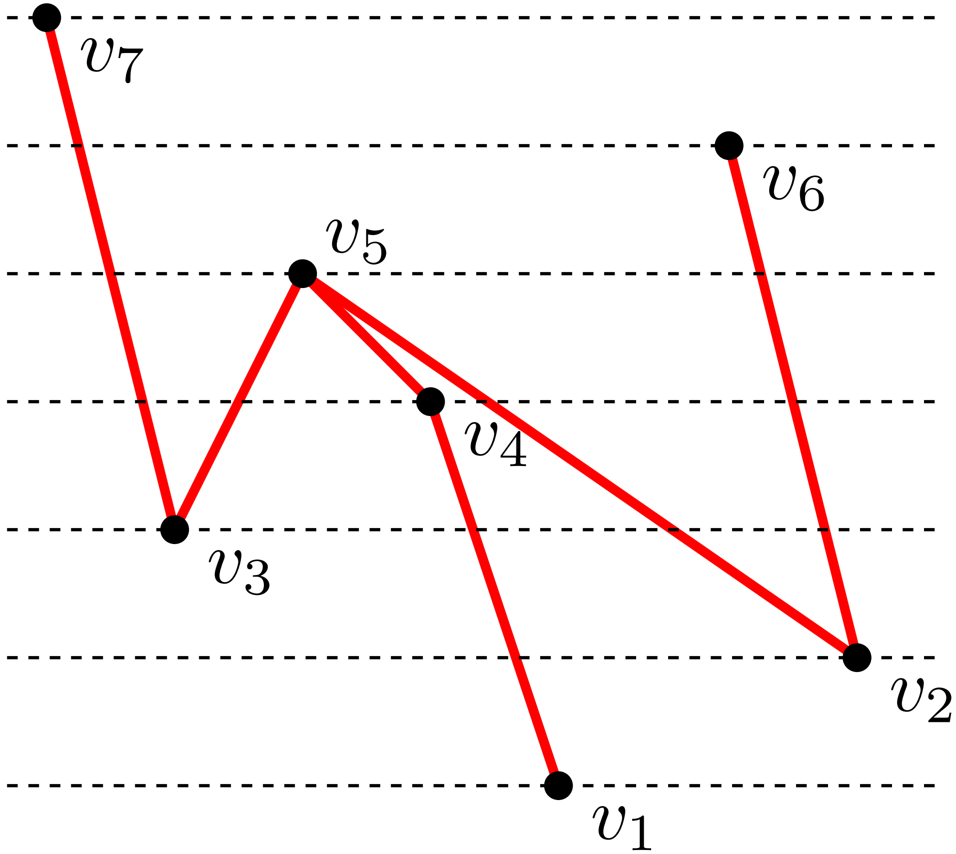
There *exists* a y-leveling such that for *any* x-leveling the resulting drawing is *planar*

An useful property of radius-2 stars

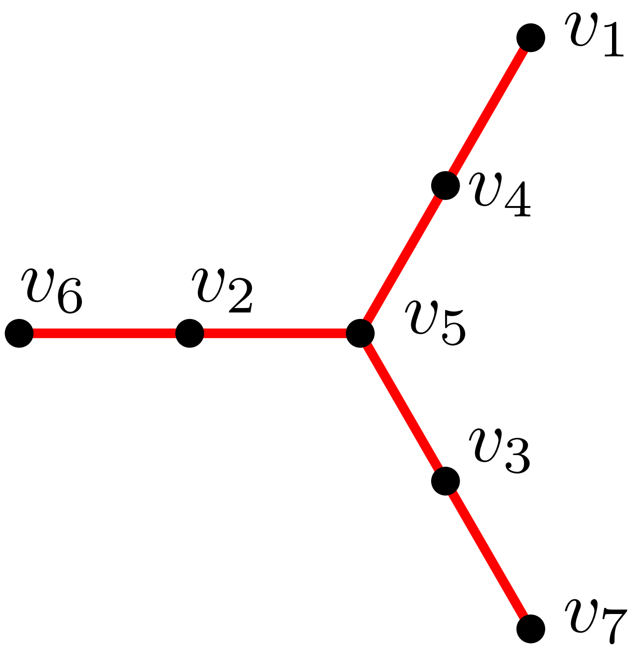


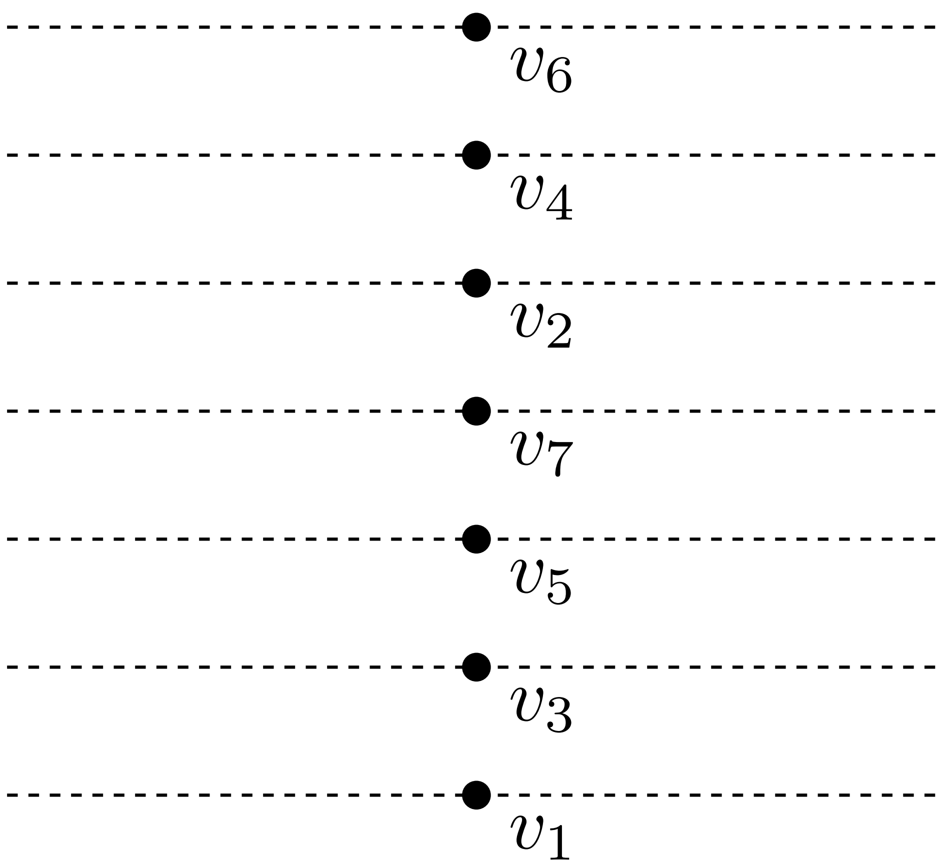
An useful property of radius-2 stars



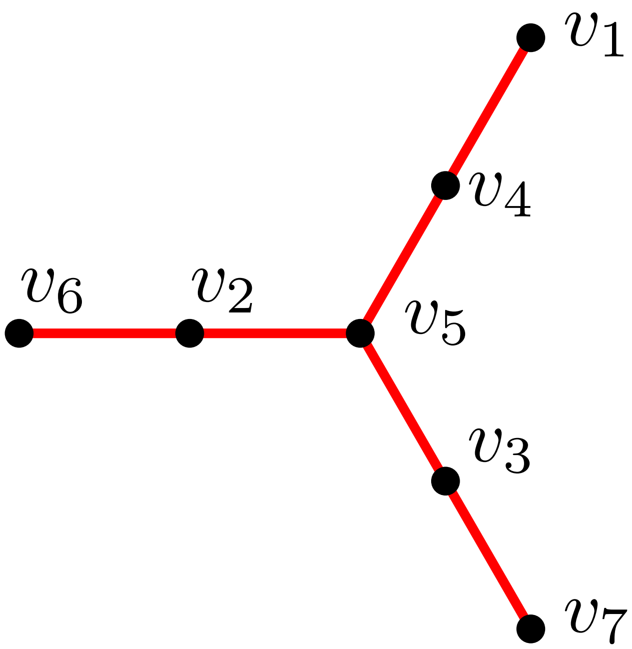


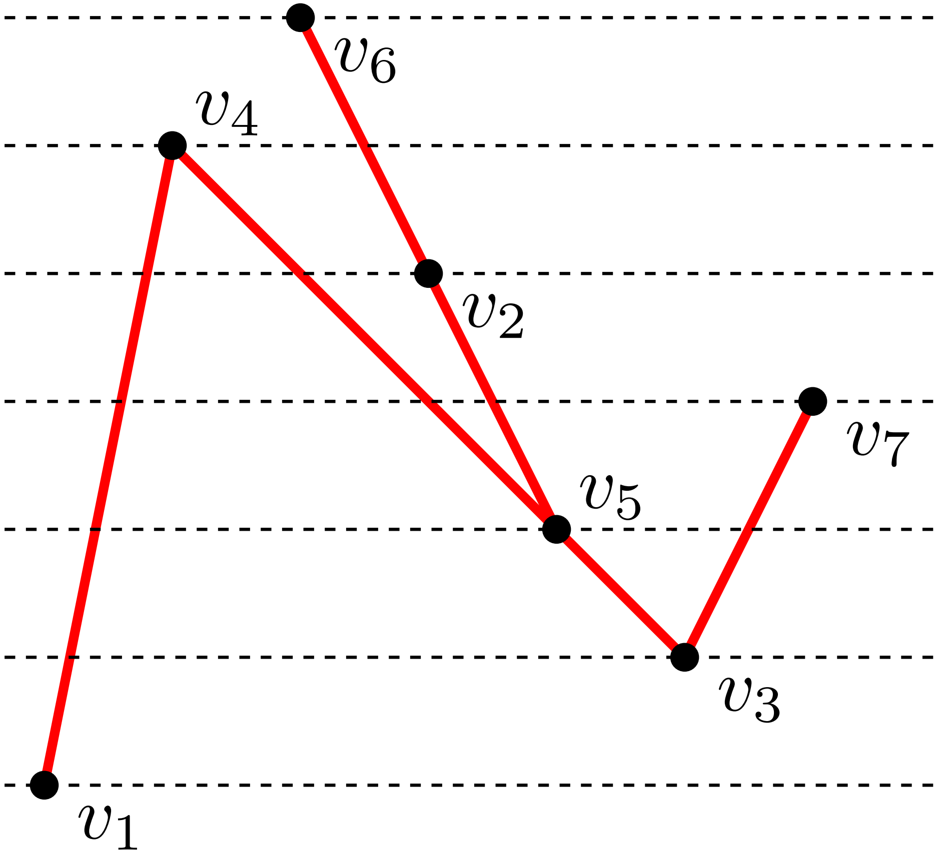
An useful property of radius-2 stars



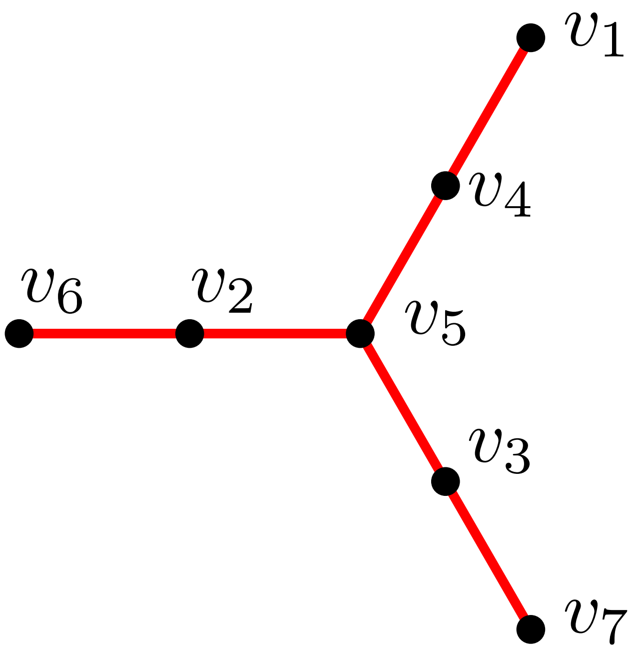


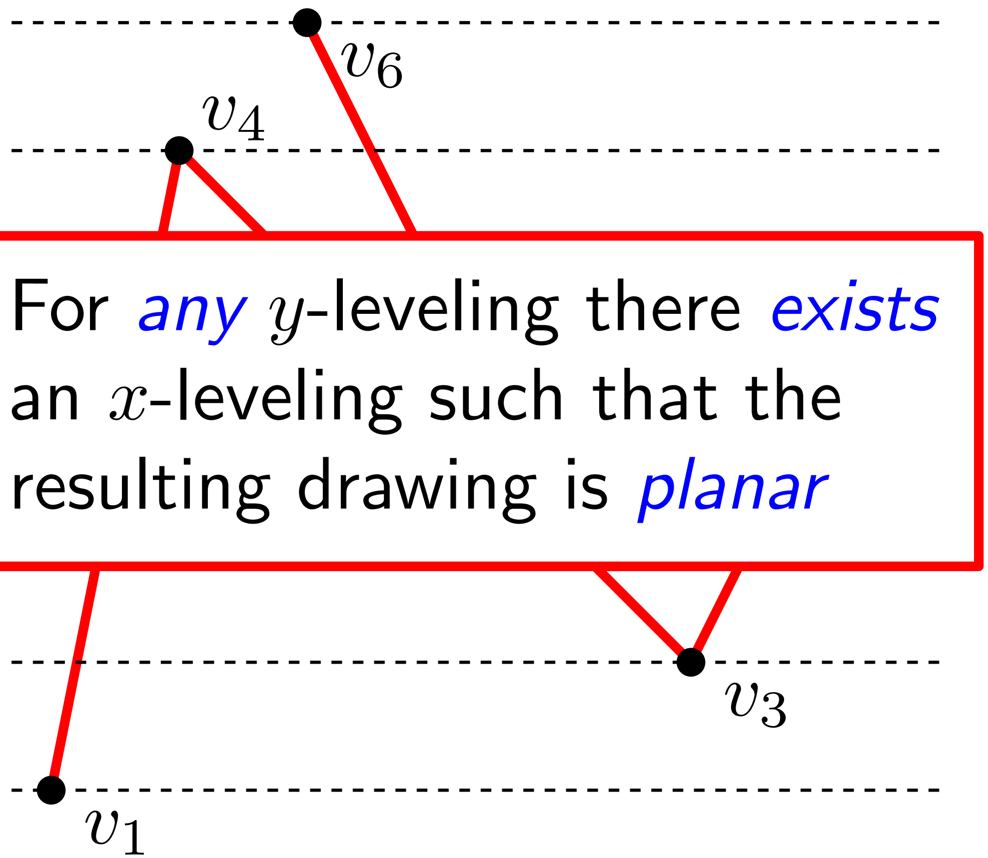
An useful property of radius-2 stars





An useful property of radius-2 stars

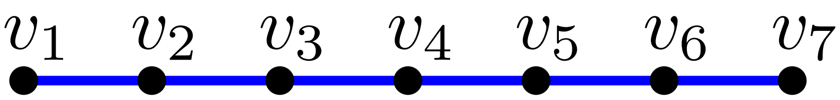




For *any* y-leveling there *exists* an x-leveling such that the resulting drawing is *planar*

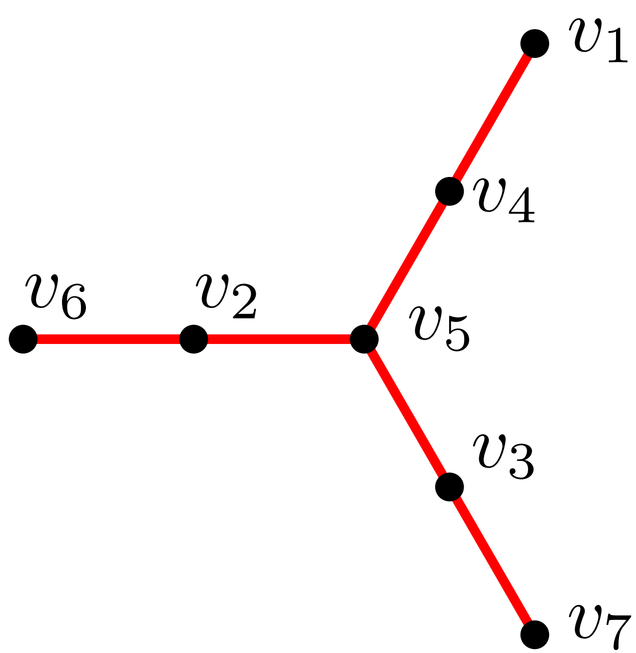
In summary

G_1: path



There *exists* a y-leveling such that for *any* x-leveling the resulting drawing is *planar*

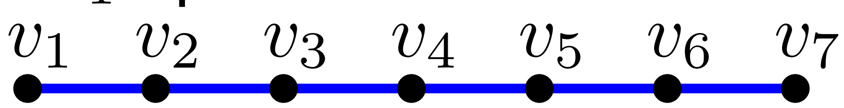
G_2: radius-2 star



For *any* y-leveling there *exists* an x-leveling such that the resulting drawing is *planar*

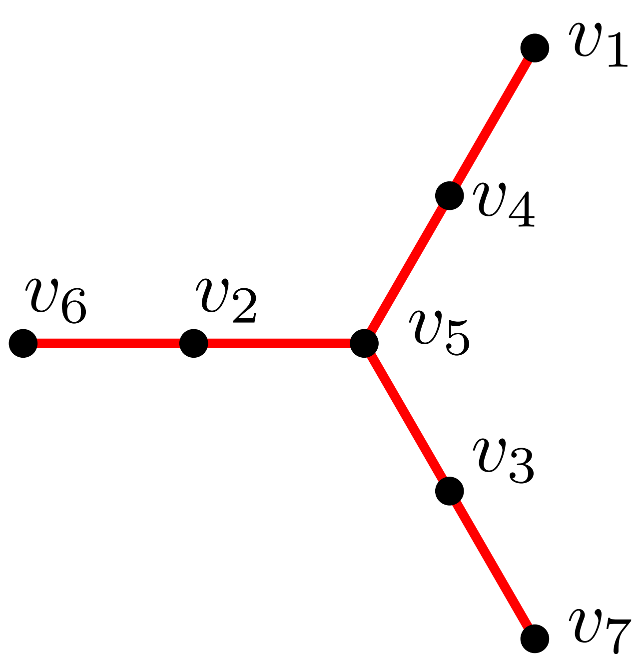
In summary

G_1: path



There *Exists* a y-leveling such that for *Any* x-leveling the resulting drawing is *Planar*

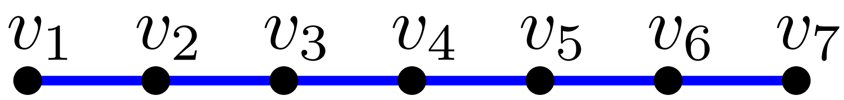
G_2: radius-2 star



For *any* y-leveling there *exists* an x-leveling such that the resulting drawing is *planar*

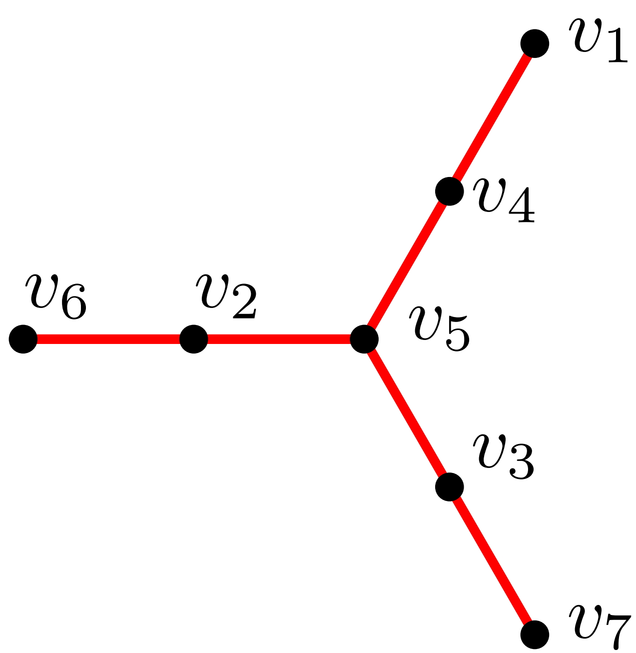
In summary

G_1: path



EAP graph

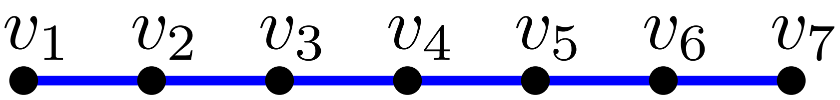
G_2: radius-2 star



For *any* y-leveling there *exists* an x-leveling such that the resulting drawing is *planar*

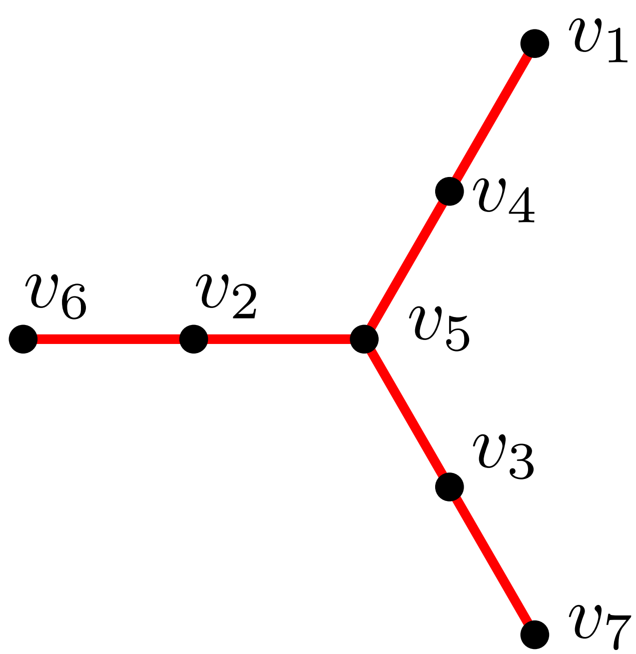
In summary

G_1: path



EAP graph

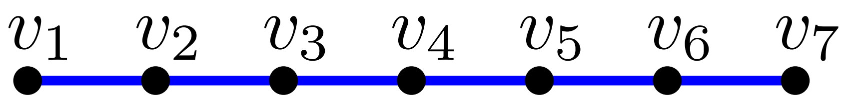
G_2: radius-2 star



For *Any* y-leveling there *Exists* an x-leveling such that the resulting drawing is *Planar*

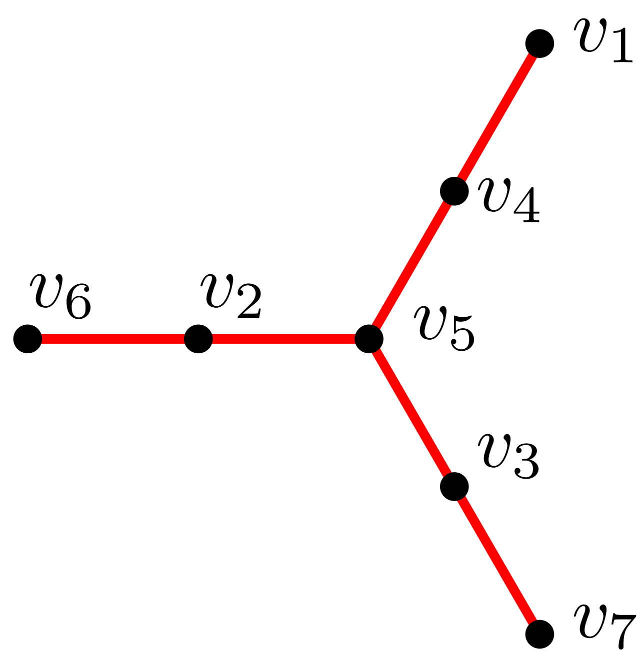
In summary

G_1: path



EAP graph

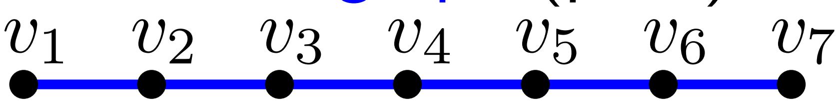
G_2: radius-2 star



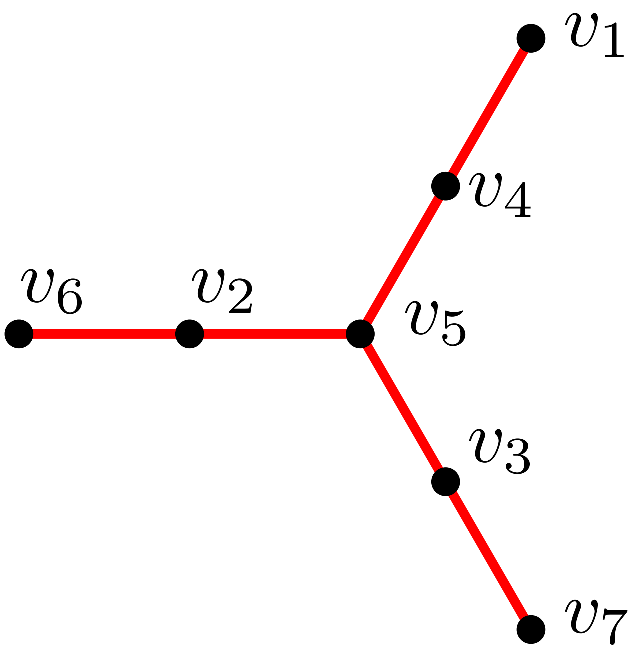
AEP graph

AEP, EAP and SGE

G_1: EAP graph (path)

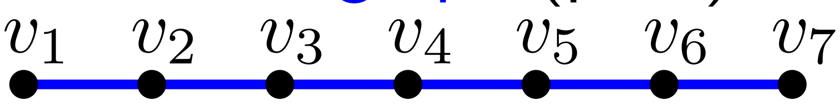


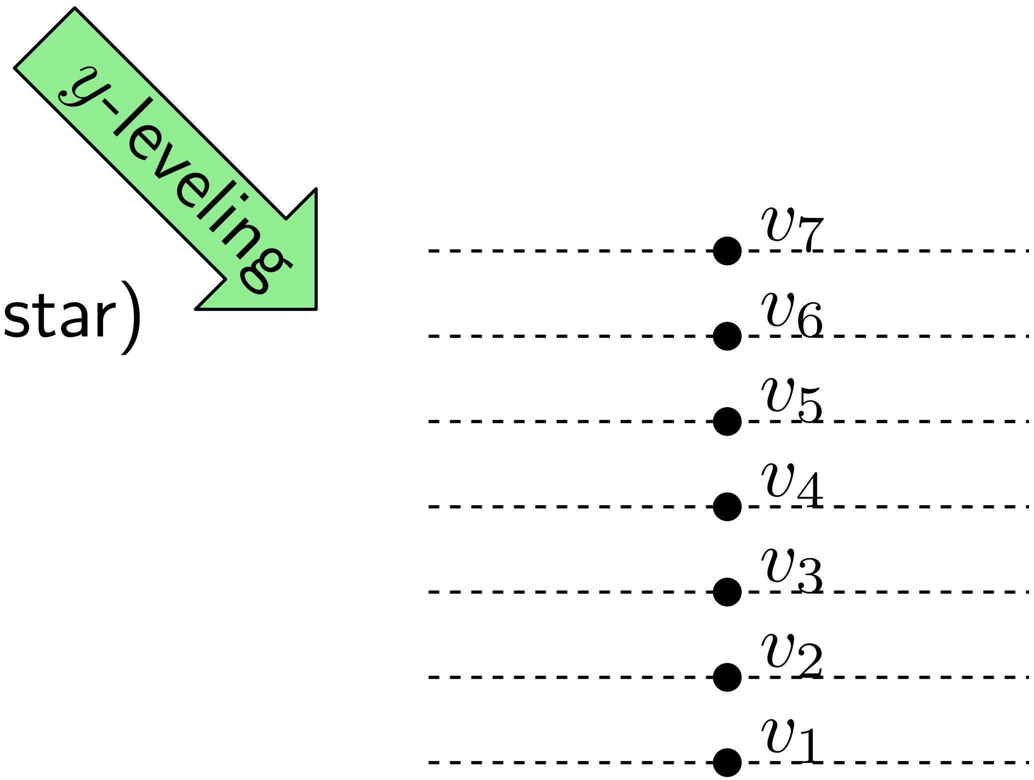
G_2: AEP graph (radius-2 star)



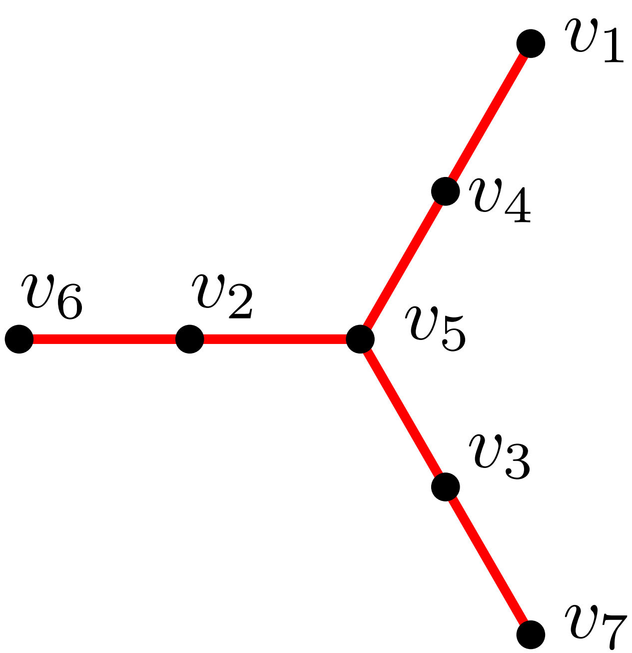
AEP, EAP and SGE

G_1: EAP graph (path)

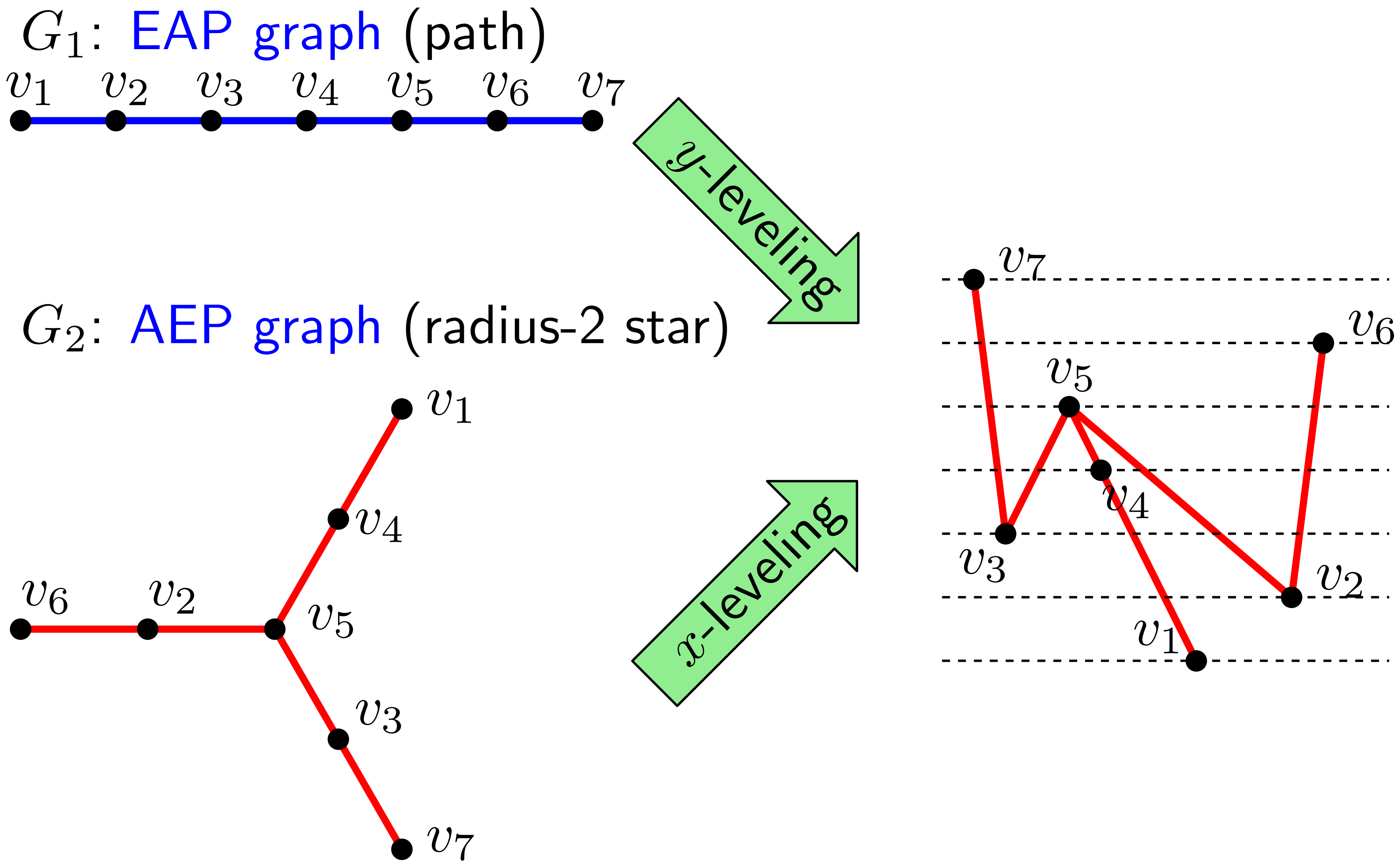




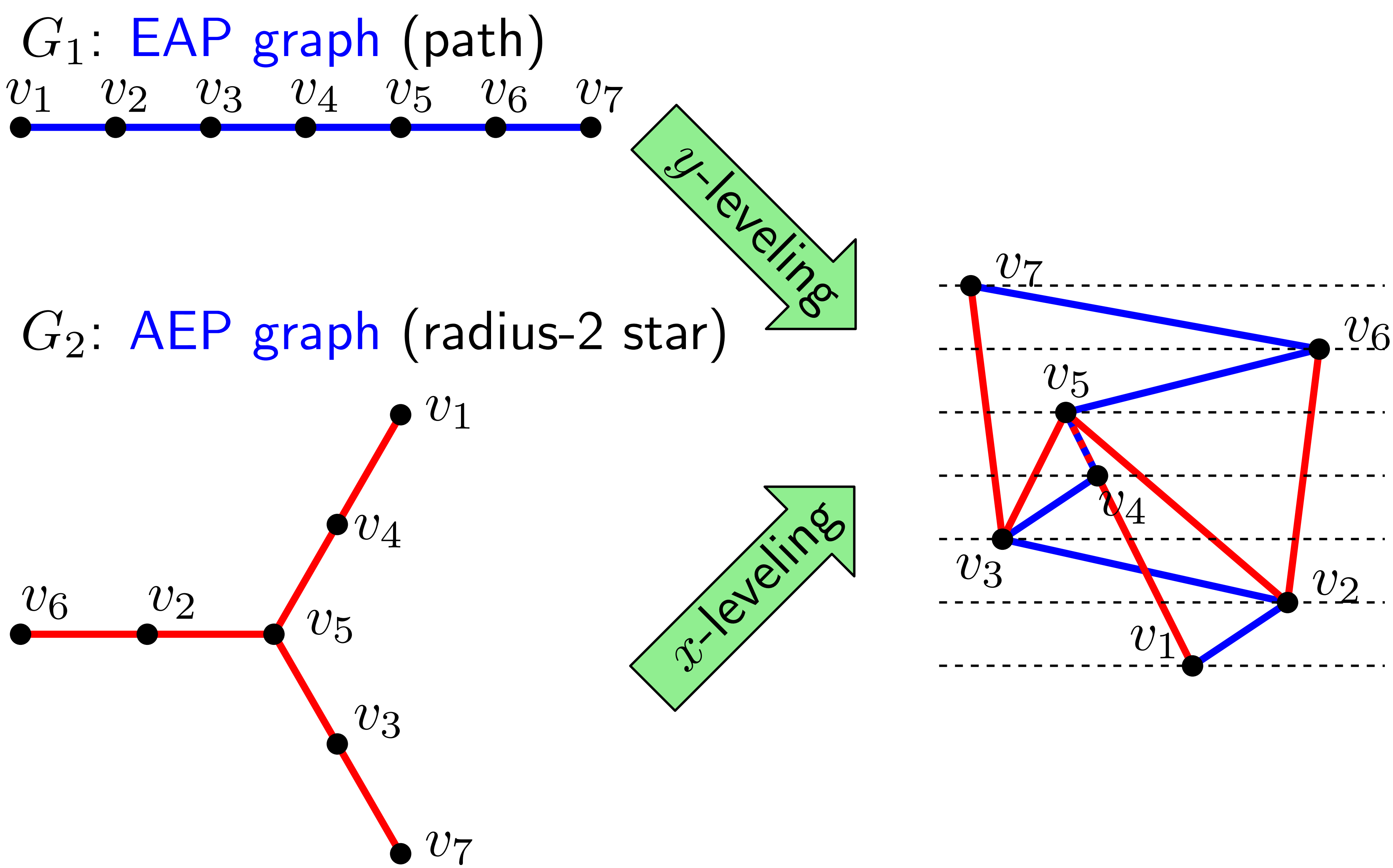
G_2: AEP graph (radius-2 star)



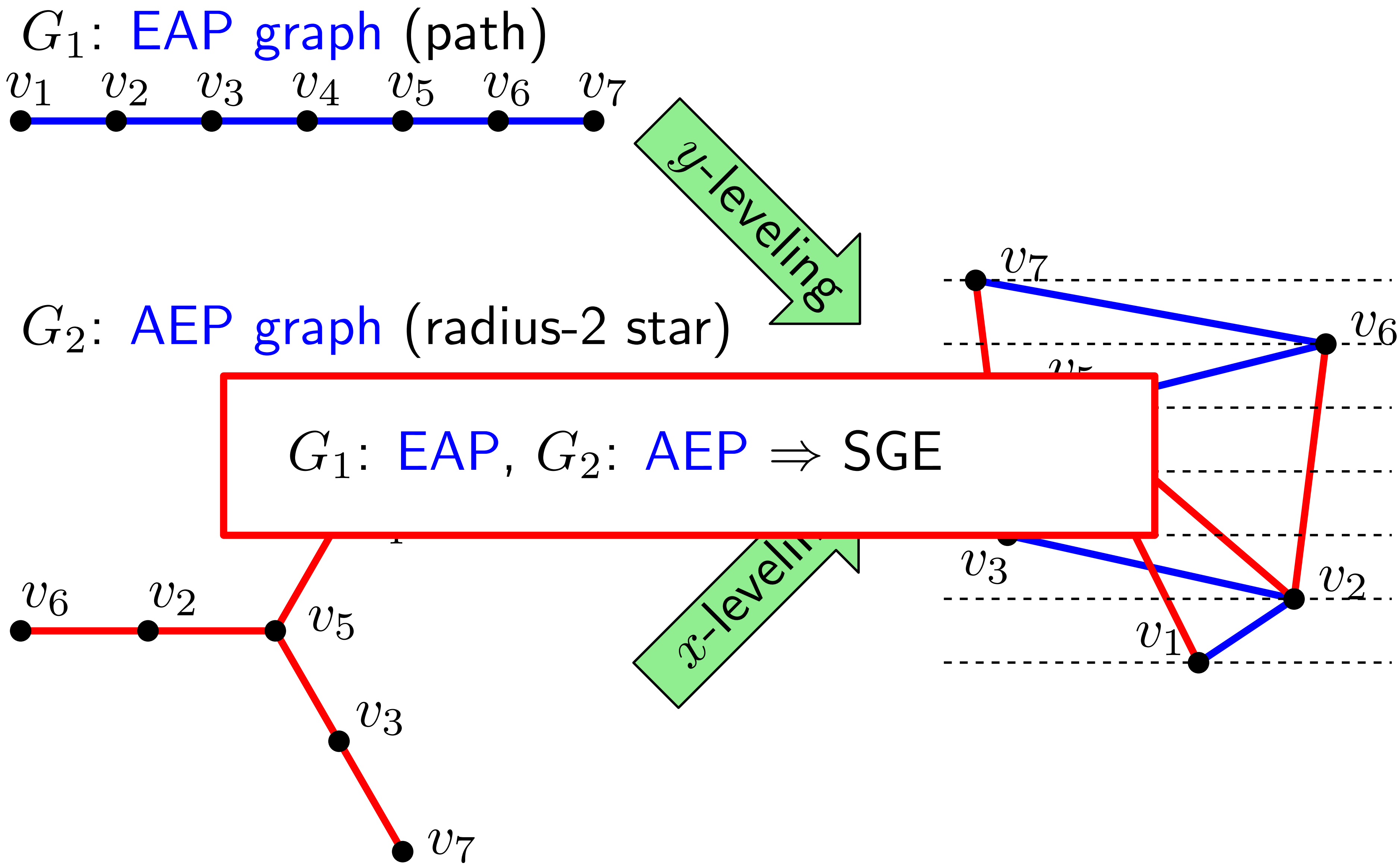
AEP, EAP and SGE



AEP, EAP and SGE



AEP, EAP and SGE



AEP graphs: who are they?

- Fowler and Kobourov studied the graphs that can be simultaneously geometrically embedded with a path using the previous technique
  Fowler, Kobourov GD 2007
- They call these graphs ULP (*Unlabeled Level Planar*) graph; they coincide with AEP graphs
- Fowler and Kobourov show that ULP=AEP graphs are the union of the following classes:
  - *radius-2 stars*
  - *extended degree-3 spider*
  - *generalized caterpillars*
- As a consequence, they obtain the following results:
  - G_1: path, G_2: radius-2 star $\Rightarrow$ SGE
  - G_1: path, G_2: ext. deg.-3 spider $\Rightarrow$ SGE
  - G_1: path, G_2: gen. caterpillar $\Rightarrow$ SGE

Our results (1/2)

- We show that G_1: EAP, G_2: AEP $\Rightarrow$ SGE
- We show that EAP $\subset$ AEP
- We characterize EAP graphs
  - they coincide with a family that we call *fat caterpillars*
  - as a consequence we obtain the following results about SGE
    - G_1: fat caterpillar, G_2: radius-2 star $\Rightarrow$ SGE
    - G_1: fat caterpillar, G_2: ext. deg.-3 spider $\Rightarrow$ SGE
    - G_1: fat caterpillar, G_2: gen. caterpillar $\Rightarrow$ SGE
- We show that
  G_1: fat caterpillar and G_2: tree of depth ≤ 2 $\Rightarrow$ SGE
  - this extends a result by Angelini et al.
    Angelini, Geyer, Kaufmann, Neuwirth JGAA 2012

Our results (2/2)

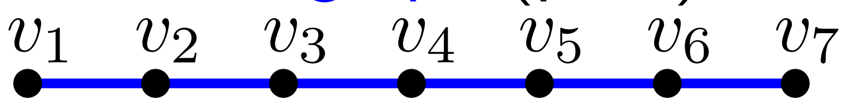
- We extends our study "beyond planarity":
  - we define *simultaneous geometric quasi-planar embedding* (SGQPE)
  - We introduce *AEQP graphs* and *EAQP graphs*
- We prove that G_1: EAQP, G_2: AEQP $\Rightarrow$ SGQPE
- We show that EAP $\subset$ AEP $\subset$ EAQP $\subset$ AEQP
- We show that all *trees* are AEQP but not all of them are EAQP
- We show that *maximal outerpillar* are EAQP
  - as a consequence we have that
    G_1: tree and G_2: maximal outerpillar $\Rightarrow$ SGQPE
    G_1: tree and G_2: path/cycle $\Rightarrow$ SGQPE

Characterization of EAP graphs

EAP $\subseteq$ AEP

Lemma 1 *Let $G \in EAP$. Then $G \in AEP$.*

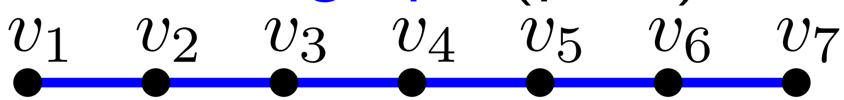
G: EAP graph (path)



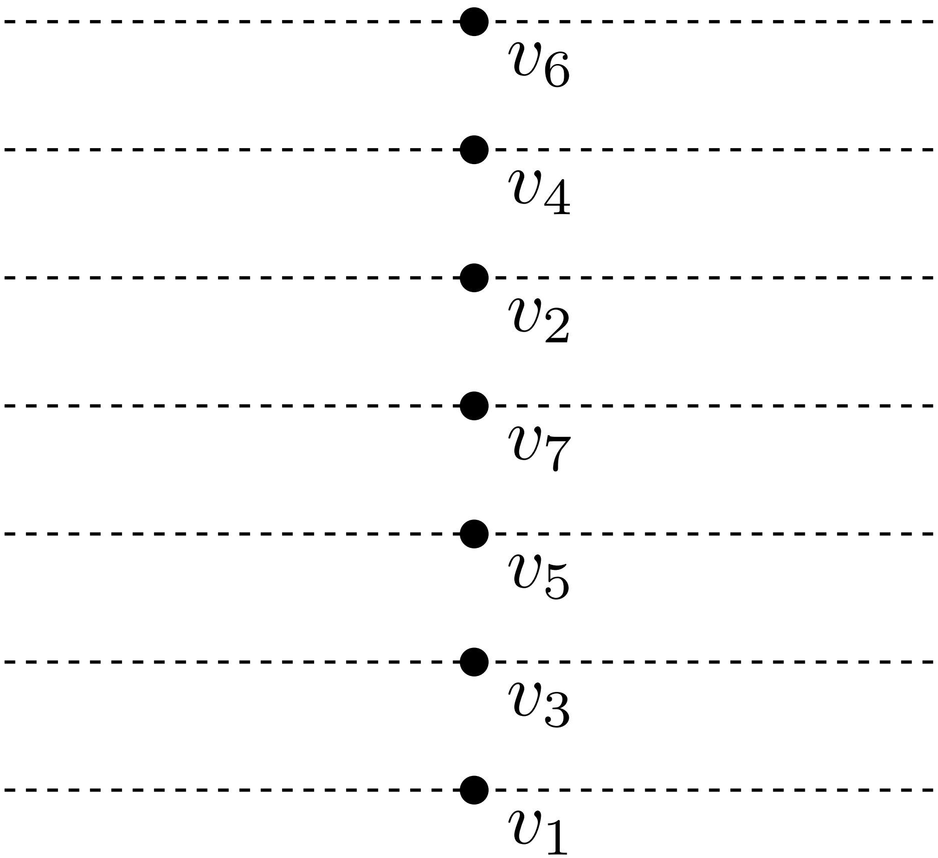
EAP $\subseteq$ AEP

Lemma 1 *Let $G \in EAP$. Then $G \in AEP$.*

G: EAP graph (path)



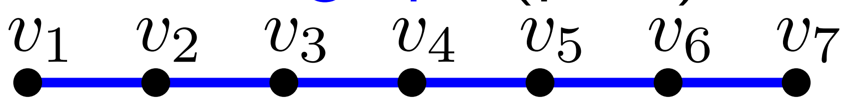
Let $\mathcal{Y}$ be a given y-leveling



EAP ⊆ AEP

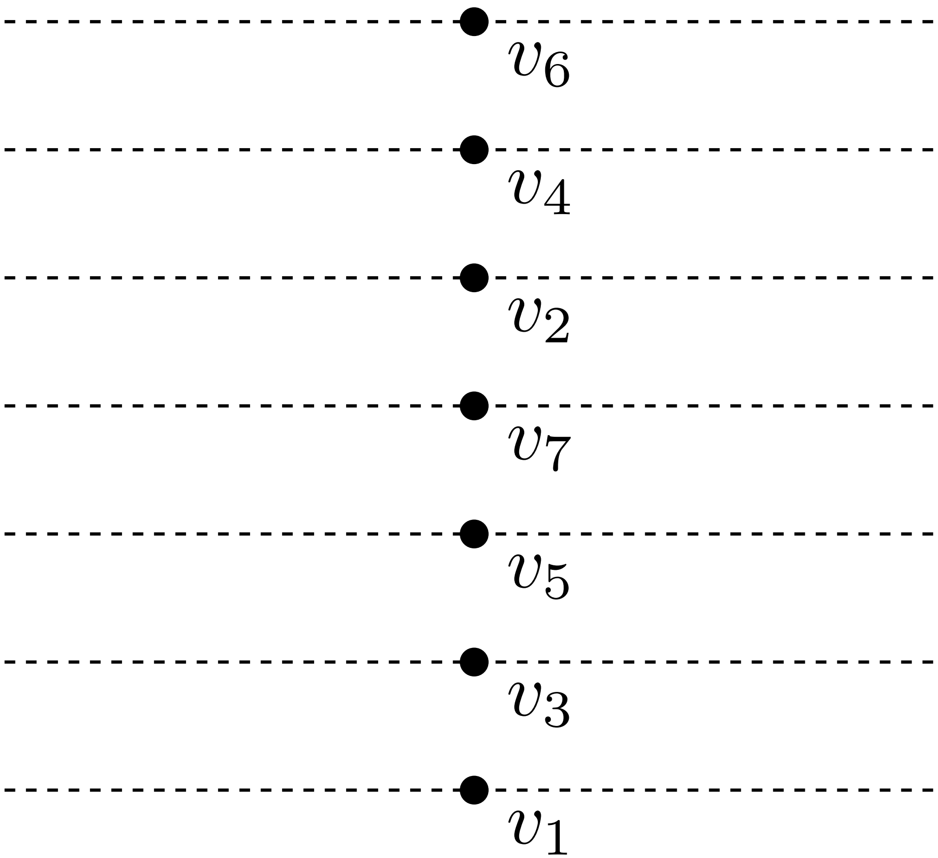
Lemma 1 *Let* $G \in EAP$*. Then* $G \in AEP$*.*

G: EAP graph (path)



Let $\mathcal{Y}$ be a given y-leveling

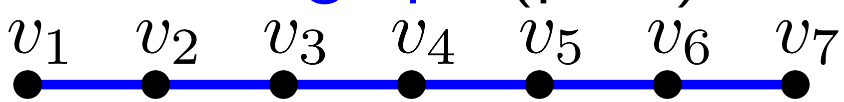
We must find an x-leveling that is OK with $\mathcal{Y}$



EAP ⊆ AEP

Lemma 1 *Let $G \in EAP$. Then $G \in AEP$.*

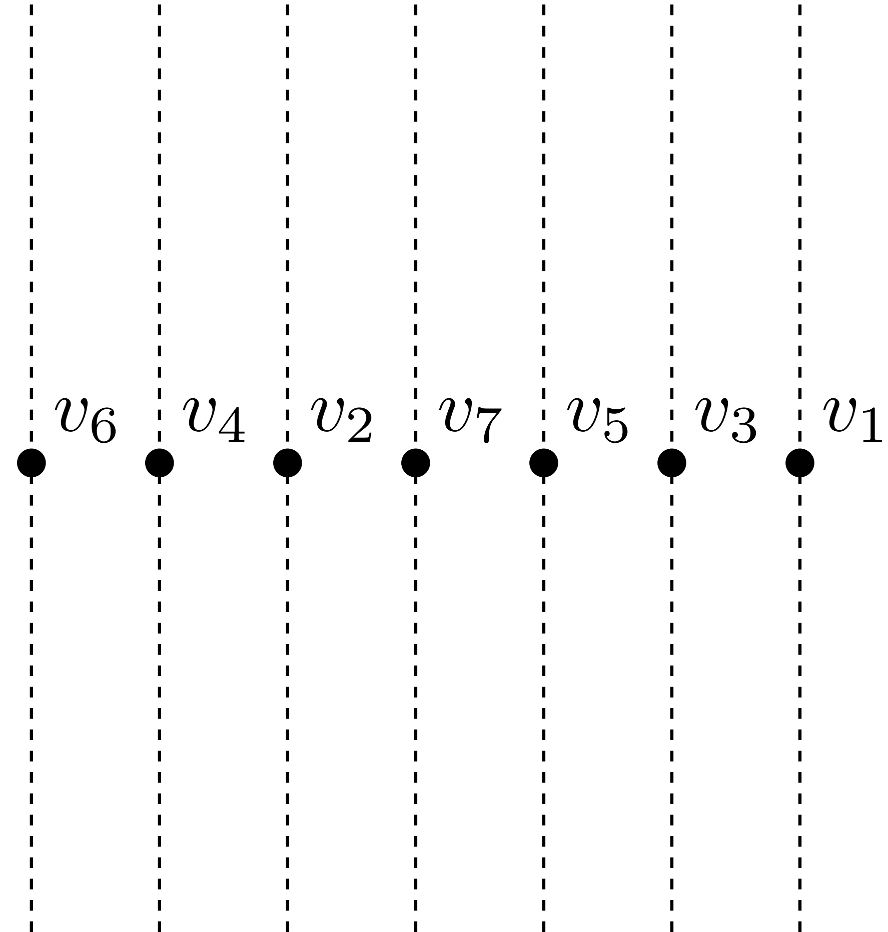
G: EAP graph (path)



Let $\mathcal{Y}$ be a given y-leveling

We must find an x-leveling that is OK with $\mathcal{Y}$

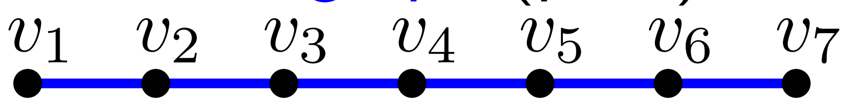
Let's rotate the plane



EAP ⊆ AEP

Lemma 1 *Let $G \in EAP$. Then $G \in AEP$.*

G: EAP graph (path)

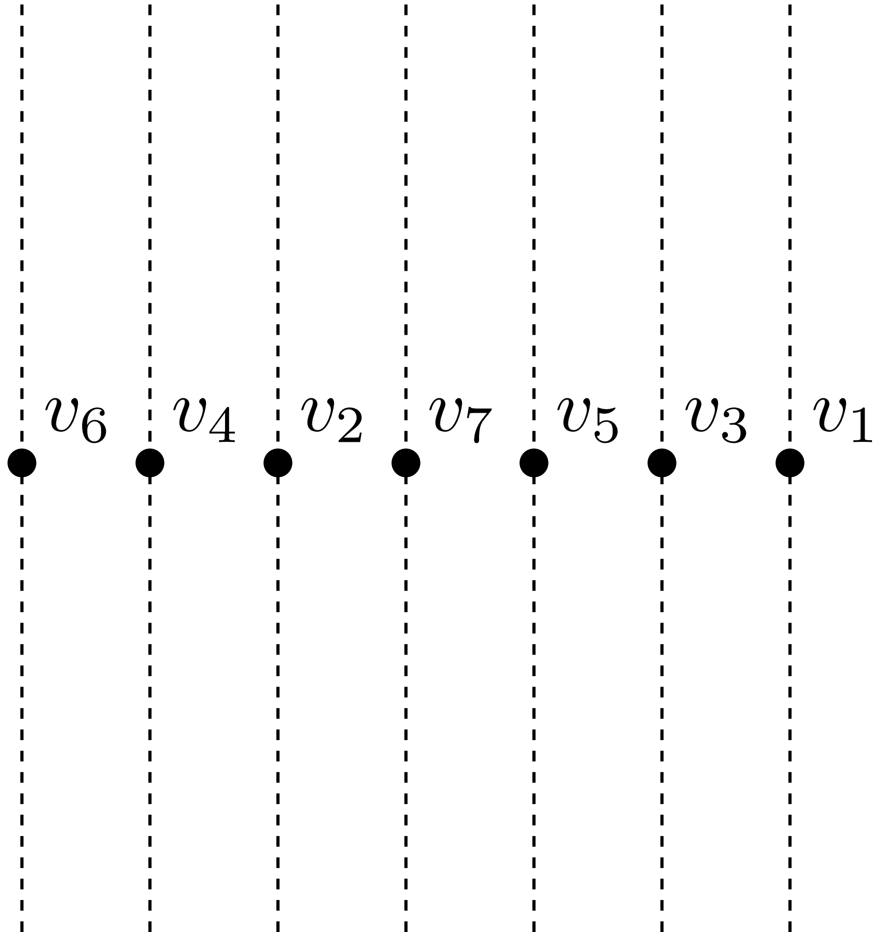


Let $\mathcal{Y}$ be a given y-leveling

We must find an x-leveling that is OK with $\mathcal{Y}$

Let's rotate the plane

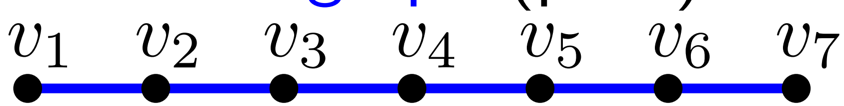
$\mathcal{Y}$ is now an x-leveling



EAP ⊆ AEP

Lemma 1 *Let $G \in EAP$. Then $G \in AEP$.*

G: EAP graph (path)



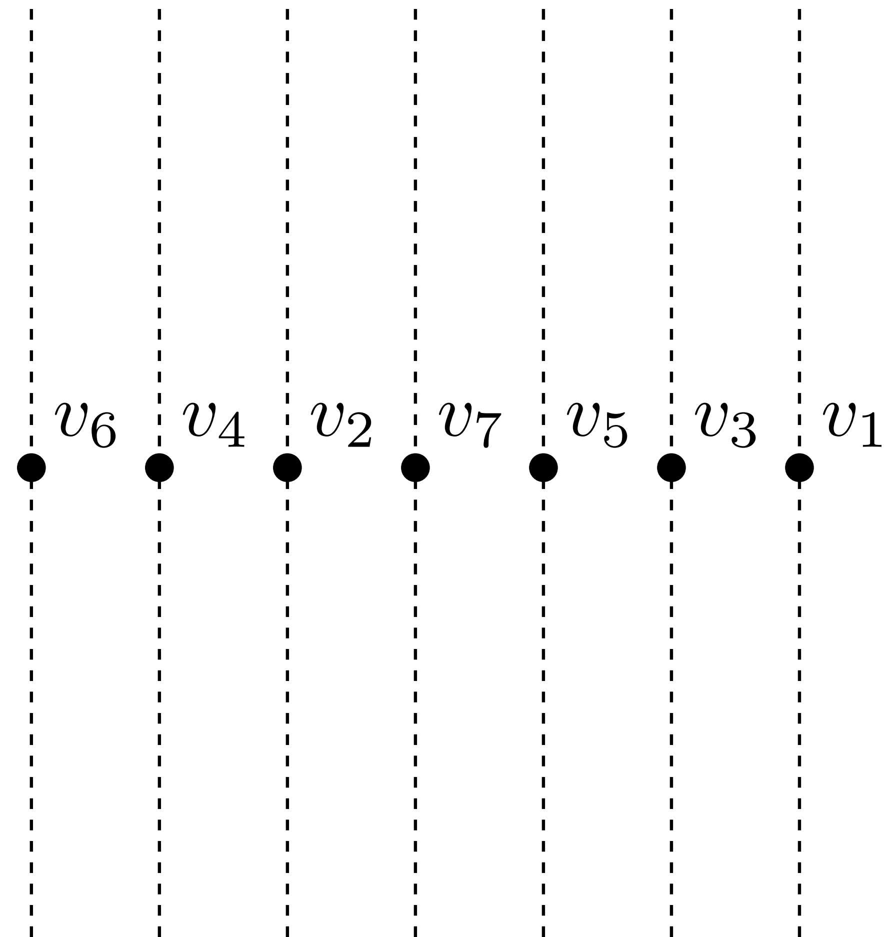
Let $\mathcal{Y}$ be a given y-leveling

We must find an x-leveling that is OK with $\mathcal{Y}$

Let's rotate the plane

$\mathcal{Y}$ is now an x-leveling

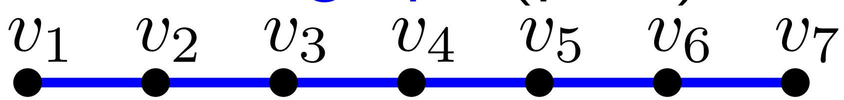
Since G is EAP, it has a y-leveling that is OK with any x-leveling



EAP ⊆ AEP

Lemma 1 *Let $G \in EAP$. Then $G \in AEP$.*

G: EAP graph (path)



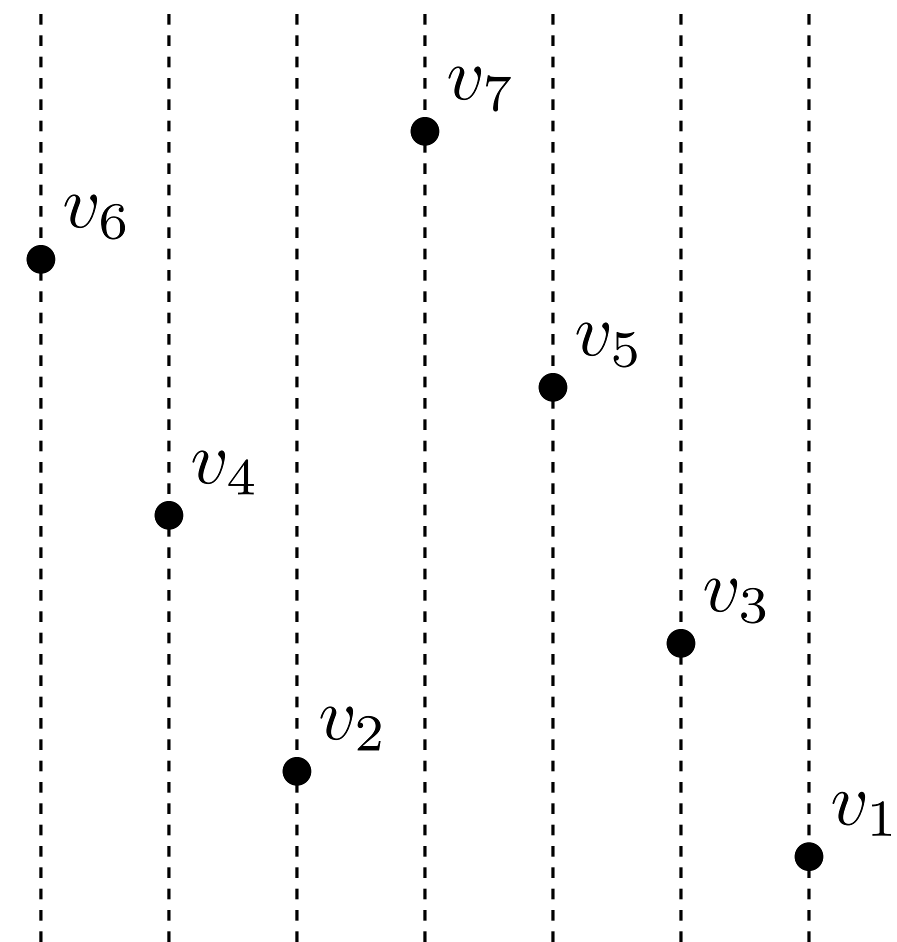
Let $\mathcal{Y}$ be a given y-leveling

We must find an x-leveling that is OK with $\mathcal{Y}$

Let's rotate the plane

$\mathcal{Y}$ is now an x-leveling

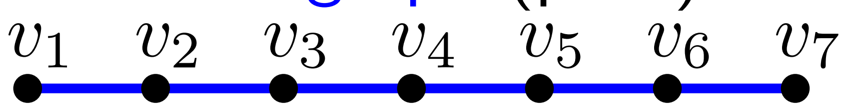
Since G is EAP, it has a y-leveling that is OK with any x-leveling



EAP $\subseteq$ AEP

Lemma 1 *Let* $G \in EAP$. *Then* $G \in AEP$.

G: EAP graph (path)



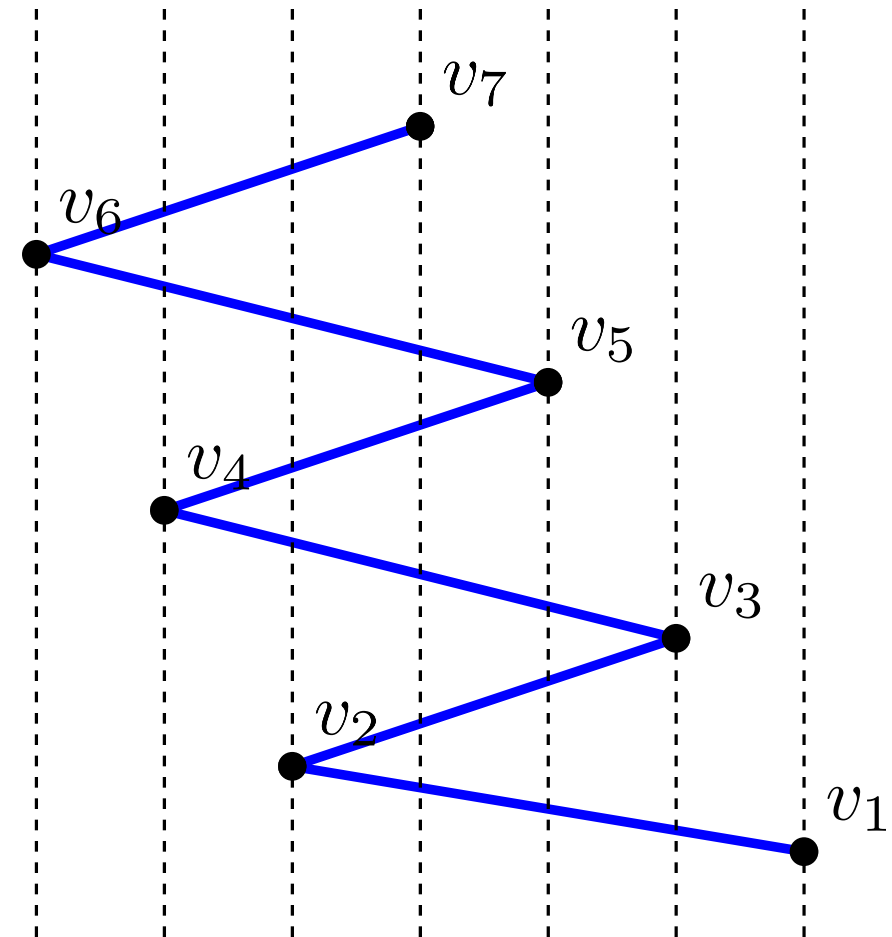
Let $\mathcal{Y}$ be a given y-leveling

We must find an x-leveling that is OK with $\mathcal{Y}$

Let's rotate the plane

$\mathcal{Y}$ is now an x-leveling

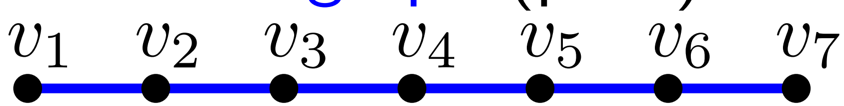
Since G is EAP, it has a y-leveling that is OK with any x-leveling



EAP $\subseteq$ AEP

Lemma 1 *Let $G \in EAP$. Then $G \in AEP$.*

G: EAP graph (path)



Let $\mathcal{Y}$ be a given y-leveling

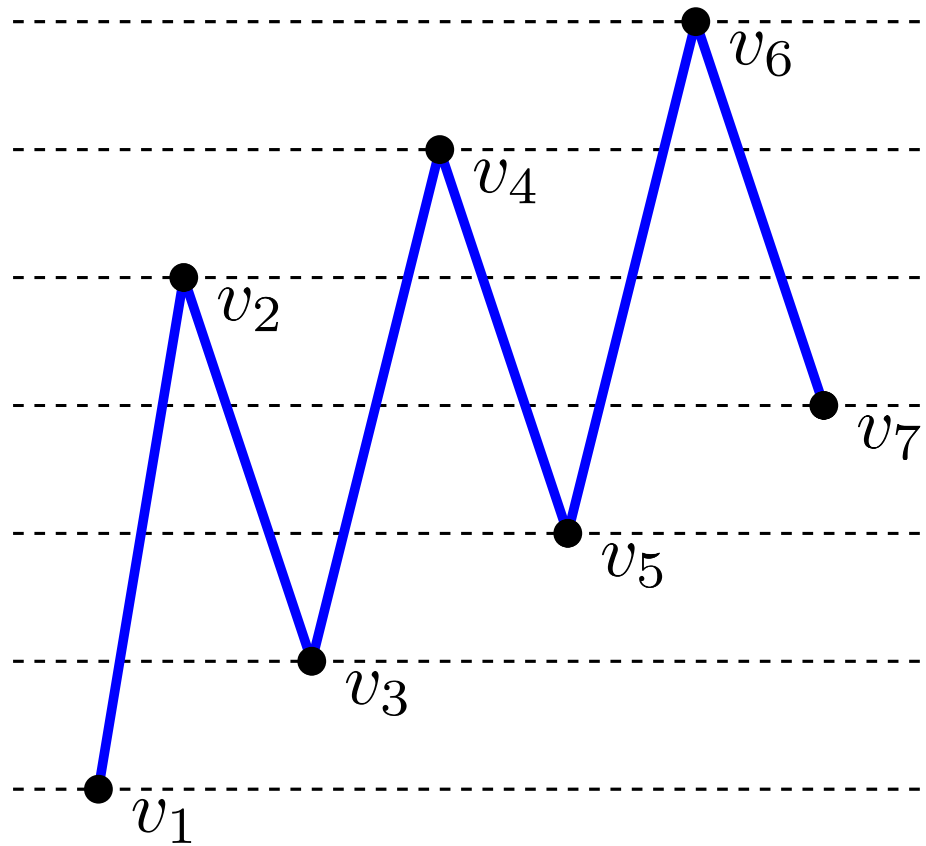
We must find an x-leveling that is OK with $\mathcal{Y}$

Let's rotate the plane

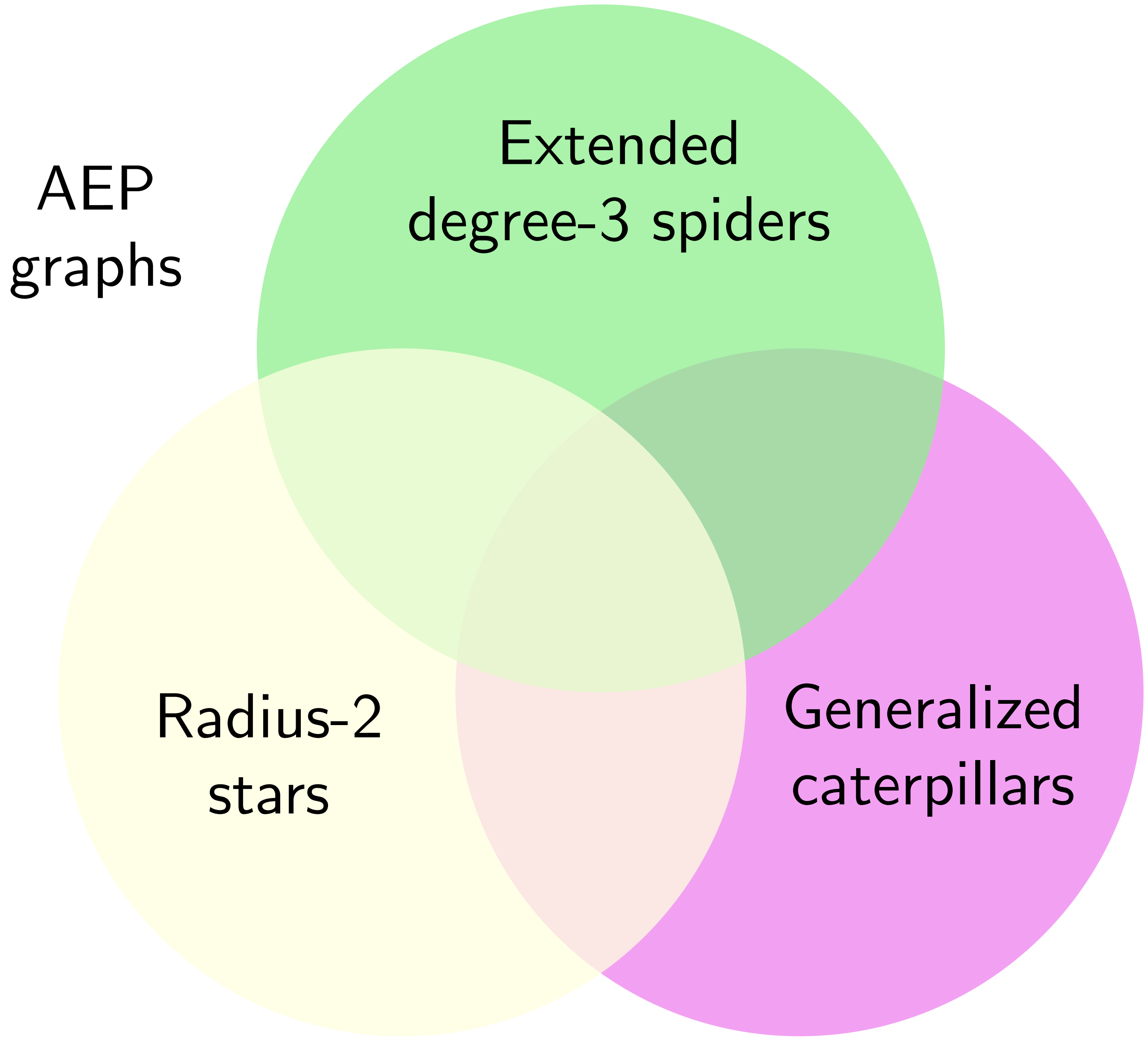
$\mathcal{Y}$ is now an x-leveling

Since G is EAP, it has a y-leveling that is OK with any x-leveling

Let's rotate back

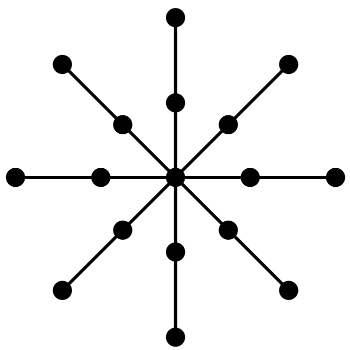


AEP graphs

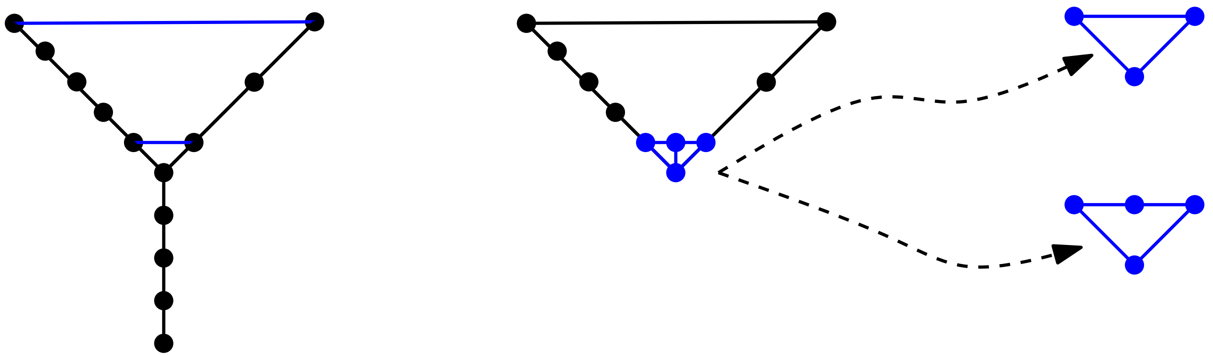


AEP
graphs
Extended
degree-3 spiders
Radius-2
stars
Generalized
caterpillars

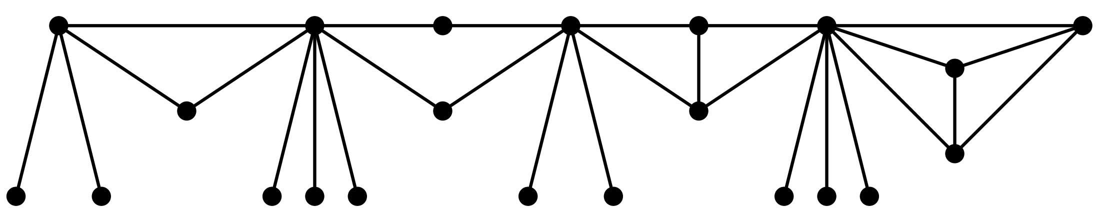
AEP graphs



Radius-2 stars

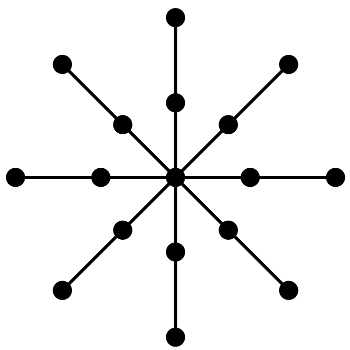


Extended degree-3 spiders

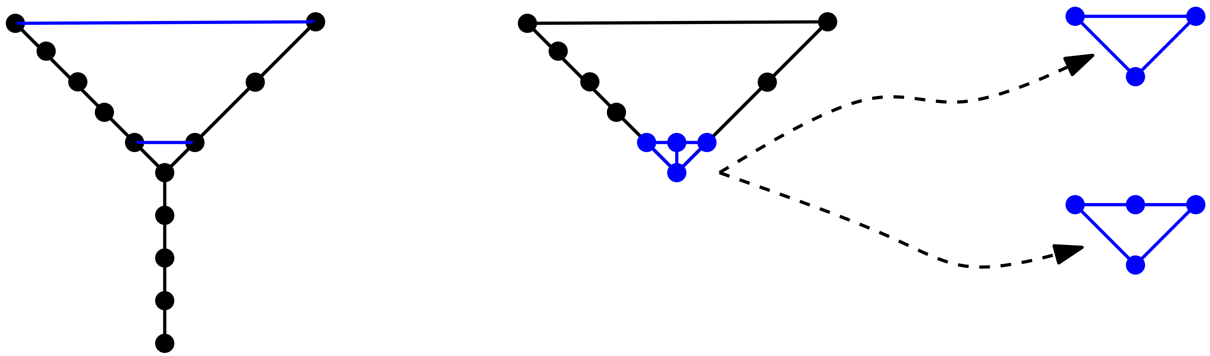


Generalized caterpillars

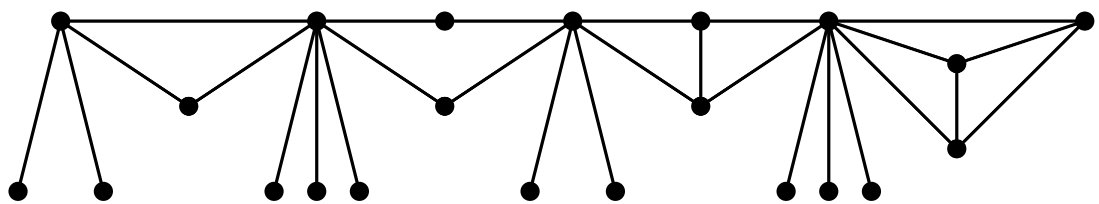
AEP graphs



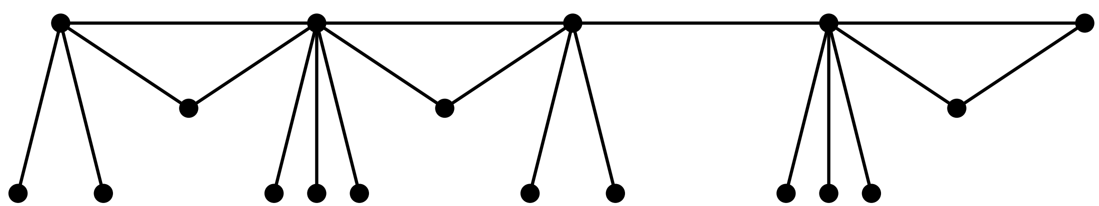
Radius-2 stars



Extended degree-3 spiders

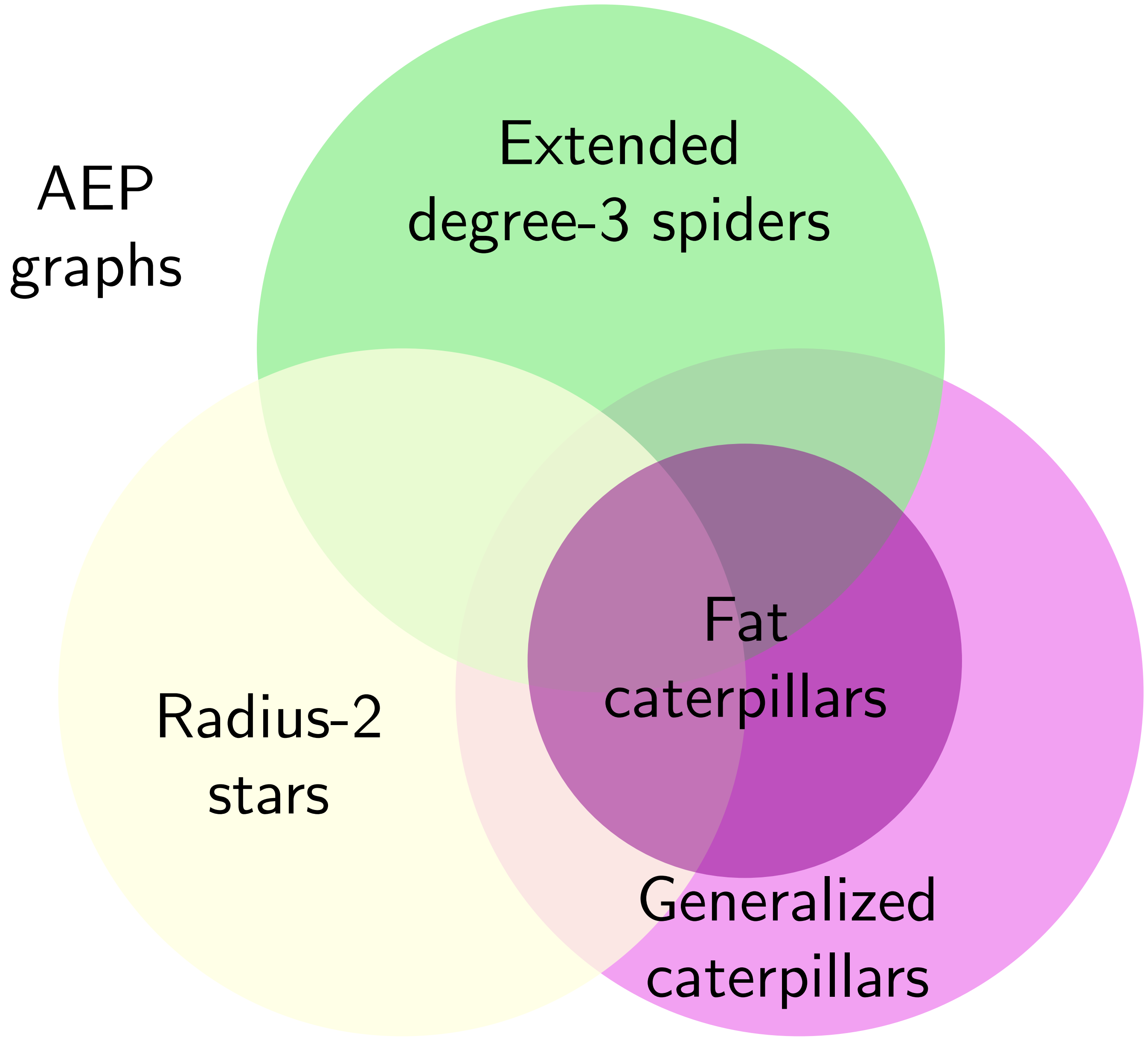


Generalized caterpillars

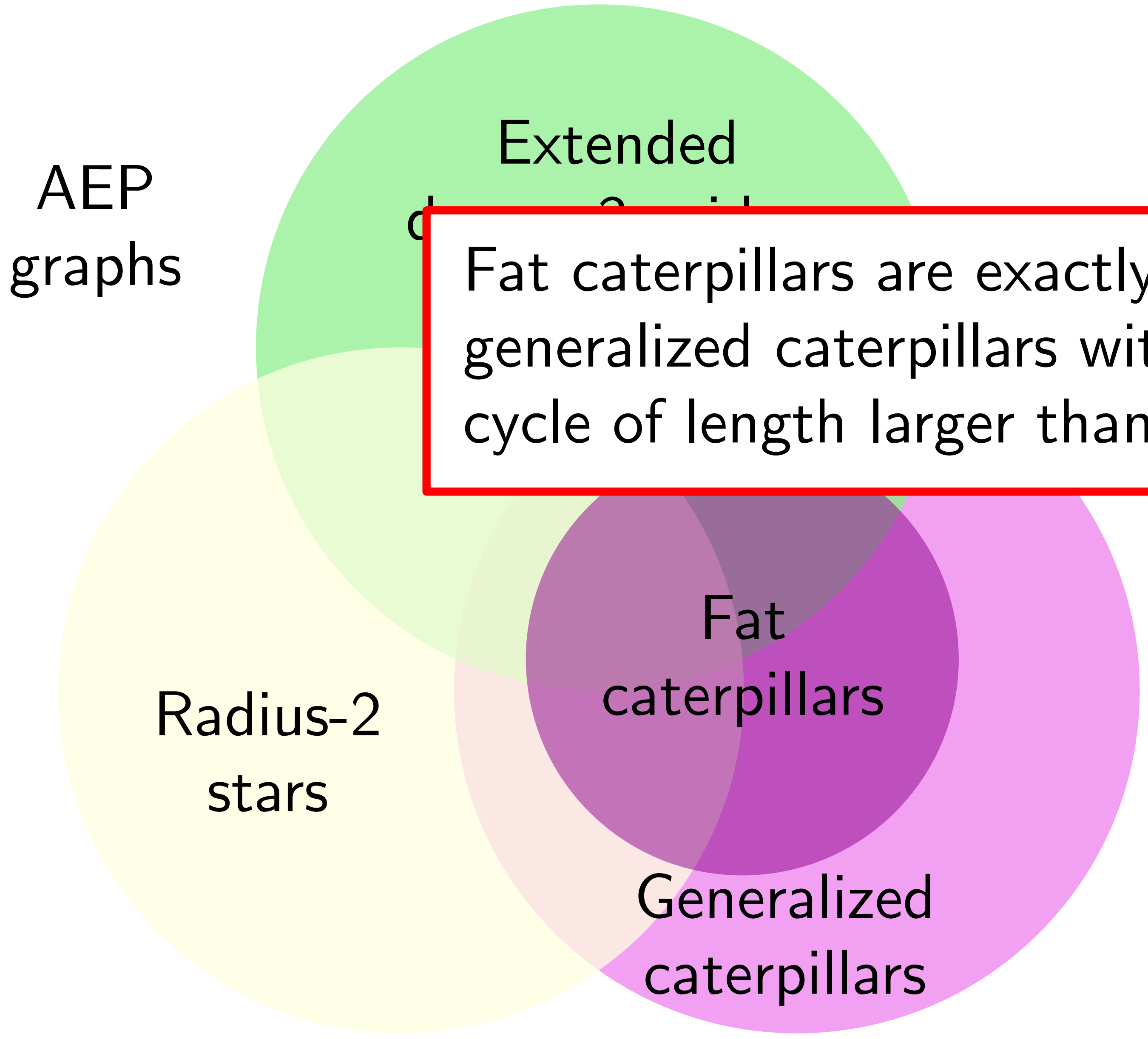


Fat caterpillars

AEP graphs


AEP
graphs
Extended
degree-3 spiders
Fat
caterpillars
Radius-2
stars
Generalized
caterpillars

AEP graphs



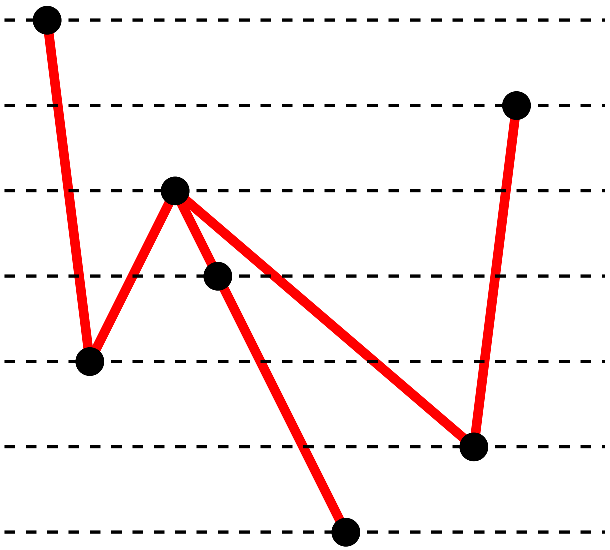
Fat caterpillars are exactly the generalized caterpillars without cycle of length larger than 3

Independent horizontal stabbing number

- The *independent horizontal stabbing number* of a s.l. drawing Γ (denoted as ihs(Γ)) is the maximum number of independent edges of Γ intersected by a horizontal line

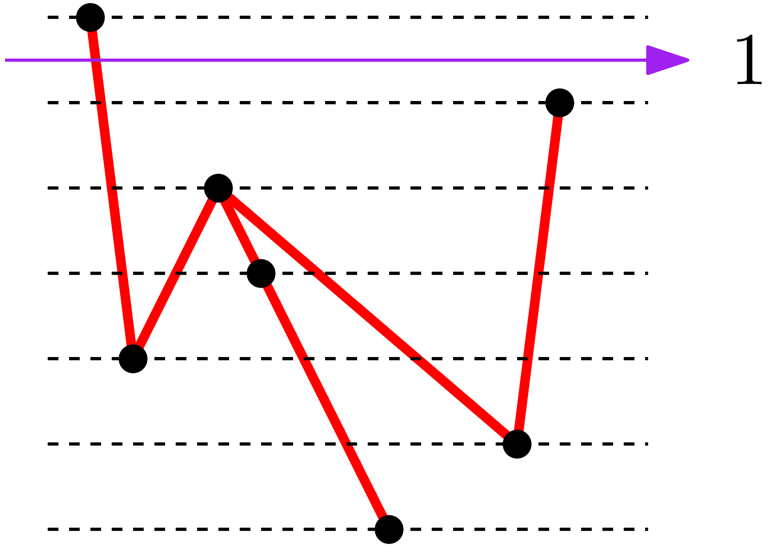
Independent horizontal stabbing number

- The *independent horizontal stabbing number* of a s.l. drawing Γ (denoted as $\text{ihs}(\Gamma)$) is the maximum number of independent edges of Γ intersected by a horizontal line



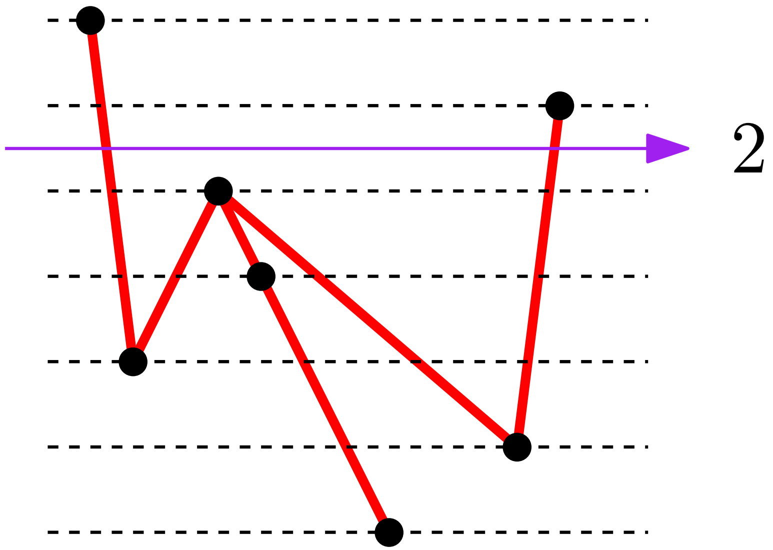
Independent horizontal stabbing number

- The *independent horizontal stabbing number* of a s.l. drawing Γ (denoted as $\text{ihs}(\Gamma)$) is the maximum number of independent edges of Γ intersected by a horizontal line



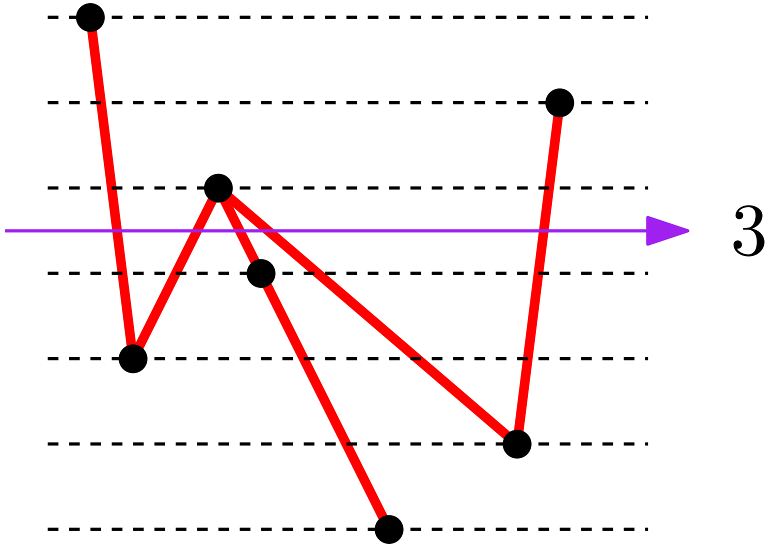
Independent horizontal stabbing number

- The *independent horizontal stabbing number* of a s.l. drawing Γ (denoted as ihs(Γ)) is the maximum number of independent edges of Γ intersected by a horizontal line



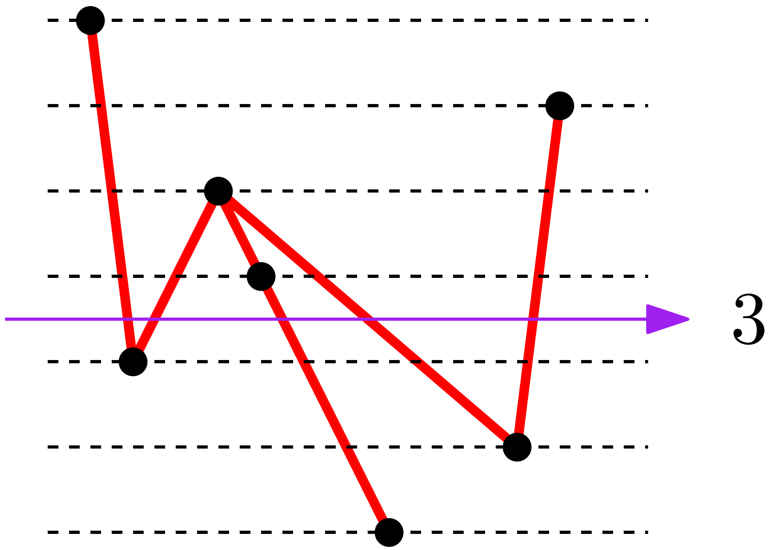
Independent horizontal stabbing number

- The *independent horizontal stabbing number* of a s.l. drawing Γ (denoted as $\mathsf{ihs}(\Gamma)$) is the maximum number of independent edges of Γ intersected by a horizontal line



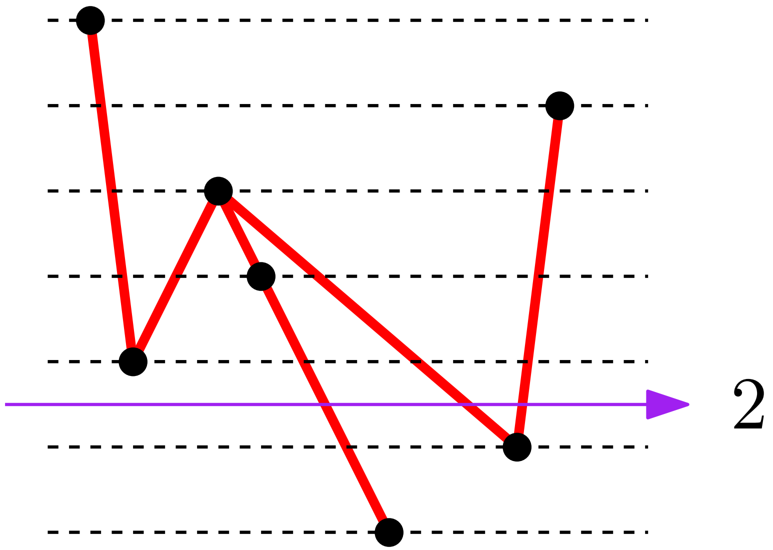
Independent horizontal stabbing number

- The *independent horizontal stabbing number* of a s.l. drawing Γ (denoted as $\mathsf{ihs}(\Gamma)$) is the maximum number of independent edges of Γ intersected by a horizontal line



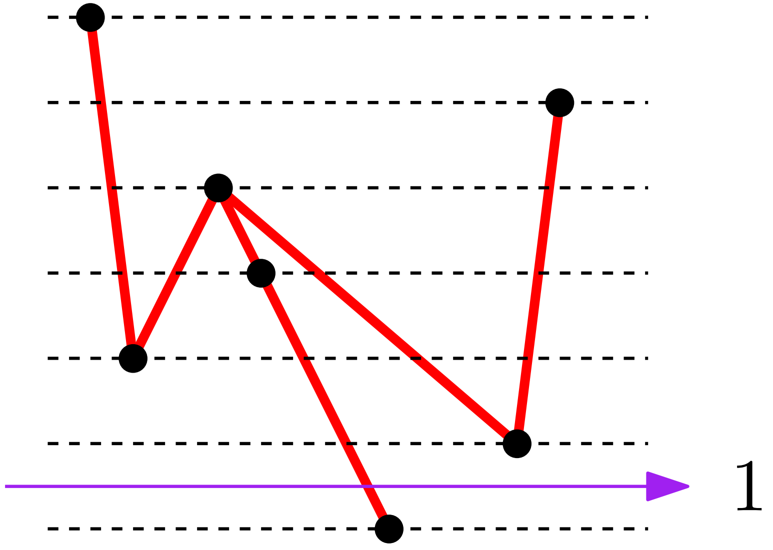
Independent horizontal stabbing number

- The *independent horizontal stabbing number* of a s.l. drawing Γ (denoted as $\text{ihs}(\Gamma)$) is the maximum number of independent edges of Γ intersected by a horizontal line



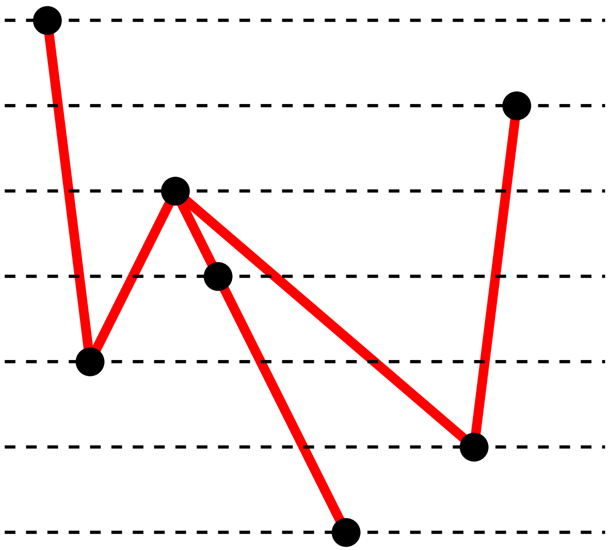
Independent horizontal stabbing number

- The *independent horizontal stabbing number* of a s.l. drawing Γ (denoted as ihs(Γ)) is the maximum number of independent edges of Γ intersected by a horizontal line



Independent horizontal stabbing number

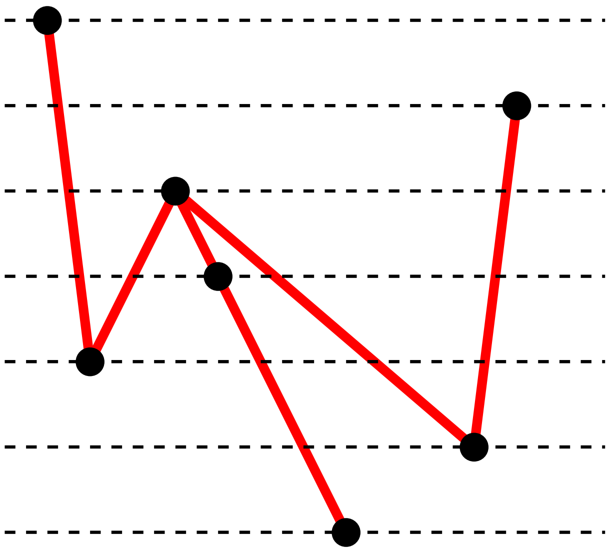
- The *independent horizontal stabbing number* of a s.l. drawing Γ (denoted as $\text{ihs}(\Gamma)$) is the maximum number of independent edges of Γ intersected by a horizontal line



$\text{ihs}(\Gamma) = 3$

Independent horizontal stabbing number

- The *independent horizontal stabbing number* of a s.l. drawing Γ (denoted as $\text{ihs}(\Gamma)$) is the maximum number of independent edges of Γ intersected by a horizontal line
- The *independent horizontal stabbing number* of a graph G (denoted as $\text{ihs}(G)$) is the minimum independent horizontal stabbing number over all straight-line drawings of G



$\text{ihs}(\Gamma) = 3$

A technical lemma

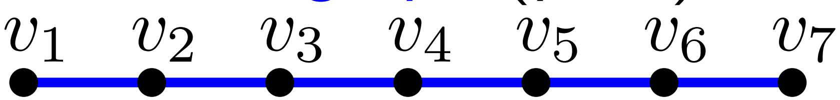
Lemma 2 *A graph* G *is an EAP graph if and only if* $\mathrm{ihs}(G) = 1$.

A technical lemma

Lemma 2 *A graph G is an EAP graph if and only if* ihs$(G) = 1$.

$(\Rightarrow)$

G: EAP graph (path)

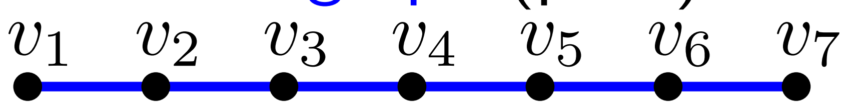


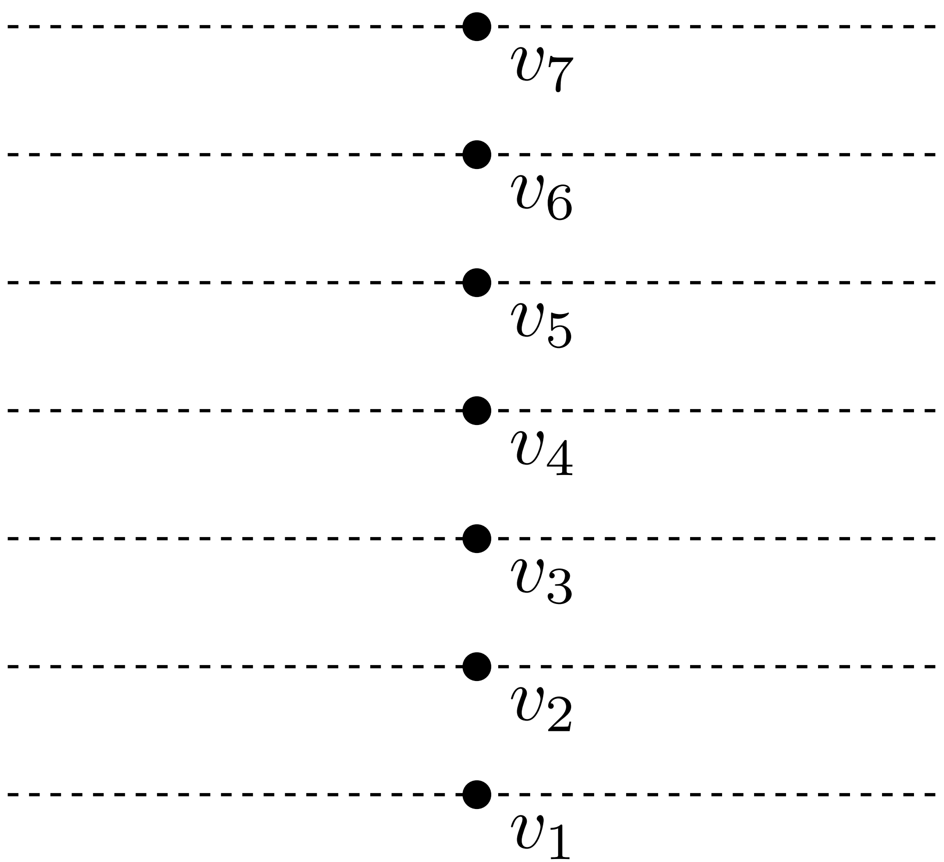
A technical lemma

Lemma 2 *A graph G is an EAP graph if and only if* $\mathsf{ihs}(G) = 1$.

$(\Rightarrow)$

G: EAP graph (path)



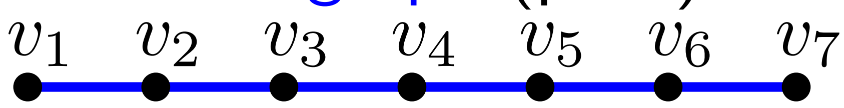


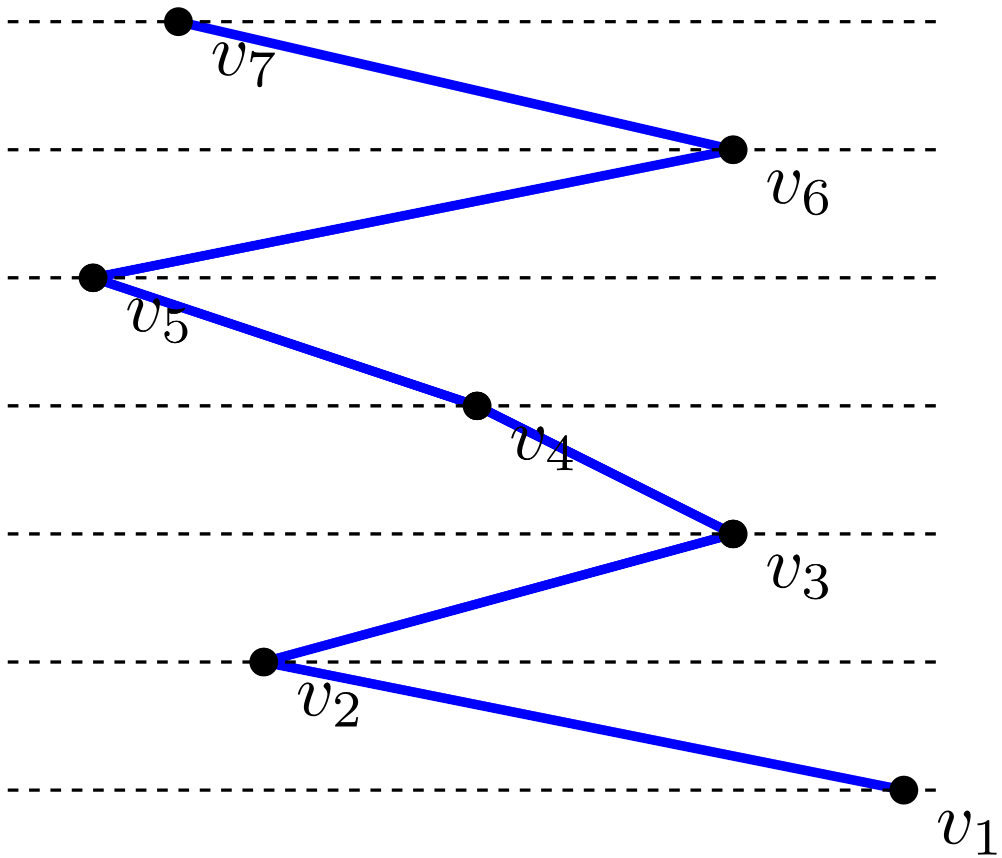
A technical lemma

Lemma 2 *A graph G is an EAP graph if and only if* $\mathsf{ihs}(G) = 1$.

$(\Rightarrow)$

G: EAP graph (path)



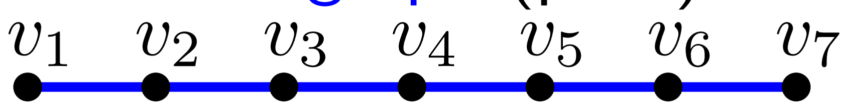


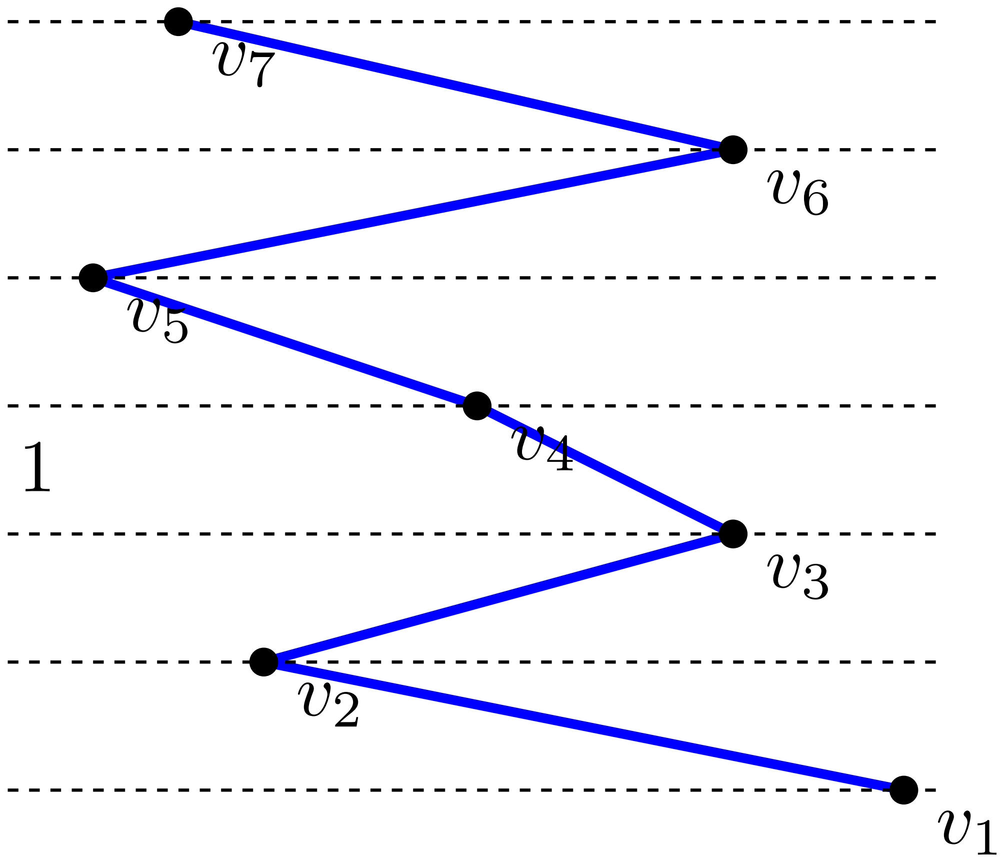
A technical lemma

Lemma 2 *A graph G is an EAP graph if and only if* $\mathrm{ihs}(G) = 1$.

$(\Rightarrow)$

G: EAP graph (path)





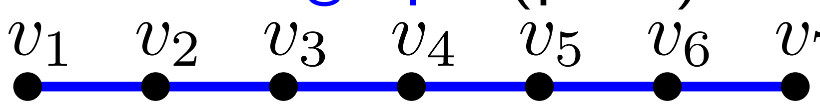
$\mathrm{ihs}(\Gamma) = 1$

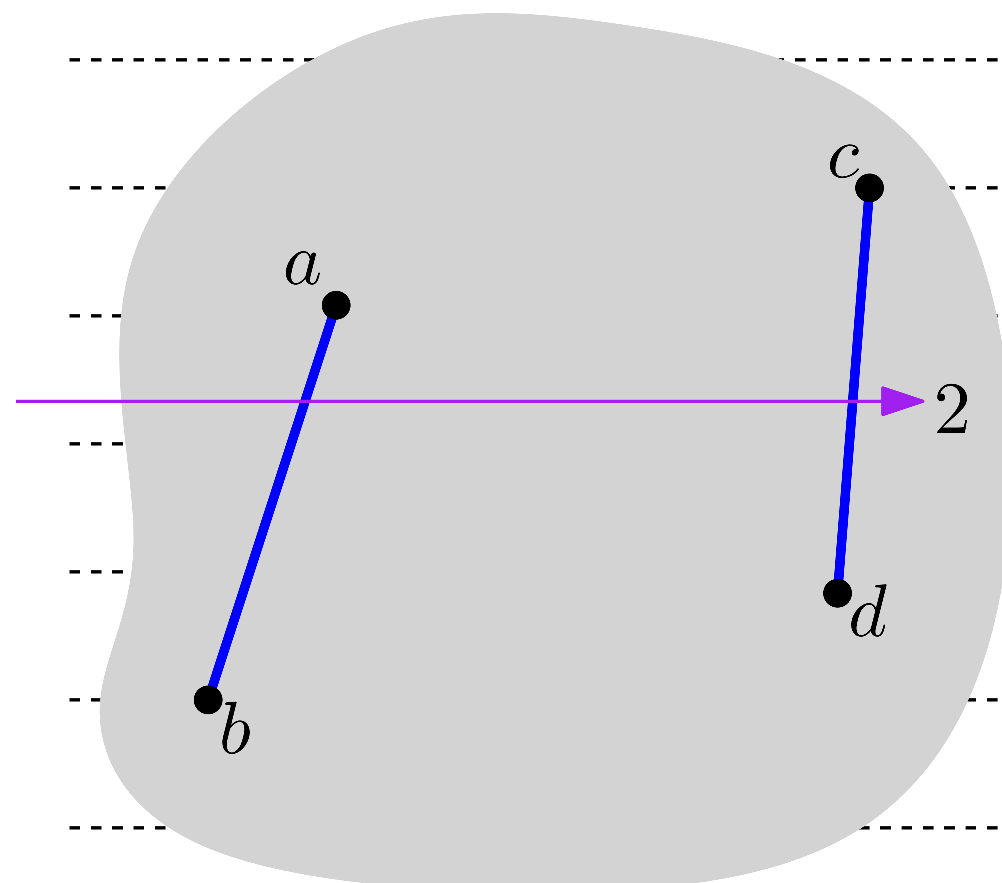
A technical lemma

Lemma 2 *A graph G is an EAP graph if and only if* $\mathsf{ihs}(G) = 1$.

$(\Rightarrow)$

G: EAP graph (path)



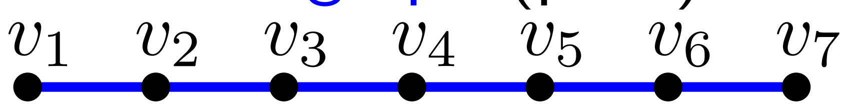


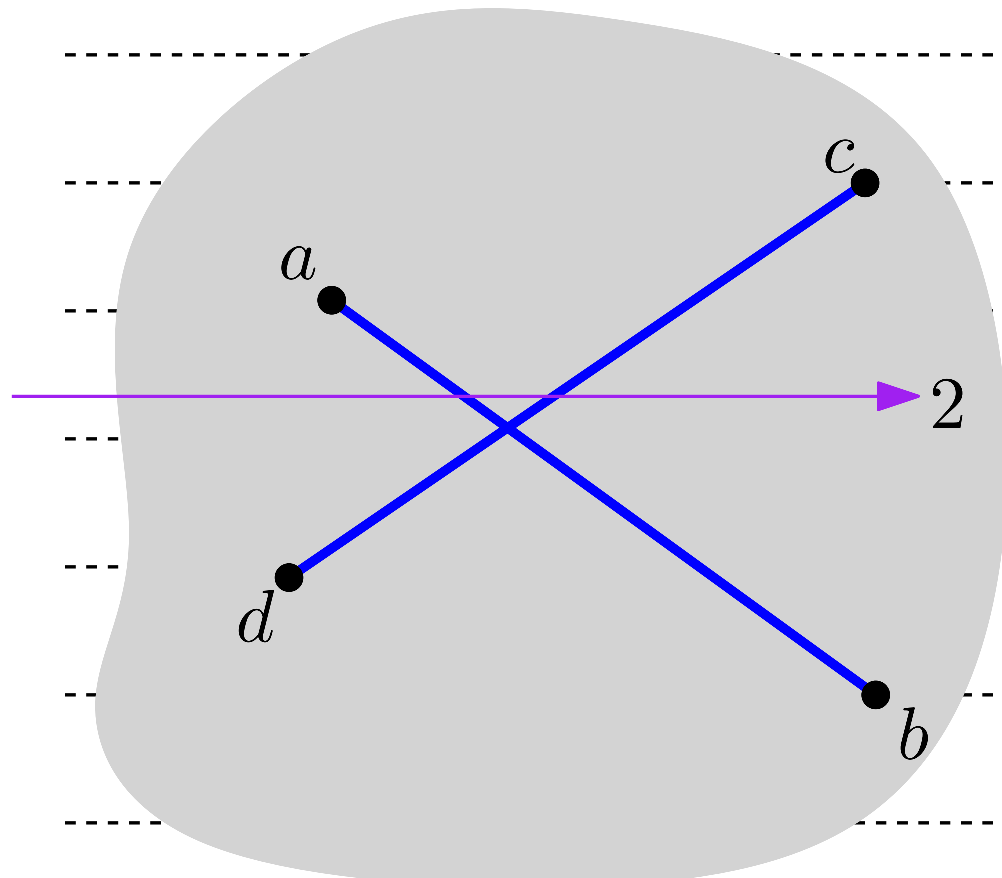
A technical lemma

Lemma 2 *A graph G is an EAP graph if and only if* $\mathsf{ihs}(G) = 1$*.*

$(\Rightarrow)$

G: EAP graph (path)



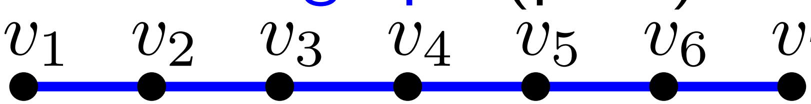


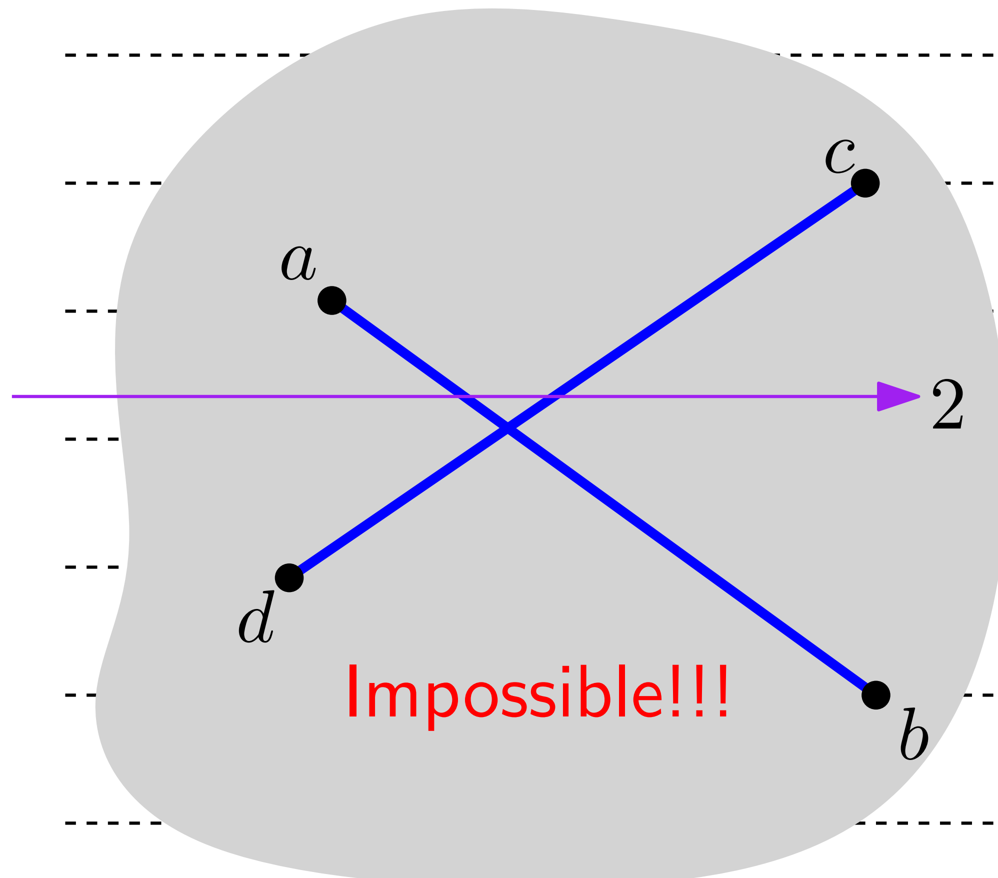
A technical lemma

Lemma 2 *A graph G is an EAP graph if and only if* $\mathsf{ihs}(G) = 1$.

$(\Rightarrow)$

G: EAP graph (path)





A technical lemma

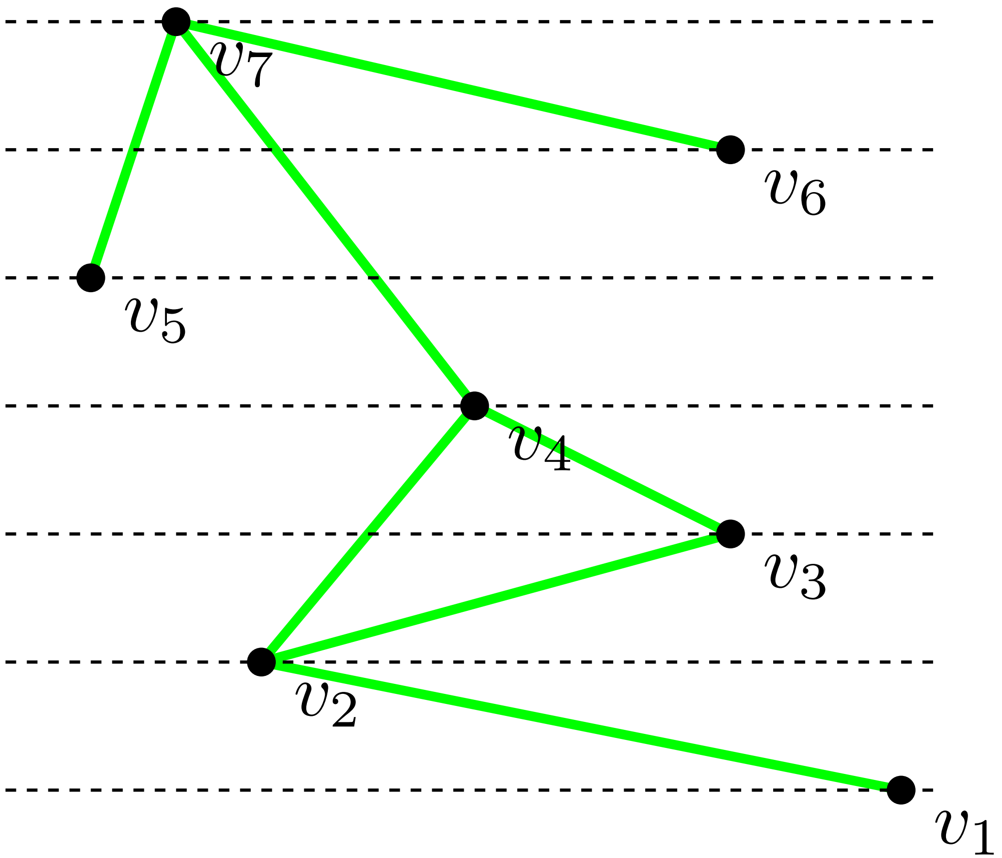
Lemma 2 *A graph* G *is an EAP graph if and only if* $\mathsf{ihs}(G) = 1$*.*

$(\Leftarrow)$

A technical lemma

Lemma 2 *A graph G is an EAP graph if and only if* $\mathrm{ihs}(G) = 1$.

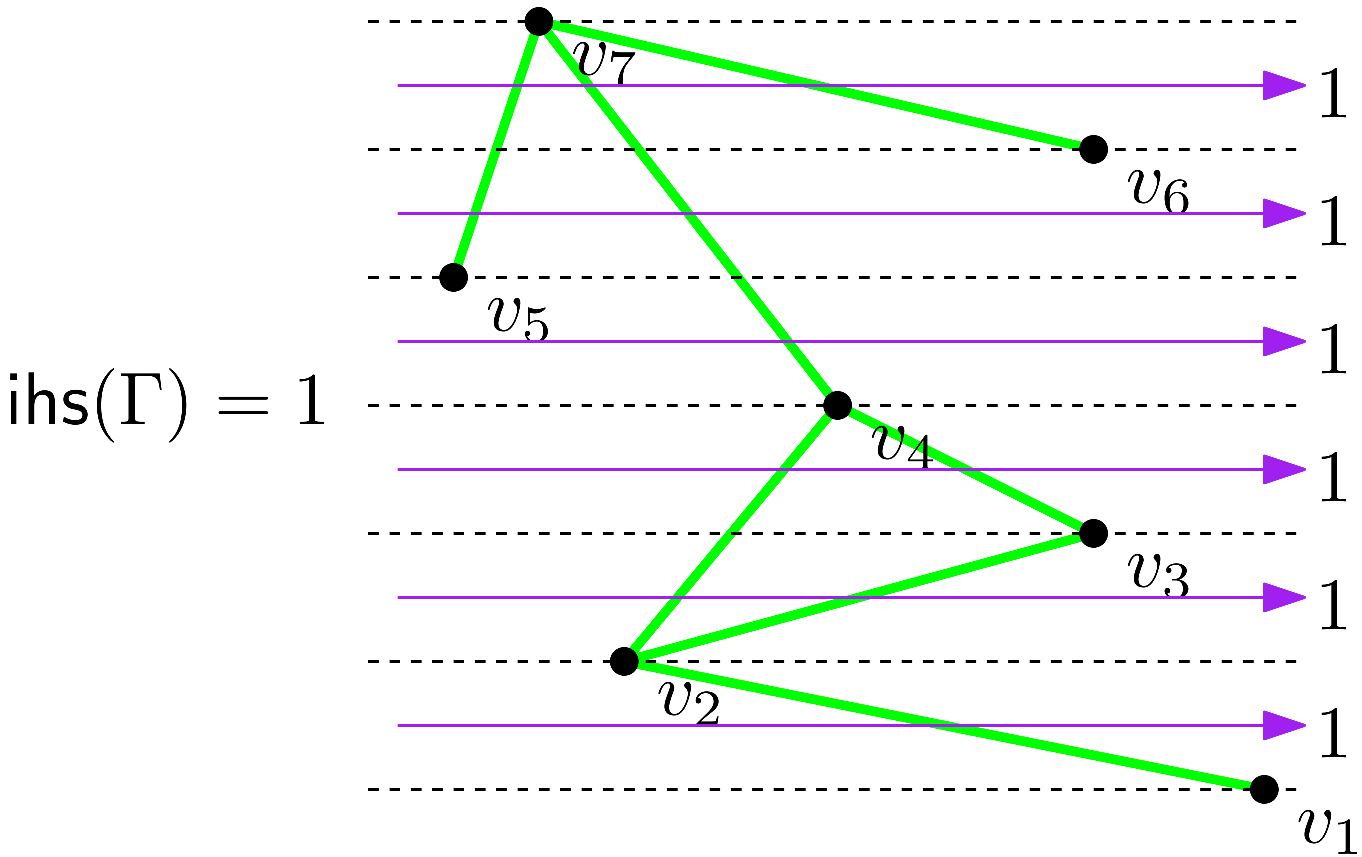
$(\Leftarrow)$



A technical lemma

Lemma 2 *A graph G is an EAP graph if and only if* $\mathsf{ihs}(G) = 1$.

$(\Leftarrow)$

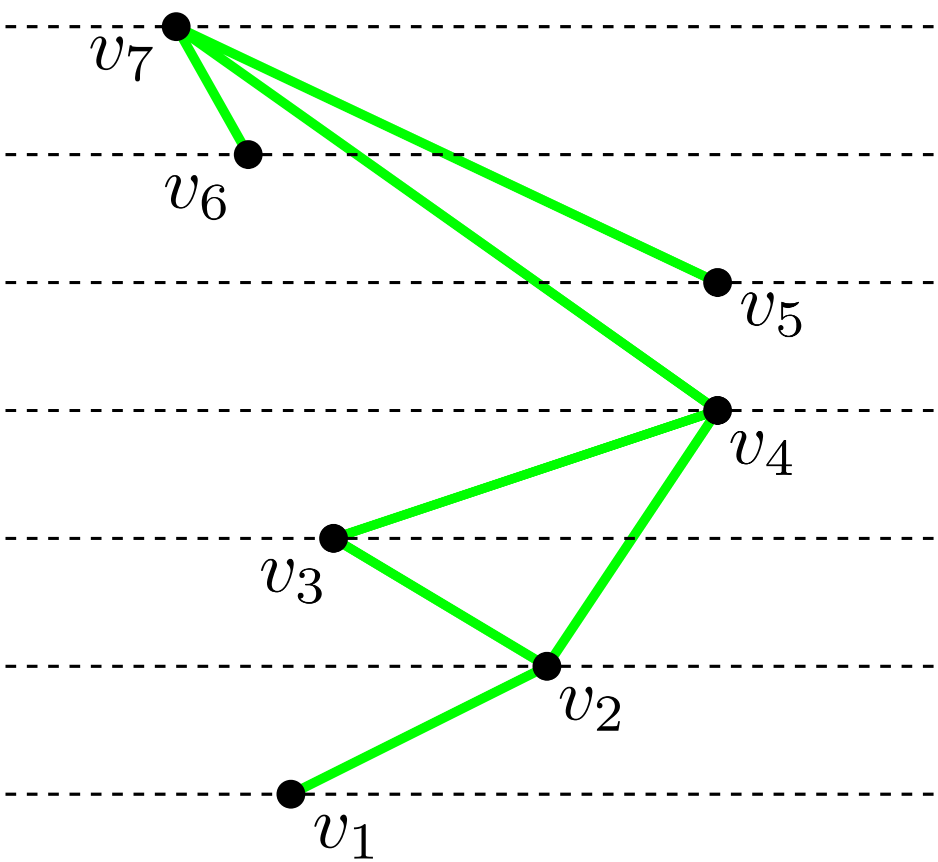


A technical lemma

Lemma 2 *A graph G is an EAP graph if and only if* $\mathsf{ihs}(G) = 1$.

$(\Leftarrow)$

$\mathsf{ihs}(\Gamma) = 1$

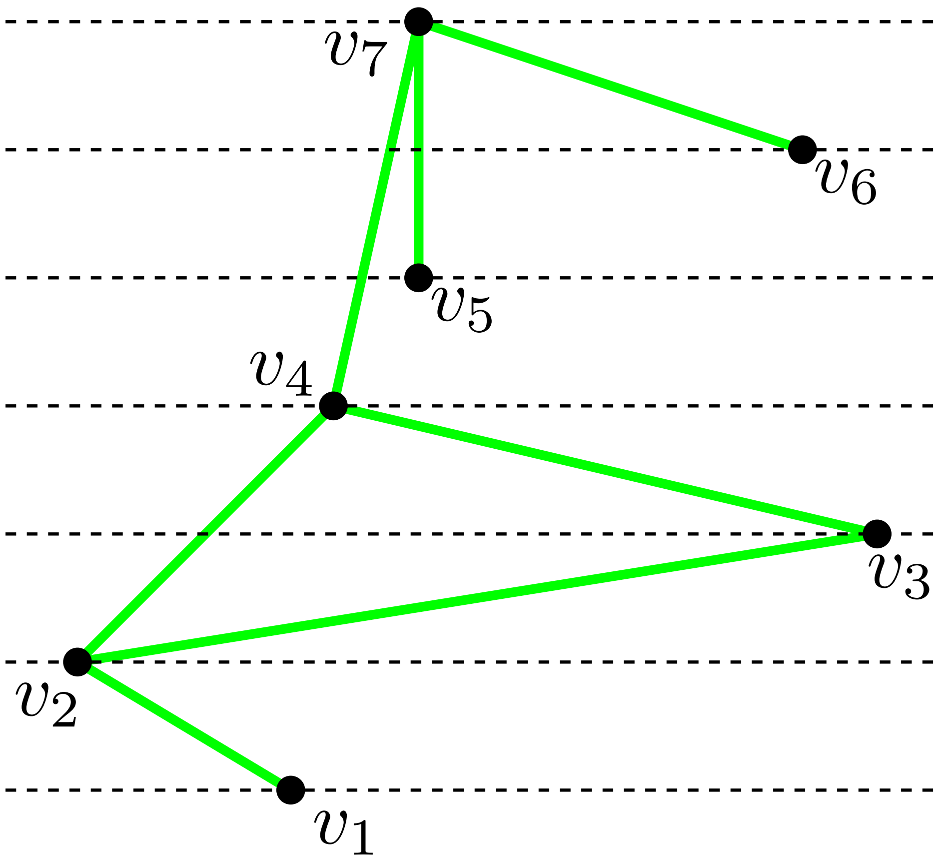


A technical lemma

Lemma 2 *A graph G is an EAP graph if and only if* $\mathrm{ihs}(G) = 1$.

$(\Leftarrow)$

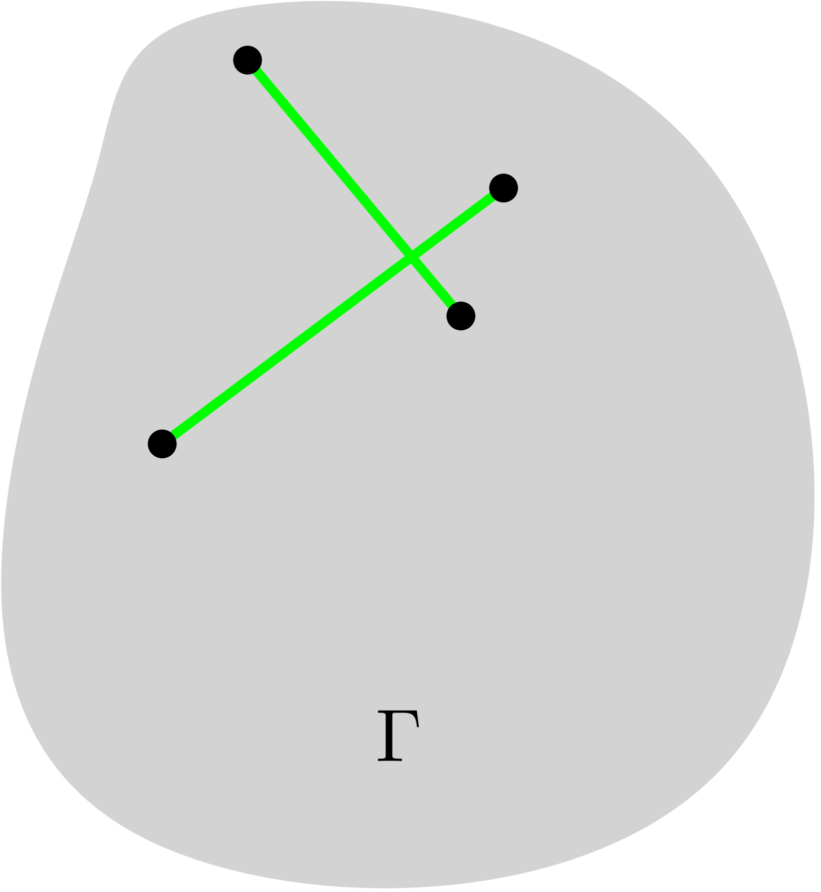
$\mathrm{ihs}(\Gamma) = 1$



A technical lemma

Lemma 2 *A graph G is an EAP graph if and only if* $\mathsf{ihs}(G) = 1$.

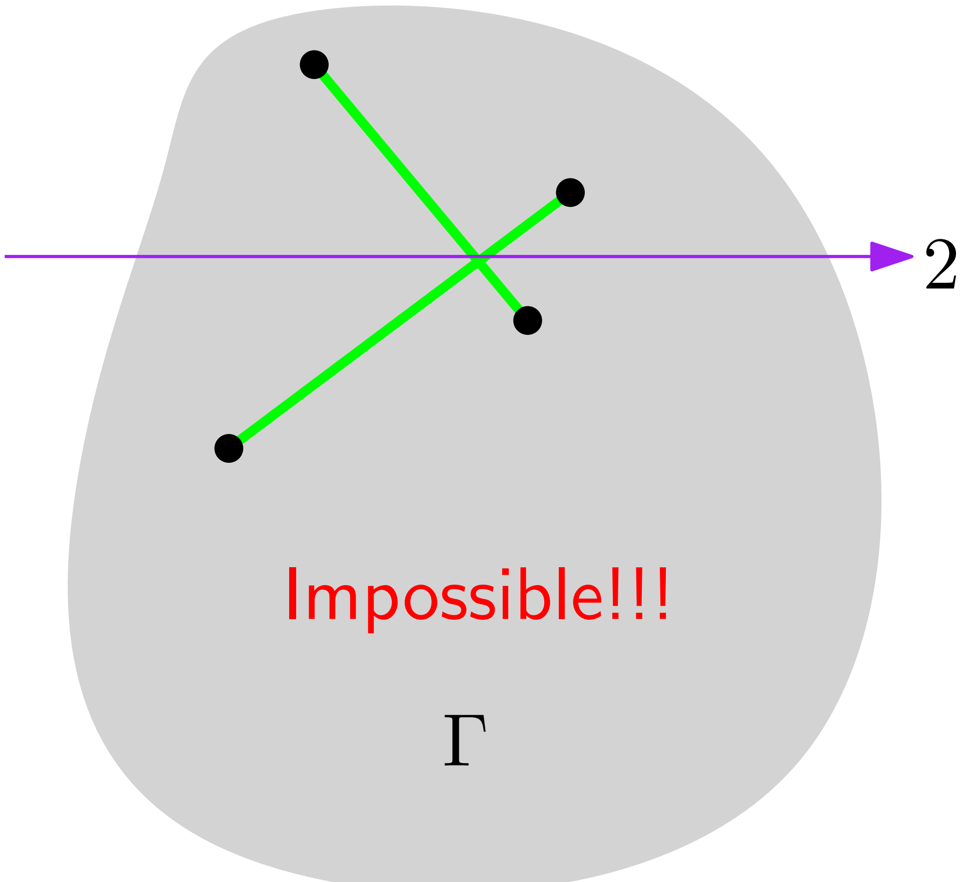
$(\Leftarrow)$



A technical lemma

Lemma 2 *A graph G is an EAP graph if and only if* $\mathsf{ihs}(G) = 1$.

$(\Leftarrow)$

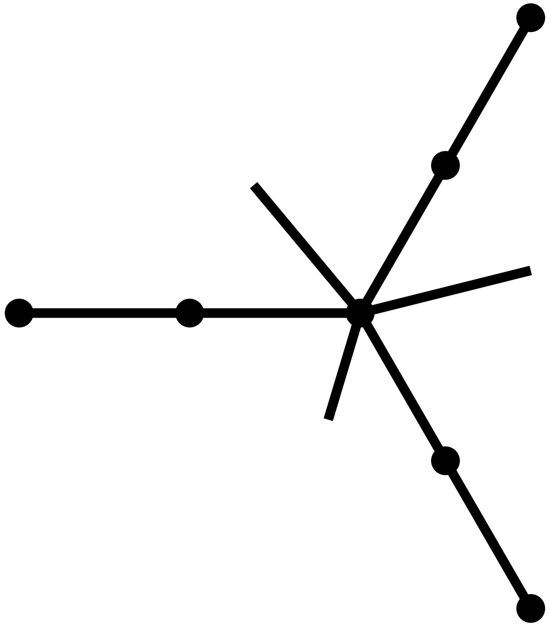


Characterization of EAP graphs

Lemma 3 *Let G be a radius-2 star that is not a generalized caterpillar, then* $\text{ihs}(G) > 1$.

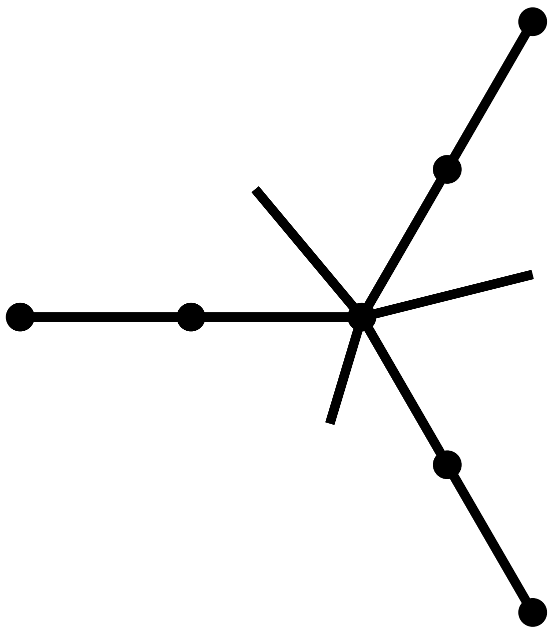
Characterization of EAP graphs

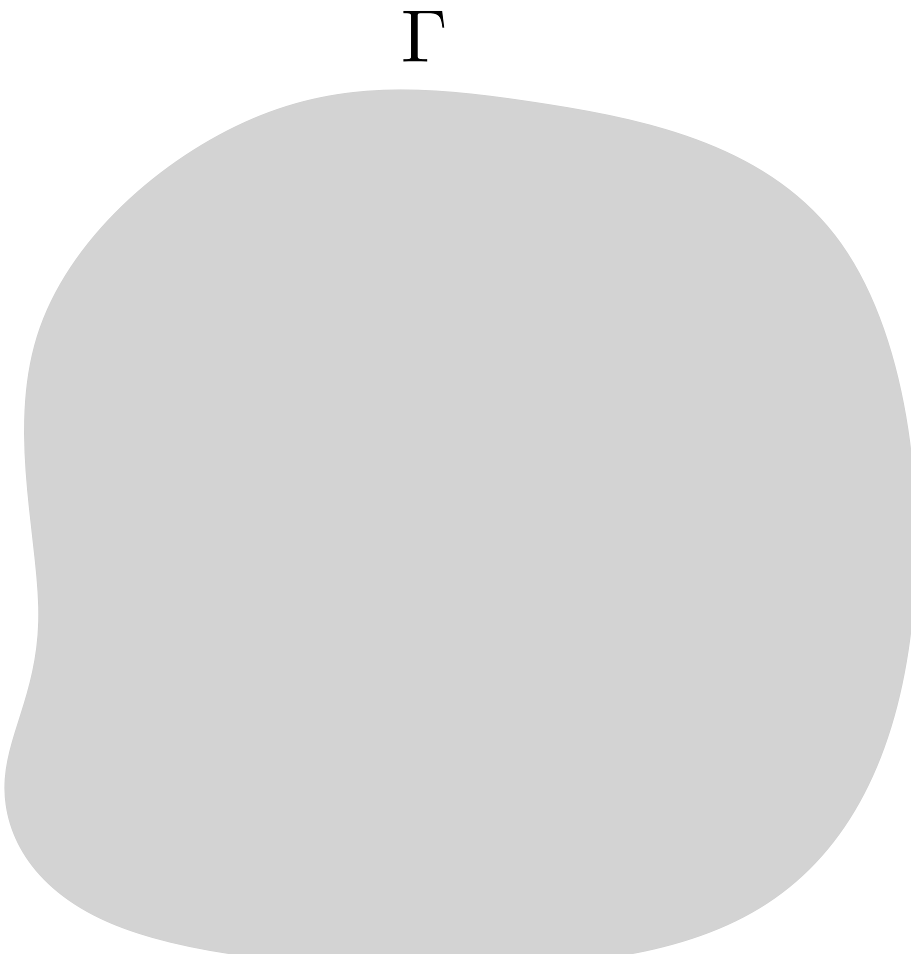
Lemma 3 *Let G be a radius-2 star that is not a generalized caterpillar, then* $\text{ihs}(G) > 1$.



Characterization of EAP graphs

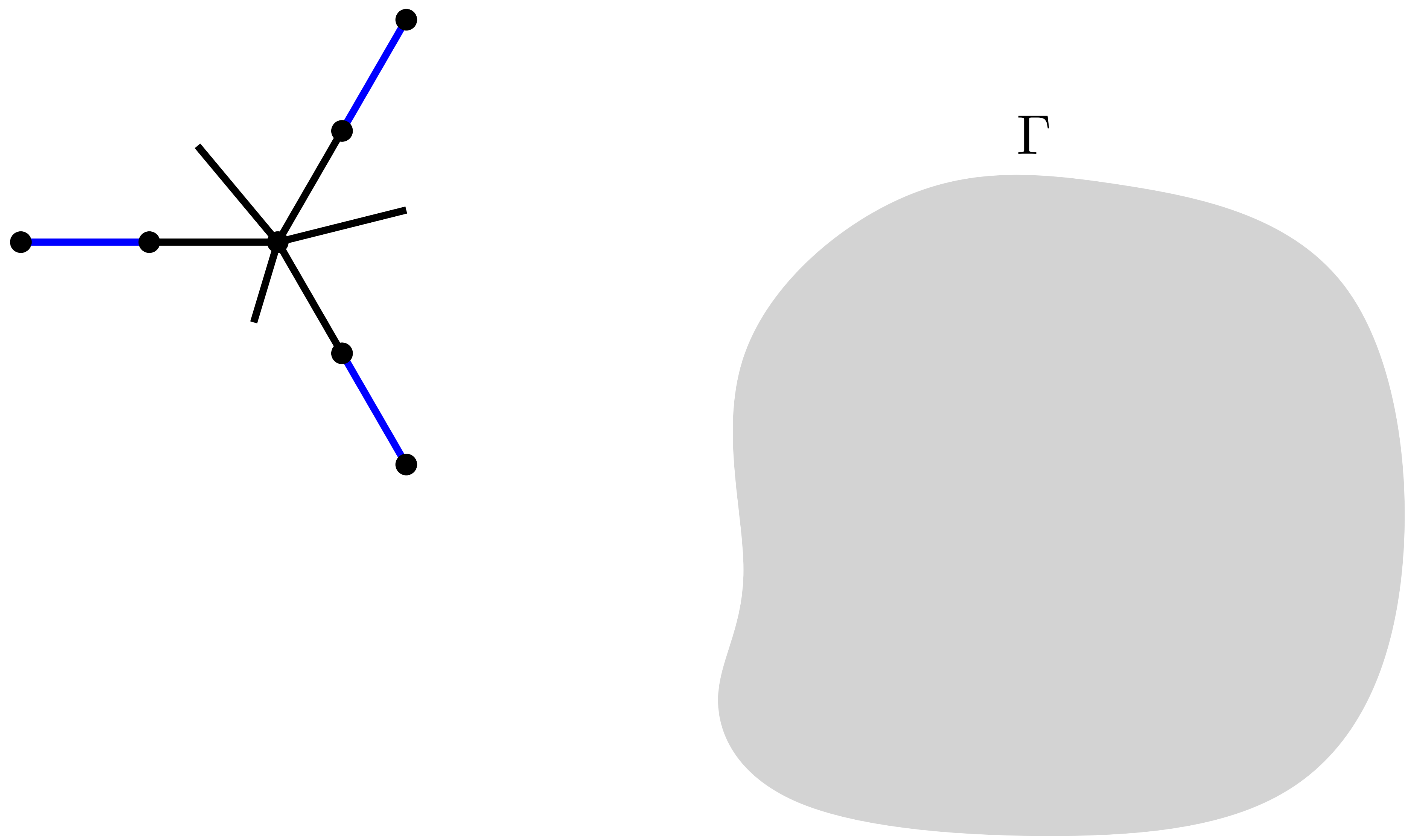
Lemma 3 *Let G be a radius-2 star that is not a generalized caterpillar, then* $\mathrm{ihs}(G) > 1$.





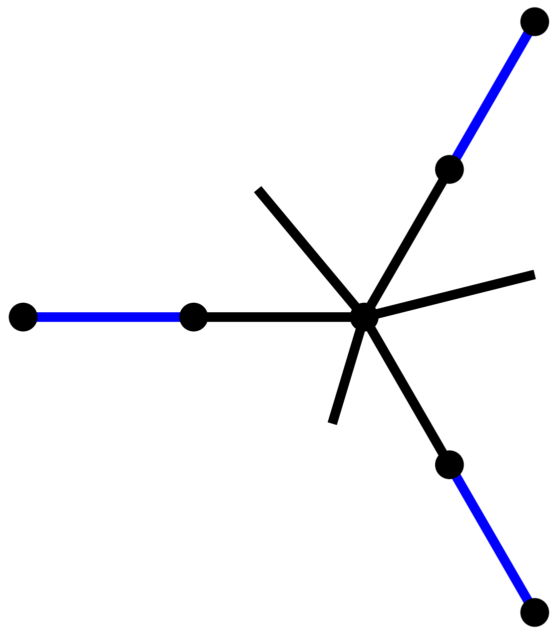
Characterization of EAP graphs

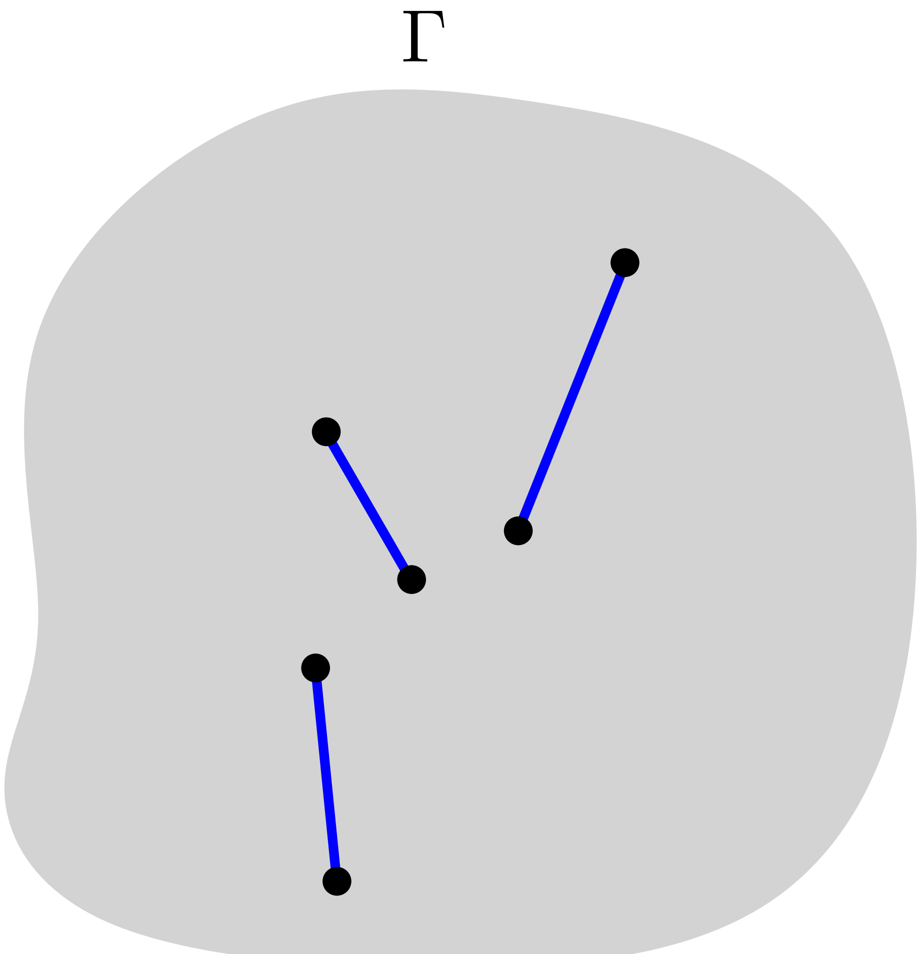
Lemma 3 *Let G be a radius-2 star that is not a generalized caterpillar, then* $\text{ihs}(G) > 1$.



Characterization of EAP graphs

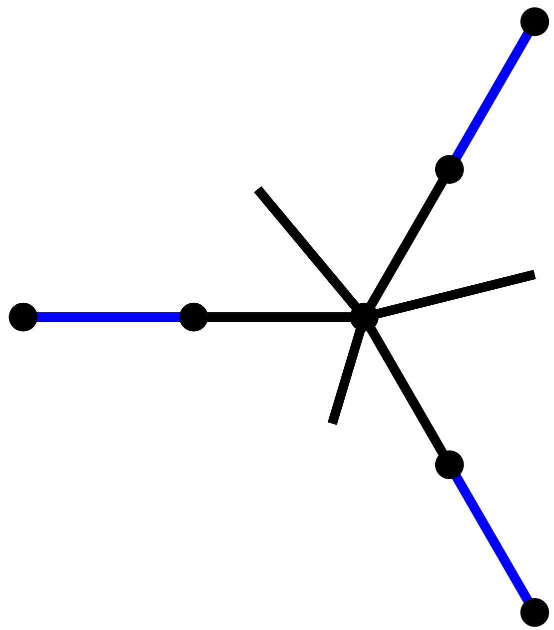
Lemma 3 *Let G be a radius-2 star that is not a generalized caterpillar, then* $\text{ihs}(G) > 1$.

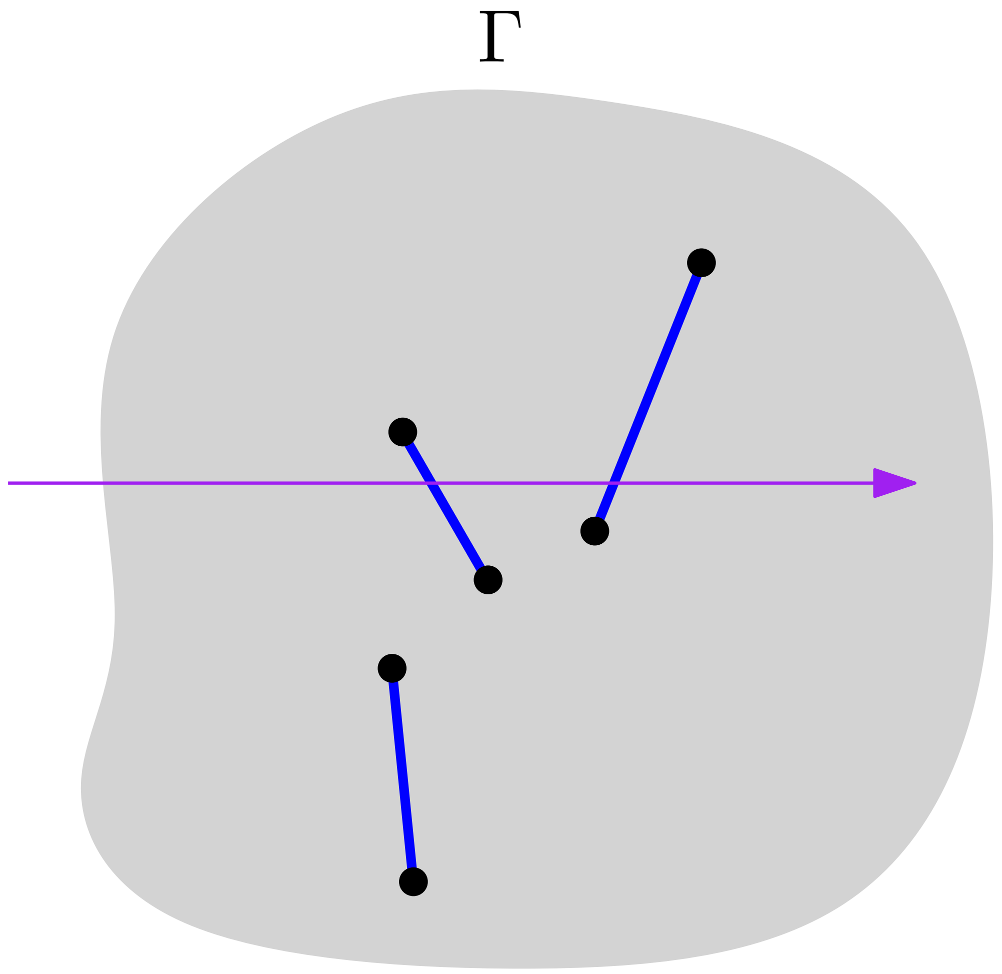




Characterization of EAP graphs

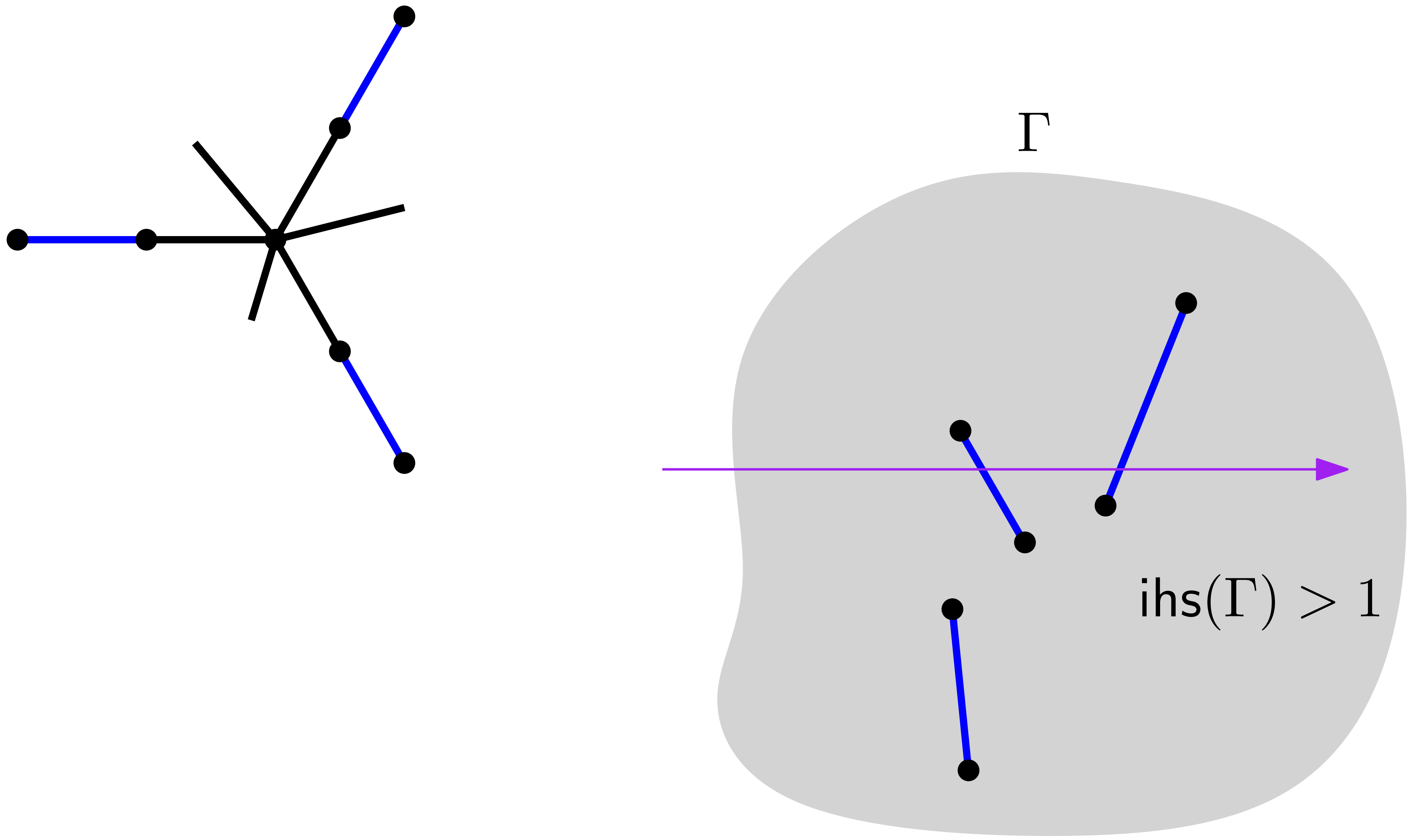
Lemma 3 *Let G be a radius-2 star that is not a generalized caterpillar, then* $\text{ihs}(G) > 1$.





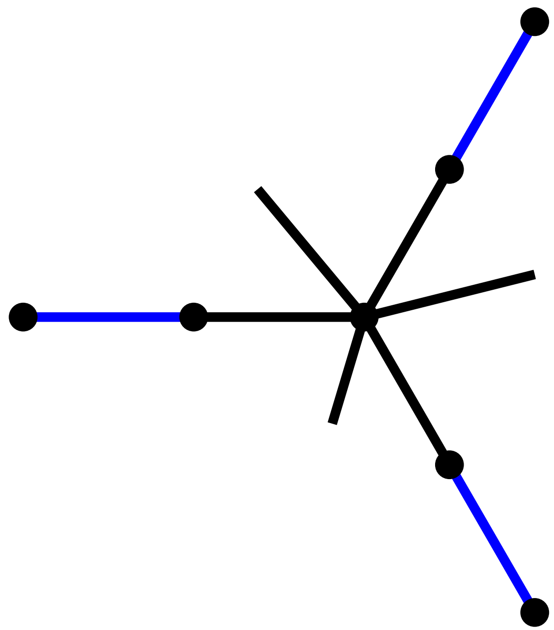
Characterization of EAP graphs

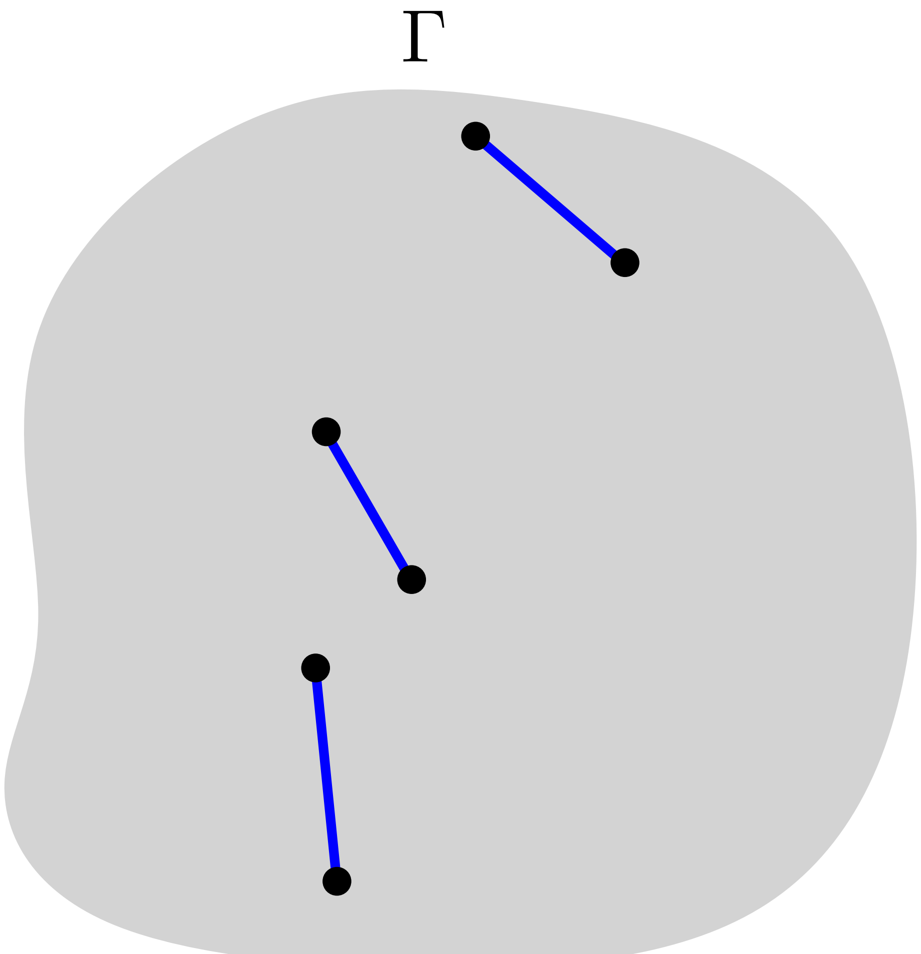
Lemma 3 *Let G be a radius-2 star that is not a generalized caterpillar, then* $\text{ihs}(G) > 1$.



Characterization of EAP graphs

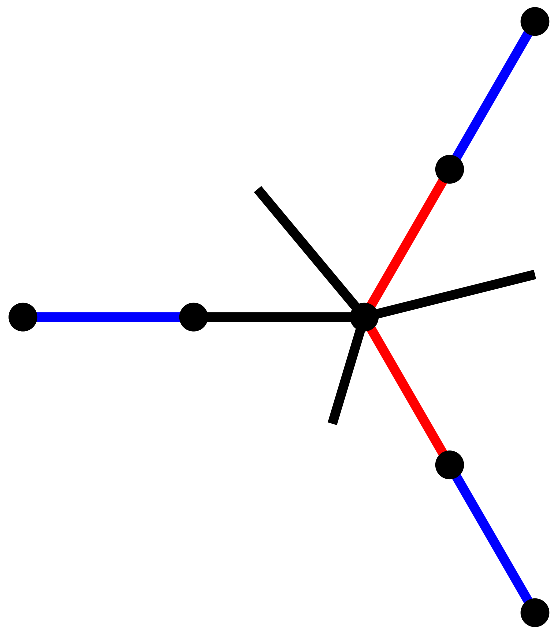
Lemma 3 *Let G be a radius-2 star that is not a generalized caterpillar, then* $\text{ihs}(G) > 1$.

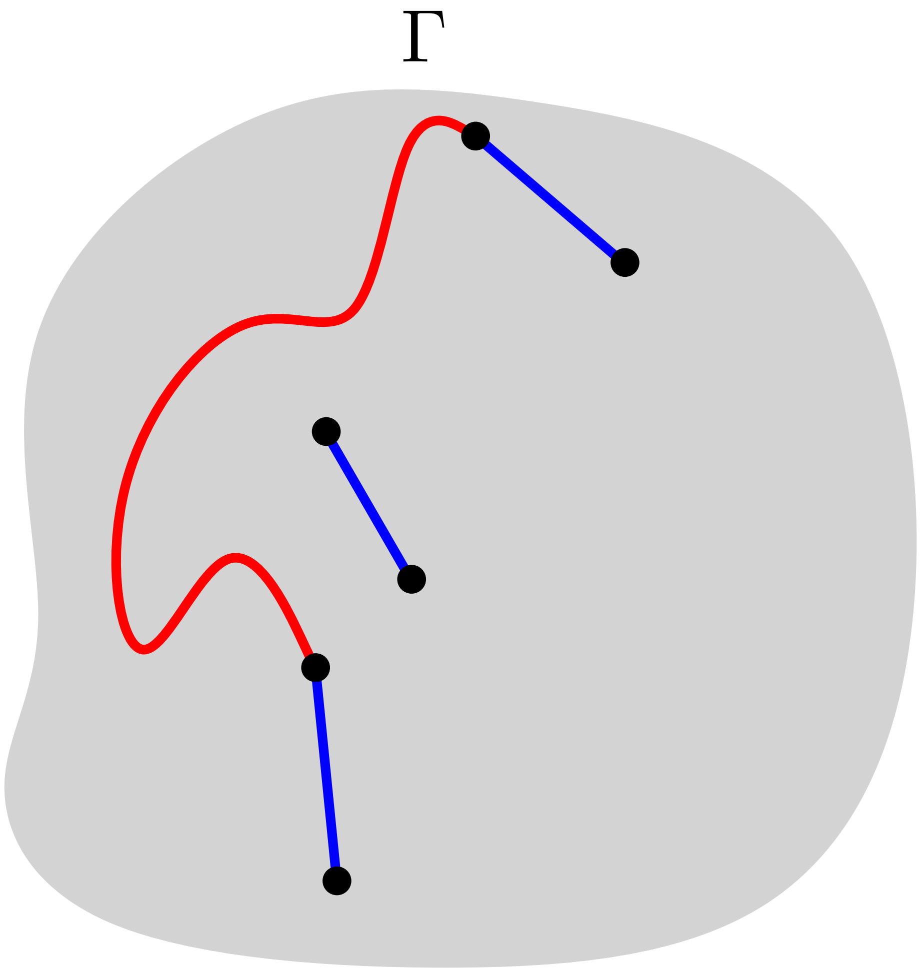




Characterization of EAP graphs

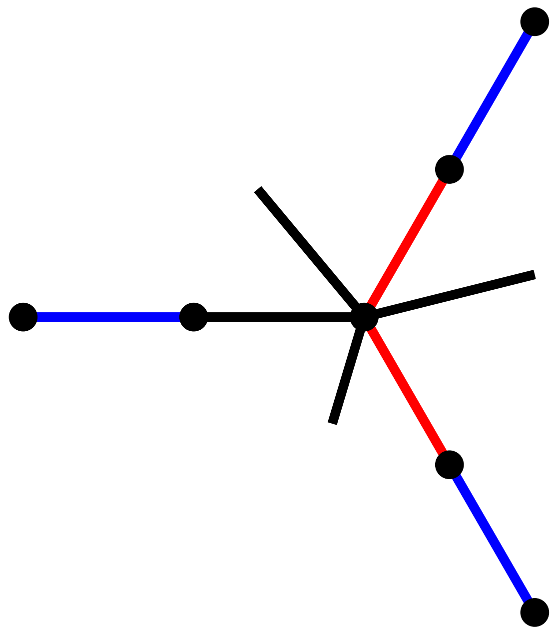
Lemma 3 *Let G be a radius-2 star that is not a generalized caterpillar, then* $\mathrm{ihs}(G) > 1$.

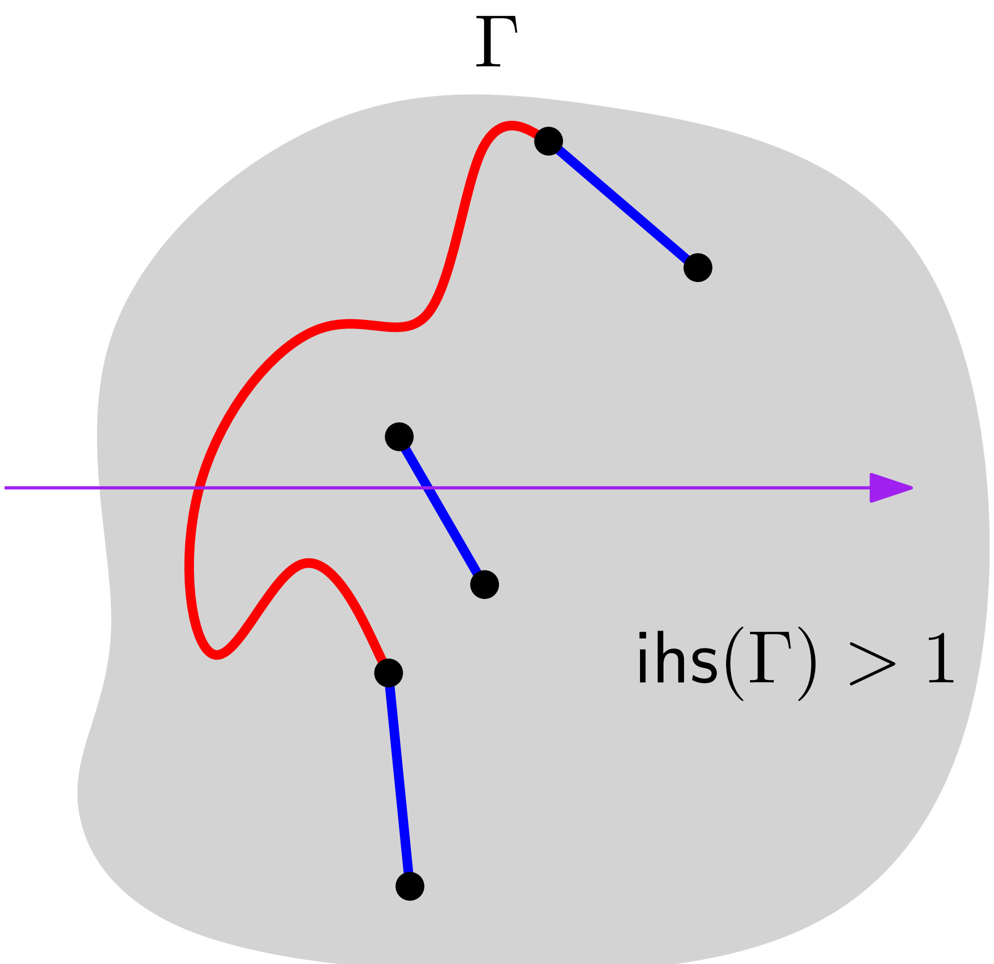




Characterization of EAP graphs

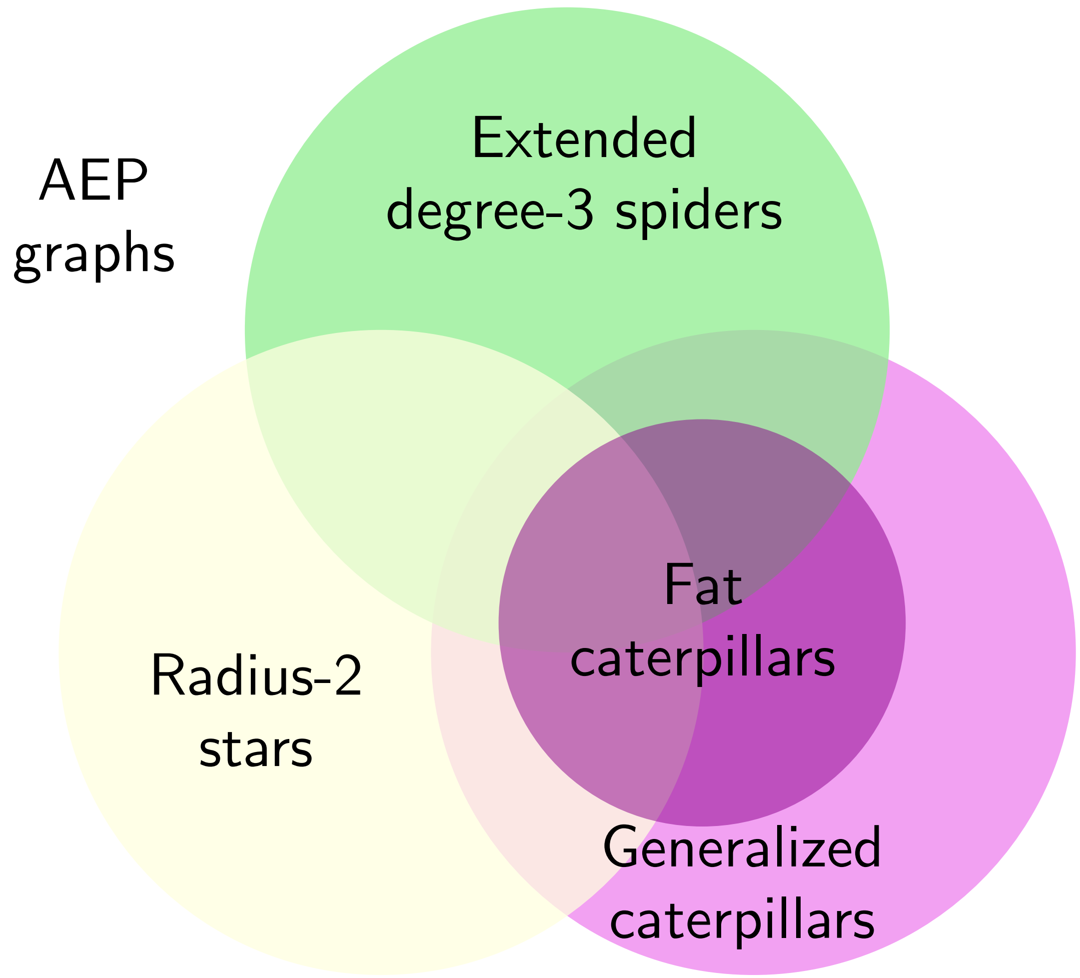
Lemma 3 *Let G be a radius-2 star that is not a generalized caterpillar, then* $\text{ihs}(G) > 1$.





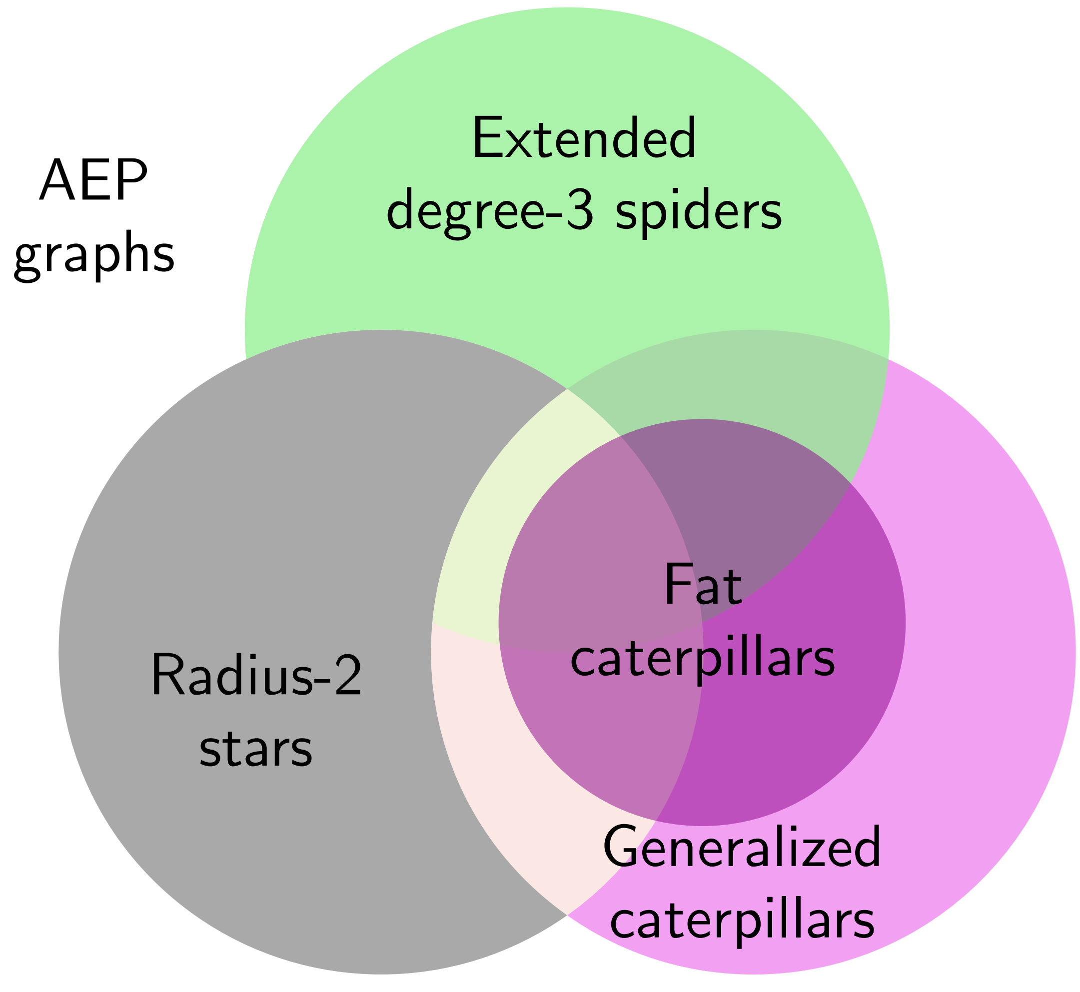
Characterization of EAP graphs

Lemma 3 *Let G be a radius-2 star that is not a generalized caterpillar, then* $\text{ihs}(G) > 1$.



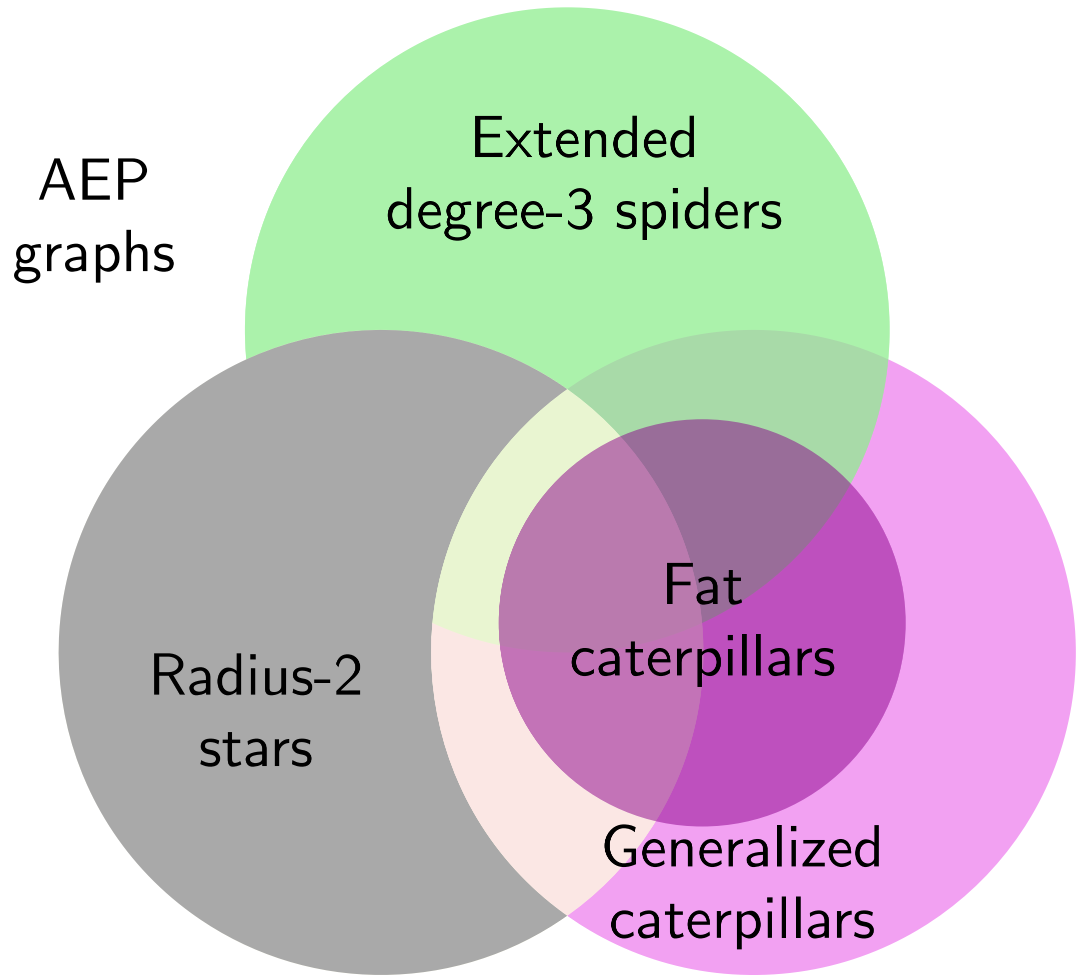
Characterization of EAP graphs

Lemma 3 *Let G be a radius-2 star that is not a generalized caterpillar, then* $\text{ihs}(G) > 1$.



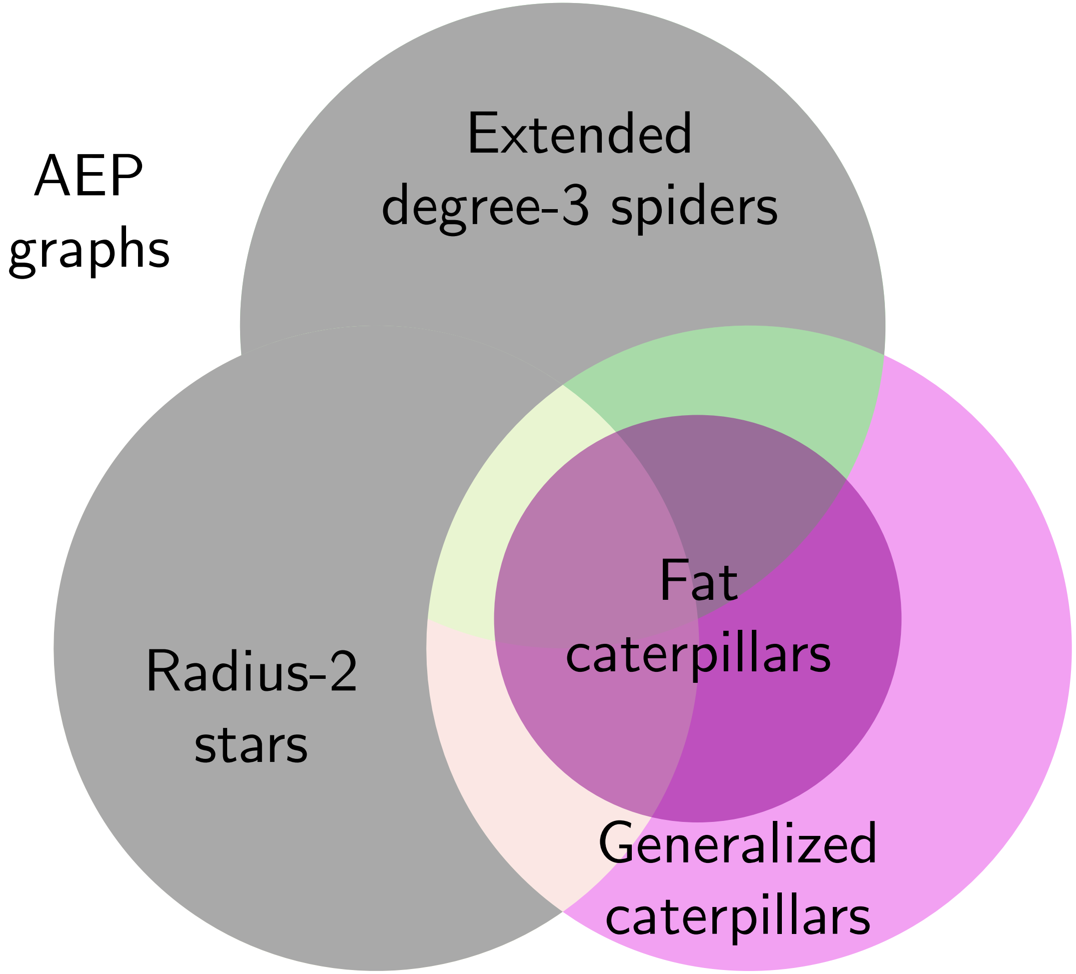
Characterization of EAP graphs

Lemma 4 *Let G be an extended degree-3 spider that is not a generalized caterpillar, then* $\text{ihs}(G) > 1$.



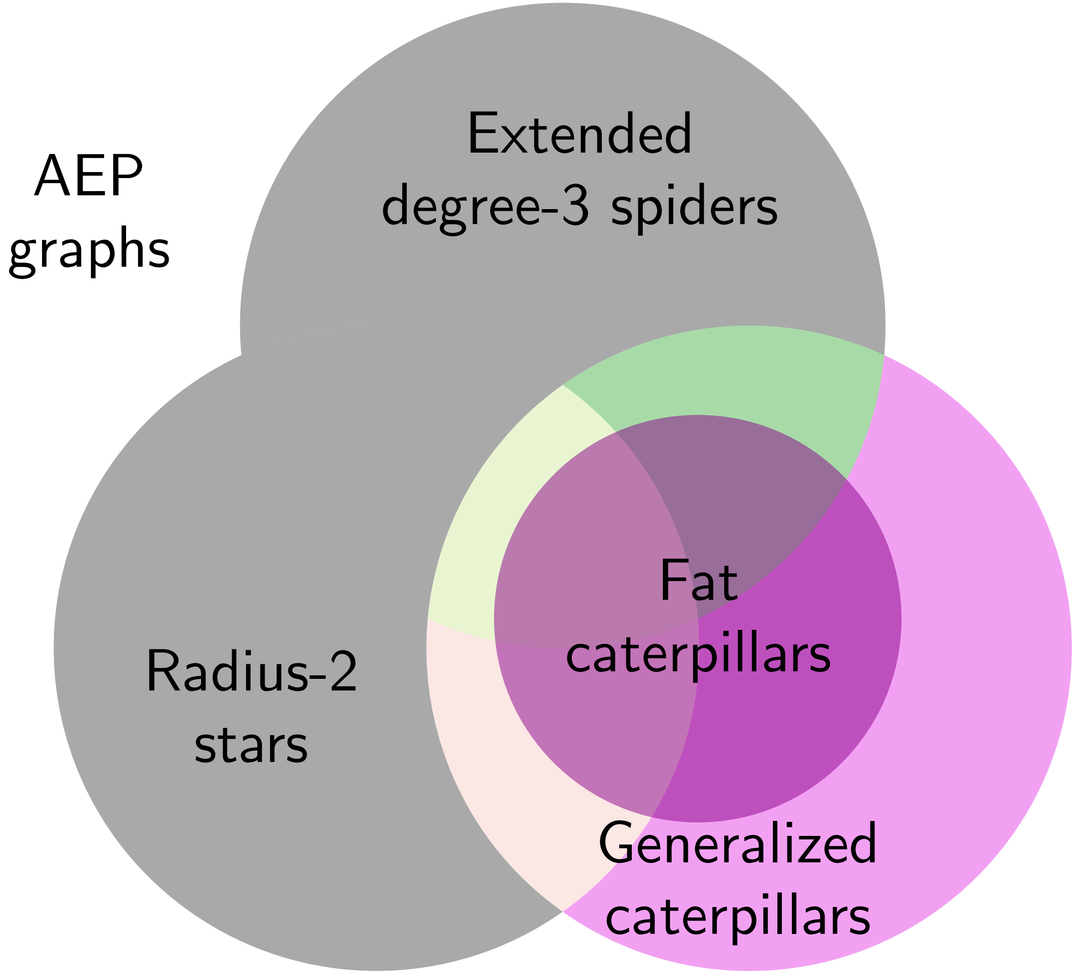
Characterization of EAP graphs

Lemma 4 *Let G be an extended degree-3 spider that is not a generalized caterpillar, then* $\text{ihs}(G) > 1$.



Characterization of EAP graphs

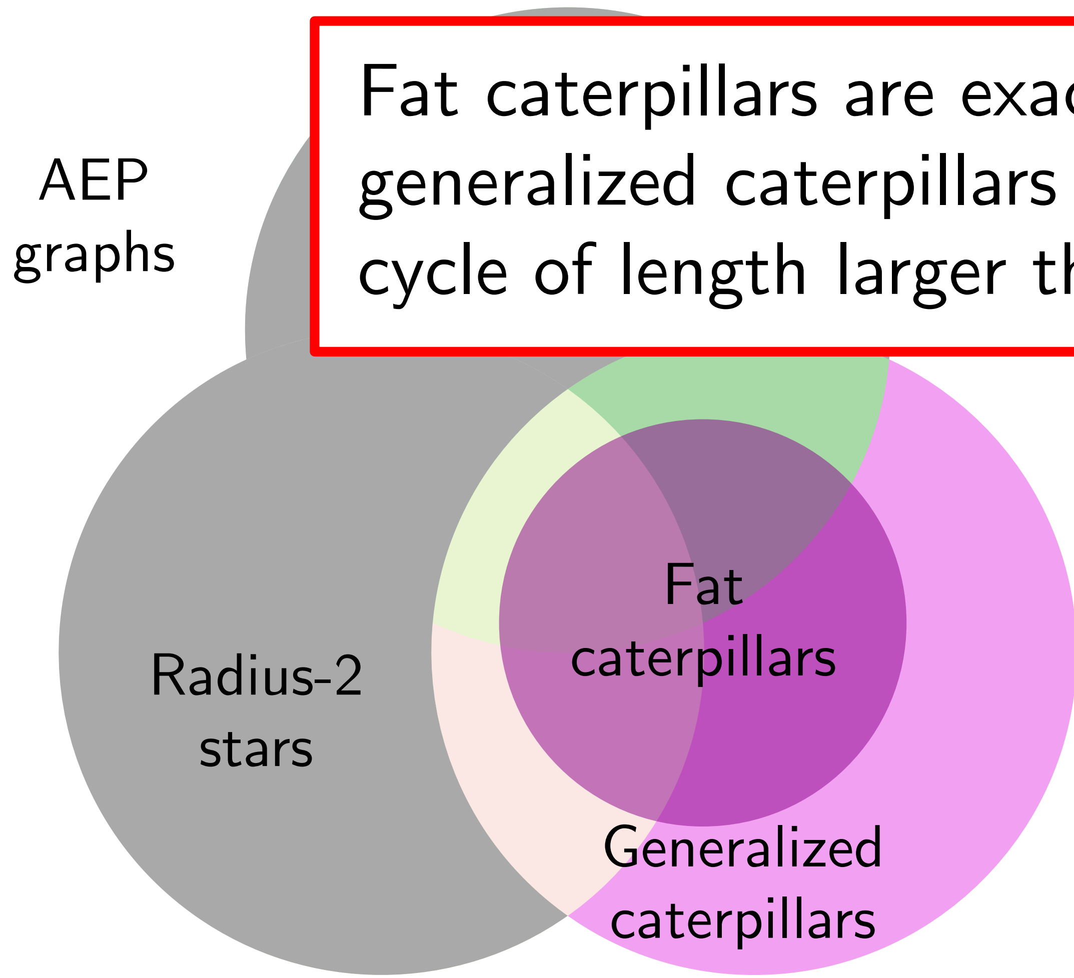
Lemma 5 *Let G be a graph that contains a cycle of length at least 4 then* $\text{ihs}(G) > 1$.



Characterization of EAP graphs

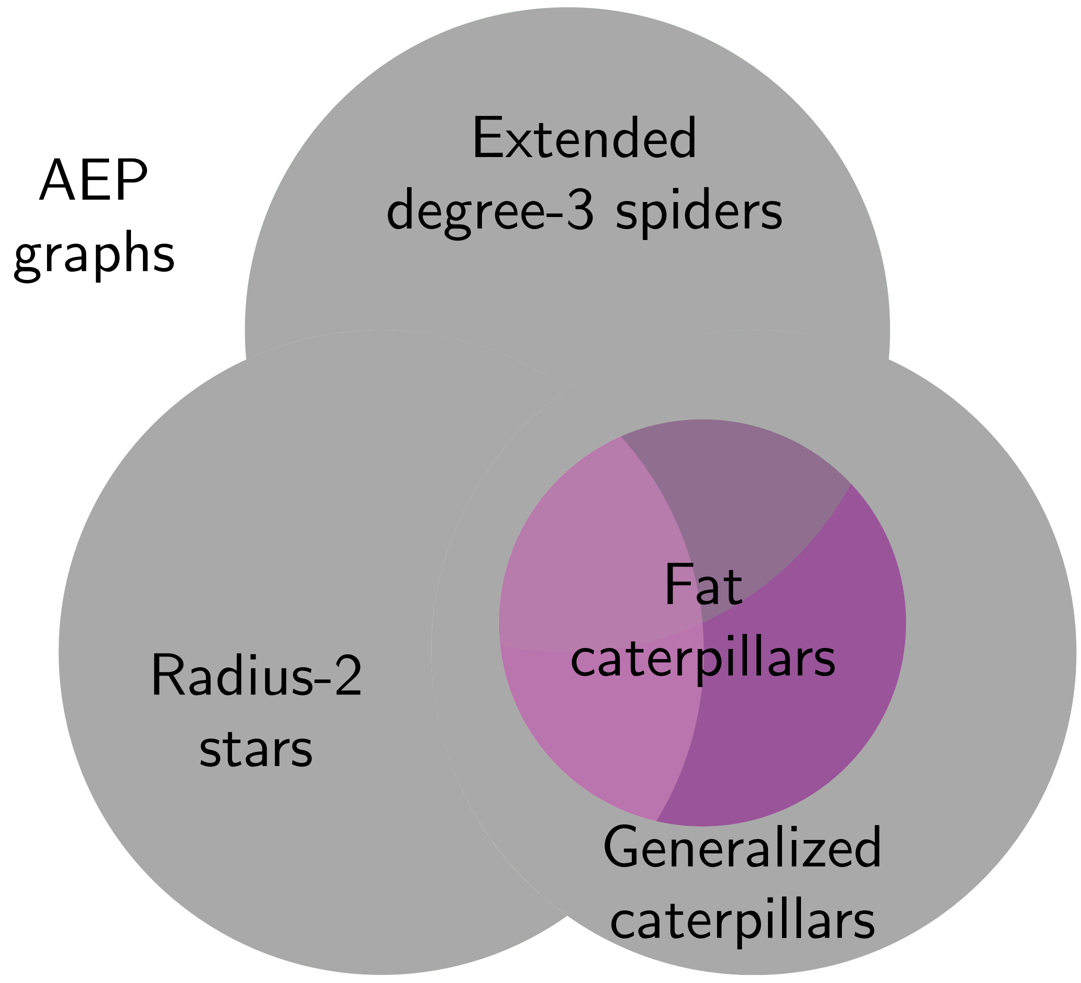
Lemma 5 *Let G be a graph that contains a cycle of length at least 4 then* $\mathrm{ihs}(G) > 1$.

Fat caterpillars are exactly the generalized caterpillars without cycle of length larger than 3



Characterization of EAP graphs

Lemma 5 *Let G be a graph that contains a cycle of length at least 4 then* $\text{ihs}(G) > 1$.

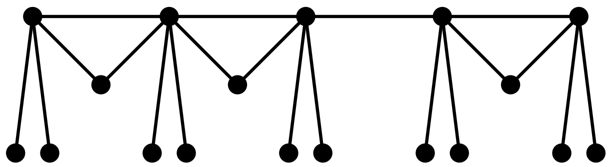


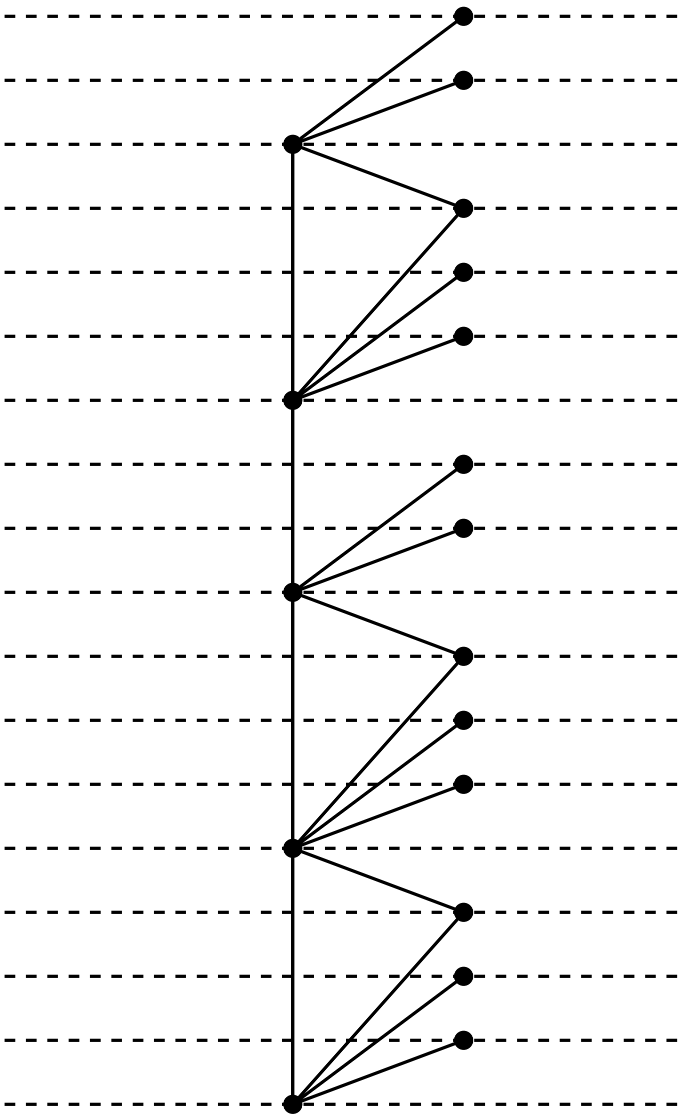
Characterization of EAP graphs

Lemma 6 *Let G be a fat caterpillar. Then* $\mathrm{ihs}(G) = 1$.

Characterization of EAP graphs

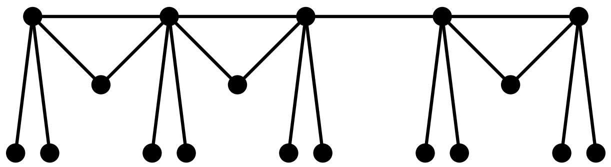
Lemma 6 *Let G be a fat caterpillar. Then* $\mathrm{ihs}(G) = 1$.

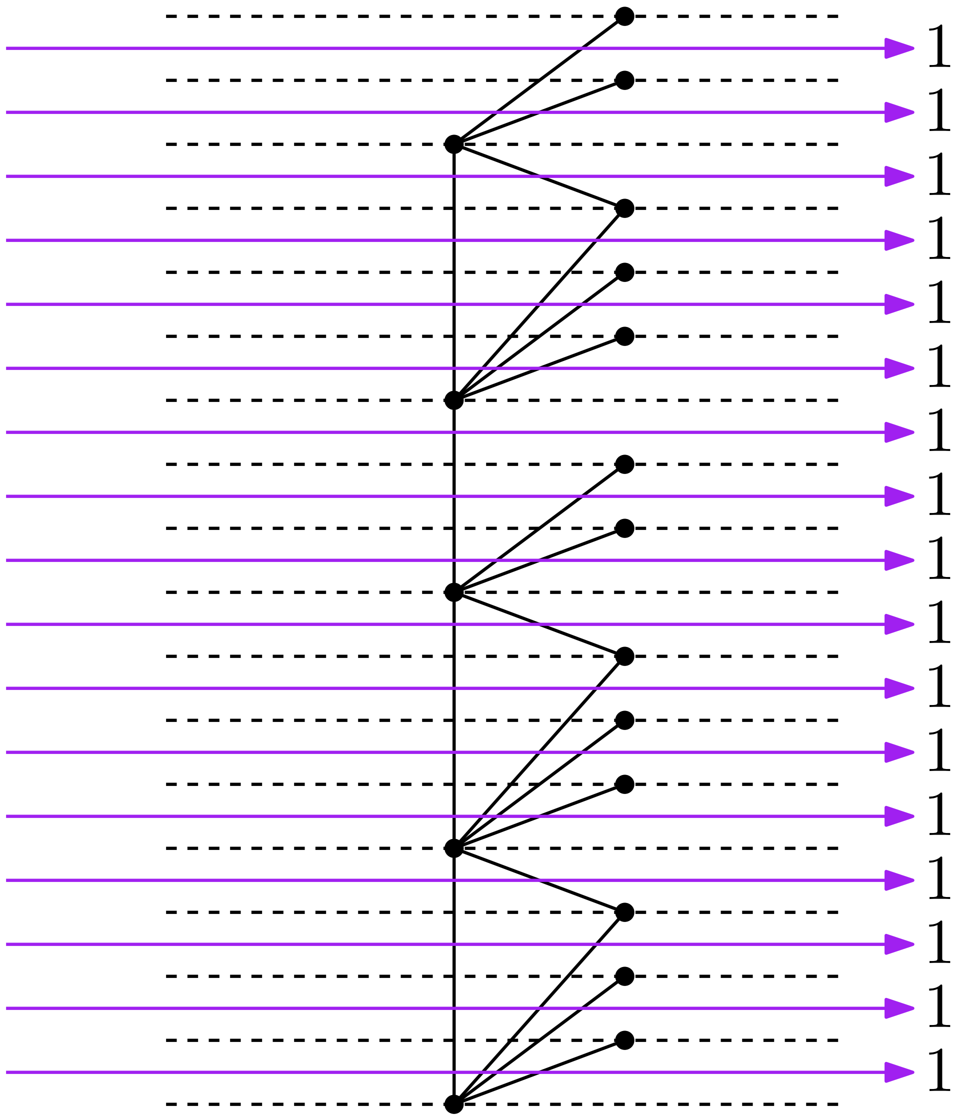




Characterization of EAP graphs

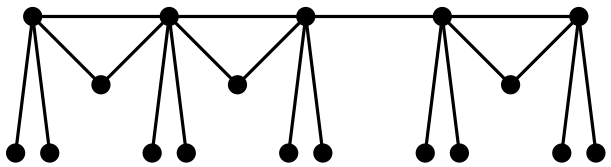
Lemma 6 *Let G be a fat caterpillar. Then* $\mathrm{ihs}(G) = 1$.



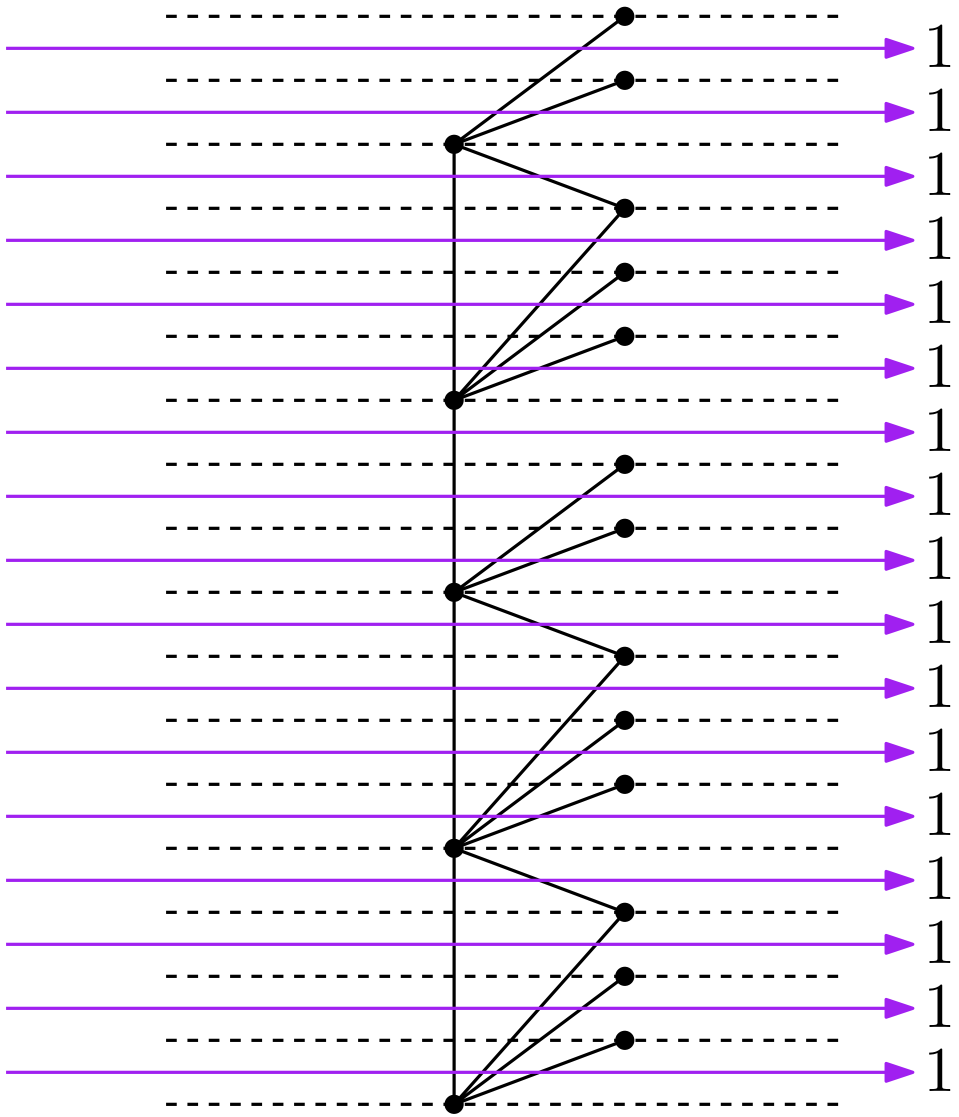


Characterization of EAP graphs

Lemma 6 *Let G be a fat caterpillar. Then* $\mathsf{ihs}(G) = 1$.

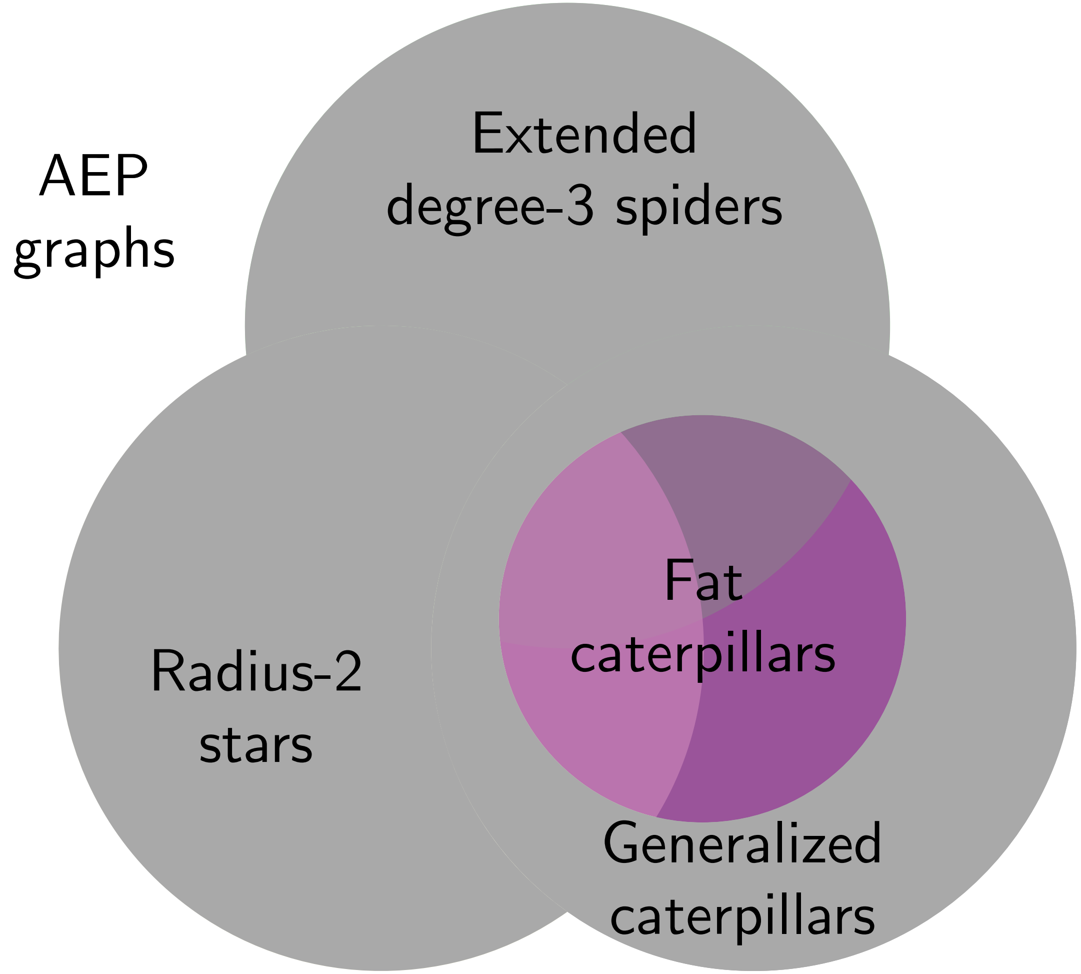


$\mathsf{ihs}(G) = 1$



Characterization of EAP graphs

Theorem 1 *A planar graph G is an EAP graph if and only if it is a fat caterpillar.*



Beyond planarity: Simultaneous Geometric Quasi Planar Embedding (SGQPE)

Simultaneous Geometric Quasi Planar Embedding

- Let $\langle G_1 = (V, E_1), G_2 = (V, E_2) \rangle$ be a pair of quasi planar graphs with the same vertex set.
- A *simultaneous geometric quasi planar embedding* (*SGQPE*) of $\langle G_1, G_2 \rangle$ is a pair of drawings $\langle \Gamma_1, \Gamma_2 \rangle$ such that:
  - Γ_i is a quasi planar straight-line drawing of G_i for $i = 1, 2$;
  - each vertex $v \in V$ is represented by the same point in Γ_1 and Γ_2.

Simultaneous Geometric Quasi Planar Embedding

- Let $\langle G_1 = (V, E_1), G_2 = (V, E_2)\rangle$ be a pair of quasi planar graphs with the same vertex set.
- A *simultaneous geometric quasi planar embedding* (*SGQPE*) of $\langle G_1, G_2\rangle$ is a pair of drawings $\langle \Gamma_1, \Gamma_2\rangle$ such that:
  - Γ_i is a quasi planar straight-line drawing of G_i for $i = 1, 2$;
  - each vertex $v \in V$ is represented by the same point in Γ_1 and Γ_2.

Recall:

- A drawing is quasi planar if it does not have three mutually crossing edges
- A graph is quasi planar if it admits a quasi planar drawing

EAQP and AEQP

EAQP graphs: There *exists* a y-leveling such that for *any* x-leveling the resulting drawing is *quasi planar*

AEQP graphs: For *any* y-leveling there *exists* a x-leveling such that the resulting drawing is *quasi planar*

Easy generalizations

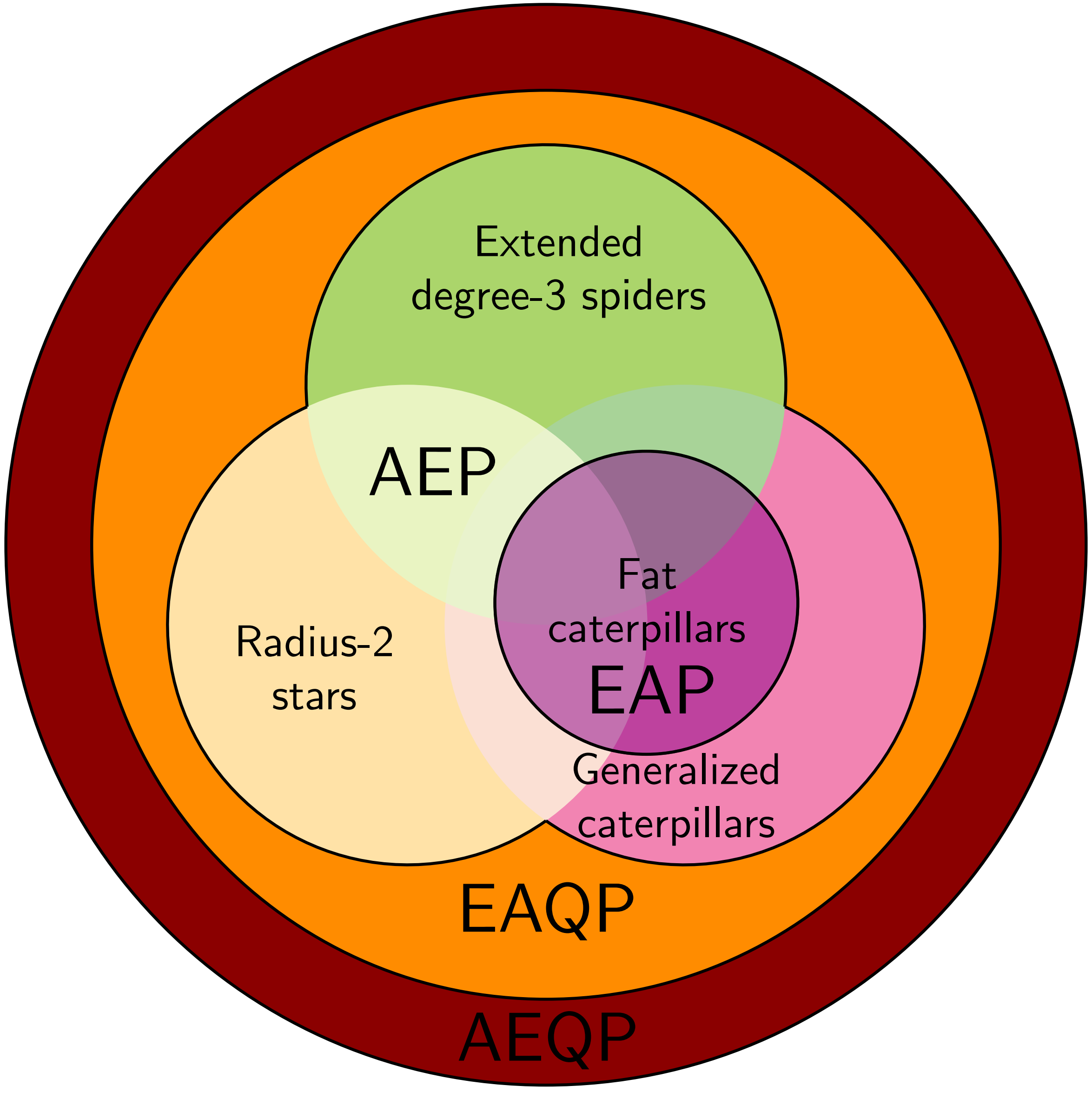
Theorem 2 *Let* $\langle G_1, G_2 \rangle$ *be a pair of graphs such that* $G_1 \in AEQP$ *and* $G_2 \in EAQP$*. Then* $\langle G_1, G_2 \rangle$ *admits a SGQPE.*

Lemma 7 *Let* $G \in EAQP$*. Then* $G \in AEQP$*.*

Lemma 8 *A graph* G *is an EAQP graph if and only if* $\mathrm{ihs}(G) \leq 2$*.*

EAP, AEP, EAQP, AEQP: relationships

Theorem 3 *EAP ⊂ AEP ⊂ EAQP ⊂ AEQP*



EAQP, AEQP: who are they?

Lemma 8 *All trees are AEQP graphs*

EAQP, AEQP: who are they?

Lemma 8 *All trees are AEQP graphs*

This can be easily derived from a paper by Didimo et al.

Didimo, Kaufmann, Liotta, Okamoto, Spillner IPL 2012

EAQP, AEQP: who are they?

Lemma 8 *All trees are AEQP graphs*

This can be easily derived from a paper by Didimo et al.
Didimo, Kaufmann, Liotta, Okamoto, Spillner IPL 2012

They prove that for any y-leveling there exists an x-leveling such that in the resulting drawing each crossing form an angle of $\frac{\pi}{2} - \varepsilon$ (for any given $\varepsilon > 0$)

EAQP, AEQP: who are they?

Lemma 8 *All trees are AEQP graphs*

This can be easily derived from a paper by Didimo et al.
Didimo, Kaufmann, Liotta, Okamoto, Spillner IPL 2012

They prove that for any y-leveling there exists an x-leveling such that in the resulting drawing each crossing form an angle of $\frac{\pi}{2} - \varepsilon$ (for any given $\varepsilon > 0$)

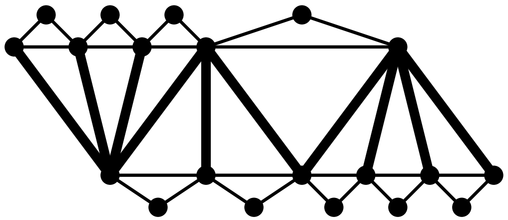
If ε is less than $\frac{\pi}{6}$ then there cannot be three mutually crossing edges as otherwise there would be a triangle whose angle sum up to more than π

EAQP, AEQP: who are they?

Lemma 9 *All maximal outerpillar are EAQP graphs*

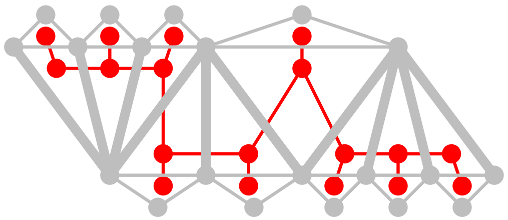
EAQP, AEQP: who are they?

Lemma 9 *All maximal outerpillar are EAQP graphs*



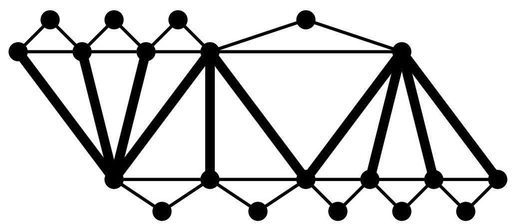
EAQP, AEQP: who are they?

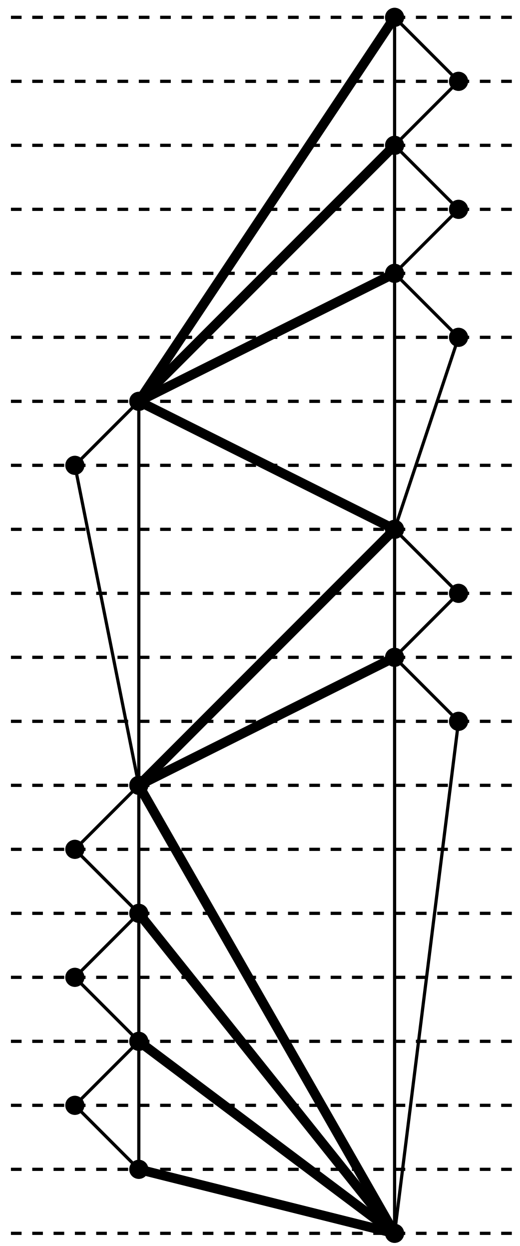
Lemma 9 *All maximal outerpillar are EAQP graphs*



EAQP, AEQP: who are they?

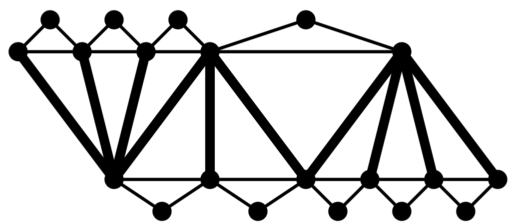
Lemma 9 *All maximal outerpillar are EAQP graphs*

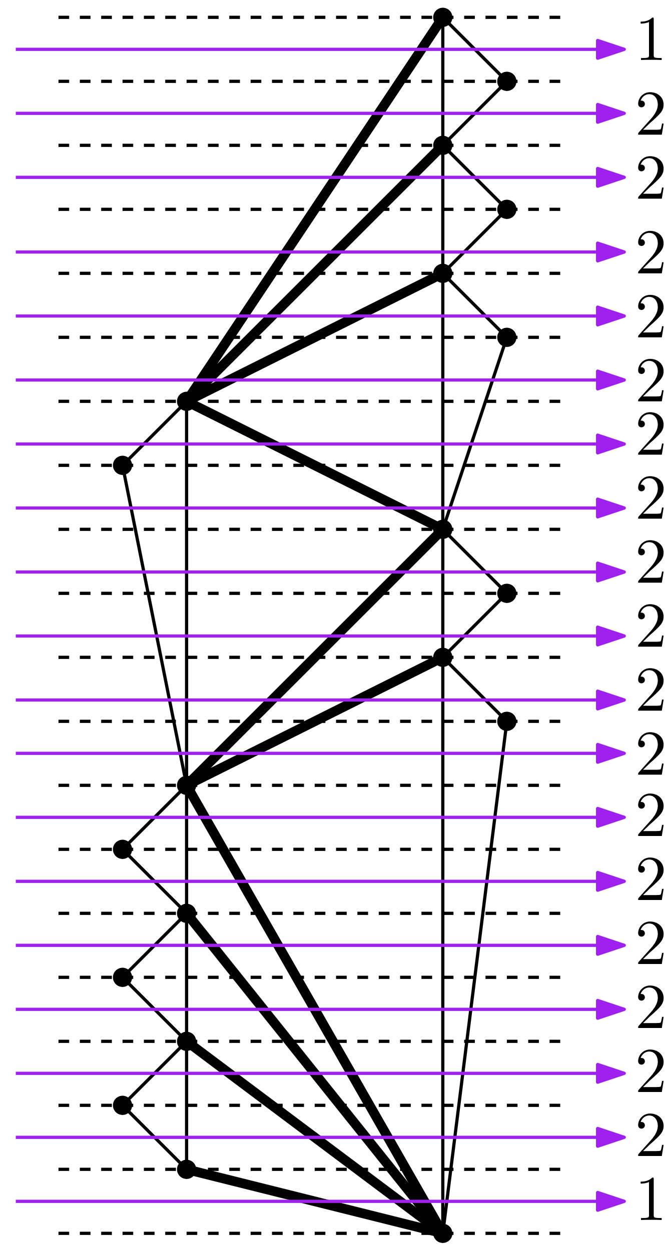




EAQP, AEQP: who are they?

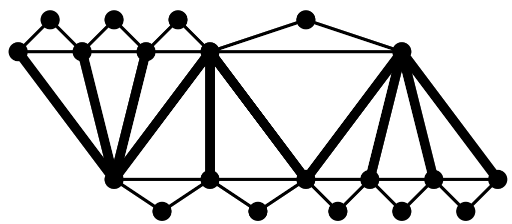
Lemma 9 *All maximal outerpillar are EAQP graphs*



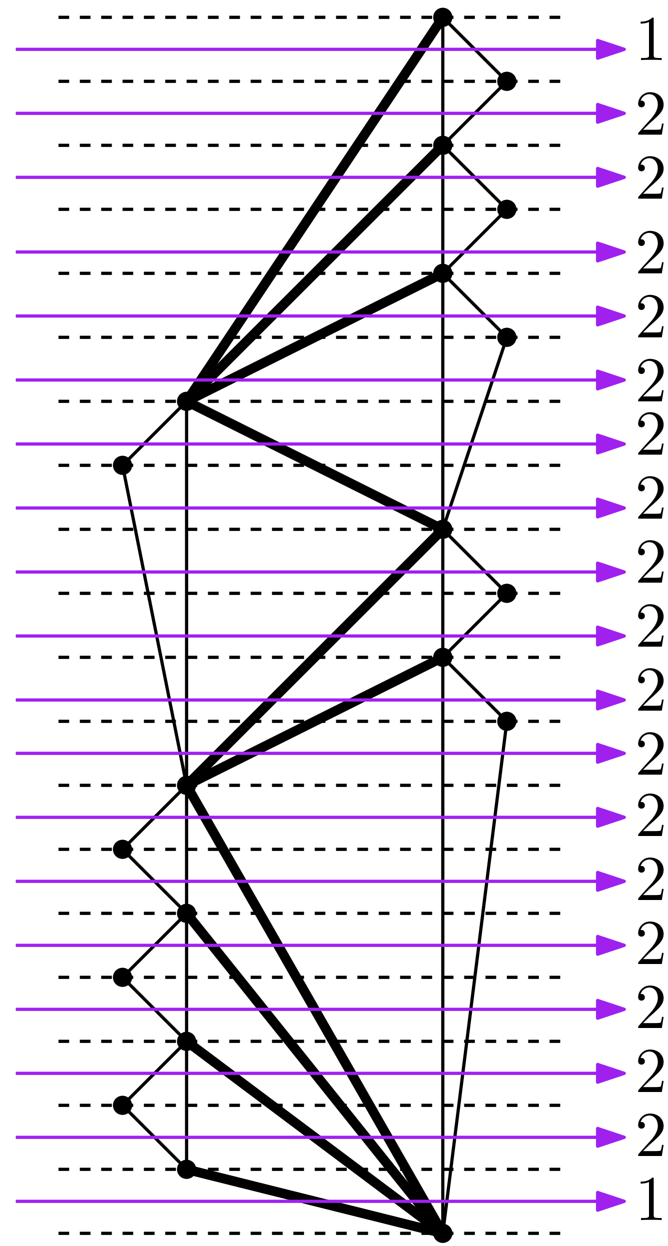


EAQP, AEQP: who are they?

Lemma 9 *All maximal outerpillar are EAQP graphs*



$\mathsf{ihs}(G) \leq 2$



Open problems

- AEP have been characterized by Fowler and Kobourov
- EAP have been characterized in our paper

Problem 1: Characterize EAQP and AEQP graphs

- Our results imply that a tree and a path/cycle admit a SGQP embedding (while they do not admit a SGE)

Problem 2: Does every pair of trees (or planar graphs) admit a SGQPE?

Problem 3: Study simultaneous embeddability for other "beyond planarity" models