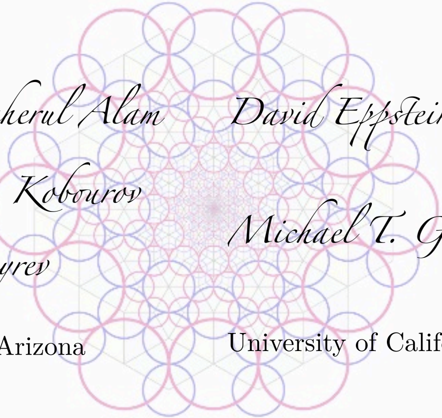
Balanced Circle Packings for Planar Graphs



Md. Jawaherul Alam

Stephen G. Kobourov

Sergey Pupyrev

University of Arizona

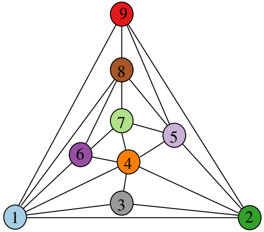
David Eppstein

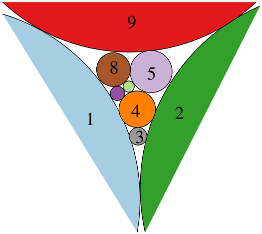
Michael T. Goodrich

University of California, Irvine

Graph Drawing
Würzburg – September 24, 2014

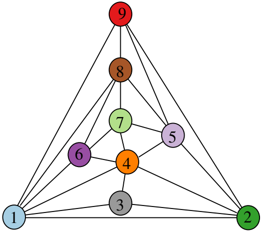
Circle Packing

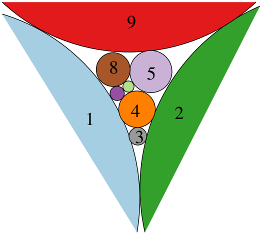




- Contact representation with circles

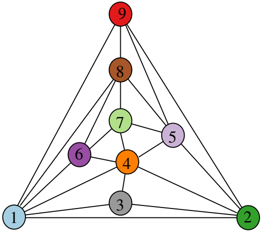
Circle Packing

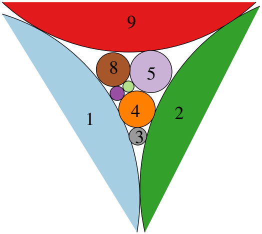




- Contact representation with circles
- Vertices are interior-disjoint circles
- Edges are contacts between circles

Circle Packing





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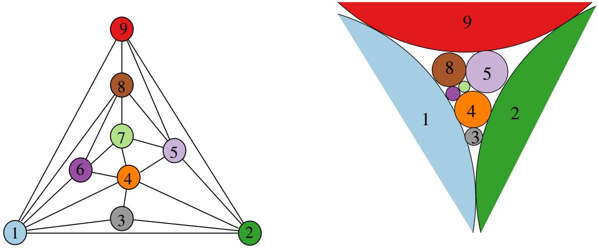
√ Any planar graph has a circle-packing [Koebe, 1936]

Circle Packing

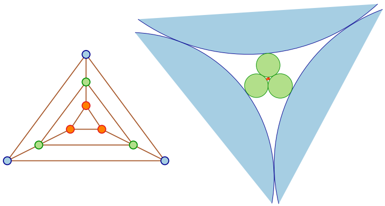


- Contact representation with circles
- Vertices are interior-disjoint circles
- Edges are contacts between circles

√ Any planar graph has a circle-packing [Koebe, 1936]

× Sizes of circles may vary exponentially

Circle Packing: Variation in Sizes



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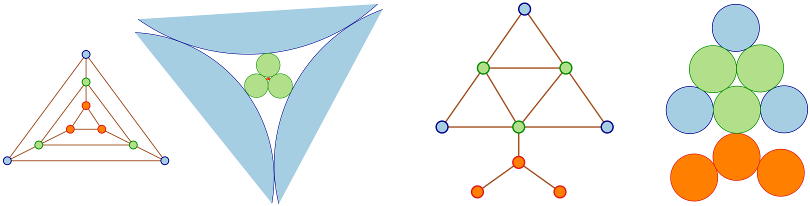

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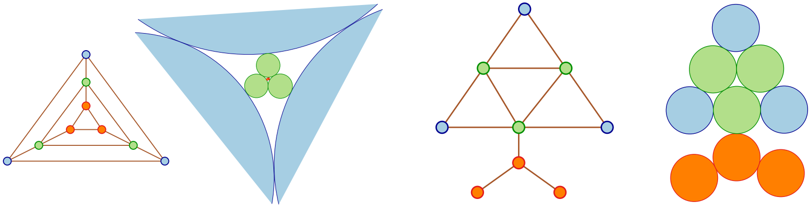


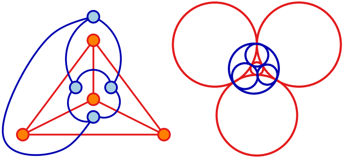
Goal: Balanced Circle-Packing

- Polynomial ratio between maximum and minimum diameter

Related Work

Circle Packing:

- Any plane graph has a circle-packing [Koebe, 1936].
- Any 3-connected plane graph has a primal-dual circle packing [Brightwell and Scheinerman, 1993].



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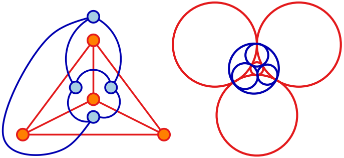

Balanced Circle Packing:

- It is NP-complete to test whether a graph admits contact representation with unit circles [Breu and Kirkpatrick, 1998].

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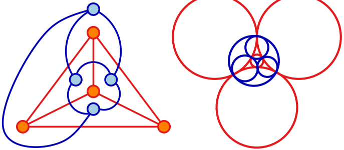


Balanced Circle Packing:

- It is NP-complete to test whether a graph admits contact representation with unit circles [Breu and Kirkpatrick, 1998].

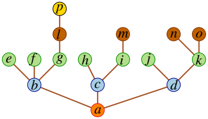
Disk Intersection Graphs:

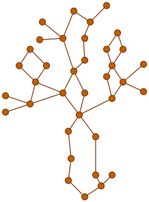
- In a realization with integer radii, radius of $2^{2^{\Theta(n)}}$ is sometimes necessary and always sufficient [McDiarmid and Müller, 2013].

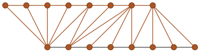
Our Result

Balanced circle packing

- √ trees.
- √ cactus graphs.
- √ outerpaths.







Our Result

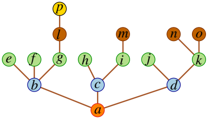
Balanced circle packing

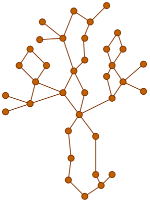
√ trees.

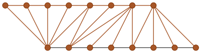
√ cactus graphs.

√ outerpaths.

√ bounded degree and $O(\log n)$ outerplanarity.







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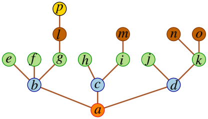

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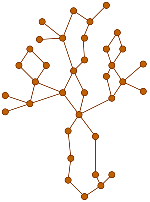
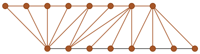


- √ bounded degree and $O(\log n)$ outerplanarity.
- × bounded degree but linear outerplanarity.
- × bounded outerplanarity but linear degree.



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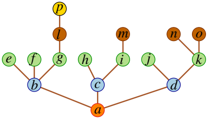

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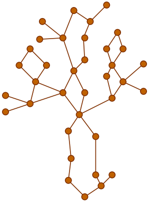
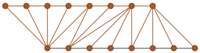


√ bounded degree and $O(\log n)$ outerplanarity.

× bounded degree but linear outerplanarity.

× bounded outerplanarity but linear degree.

√ bounded tree-depth.



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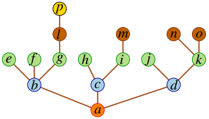

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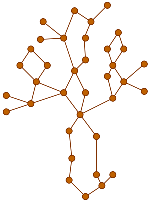

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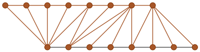
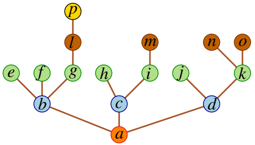


Balanced Packing for Trees



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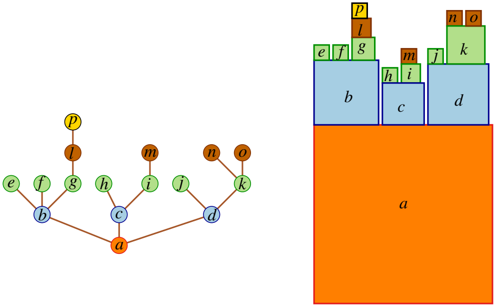
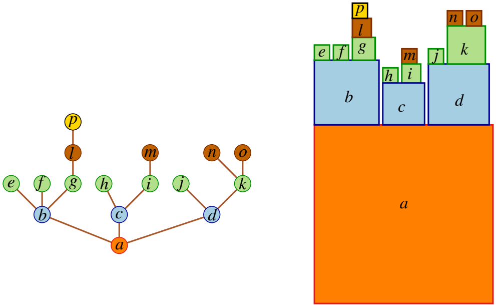


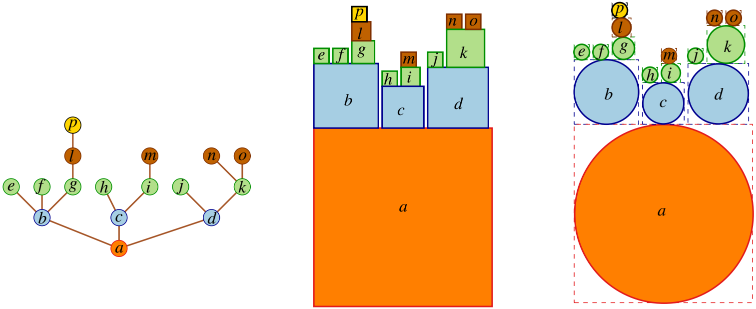
- Compute balanced square-contact representation

Balanced Packing for Trees



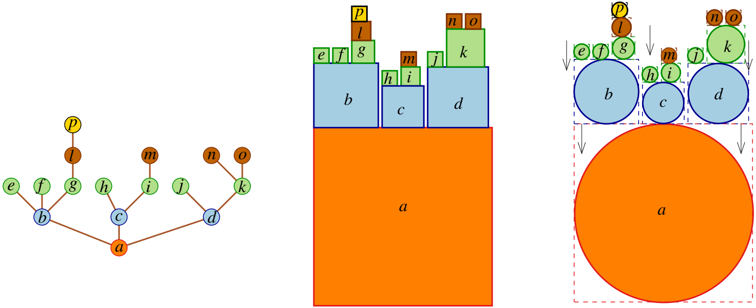
- Compute balanced square-contact representation
  - length is (roughly) proportional to the number of leaves in subtree

Balanced Packing for Trees



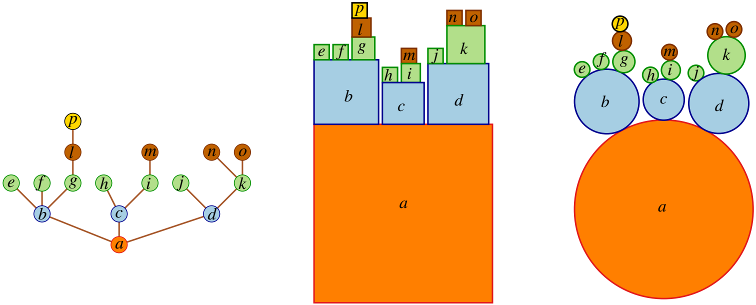
- Compute balanced square-contact representation
- Draw Inscribing circles inside the squares

Balanced Packing for Trees



- Compute balanced square-contact representation
- Draw Inscribing circles inside the squares
- Translate downwards

Balanced Packing for Trees



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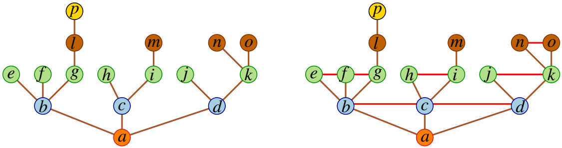
Augmented Fan-Trees



- Add a path between the children of every vertex

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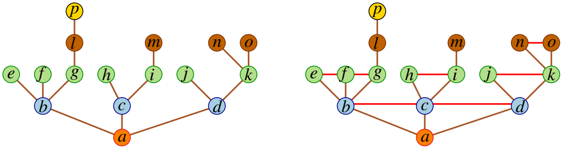
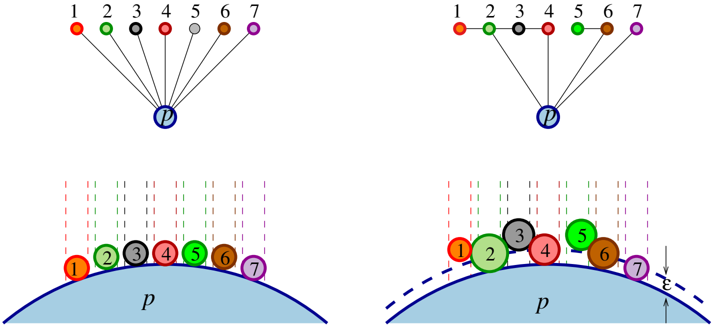


- Add a path between the children of every vertex

Claim:

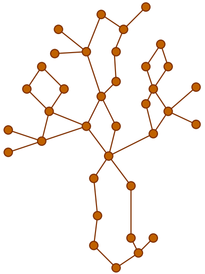
Any subgraph of an augmented fan-tree has a balanced packing

Packing for Subgraphs of Augmented Fan-Trees



- Follow the algorithm for balanced packing of the tree
- Modify the circles for the children of each vertex

Balanced Packing for Cactus Graphs



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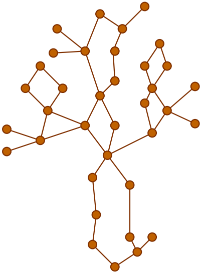

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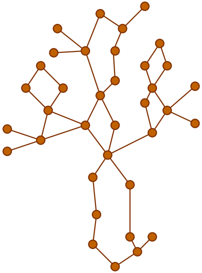
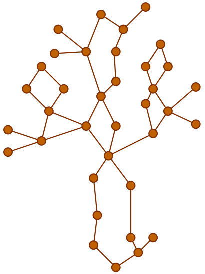


Each biconnected component is a cycle or a single edge

Balanced Packing for Cactus Graphs



Each biconnected component is a cycle or a single edge

- Each cactus graph is a subgraph of an augmented fan-tree

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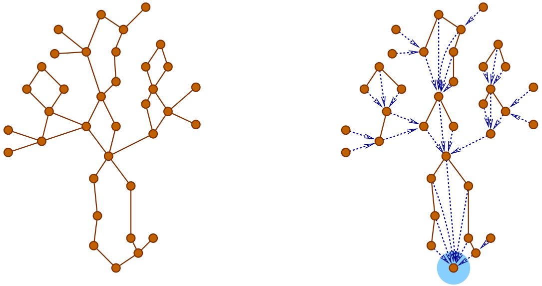

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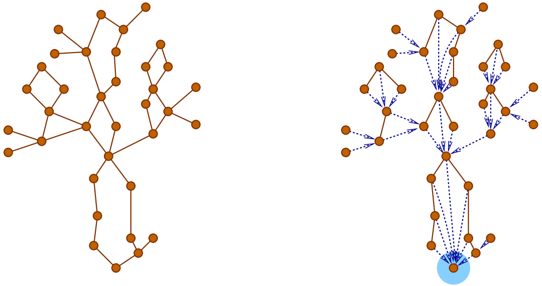

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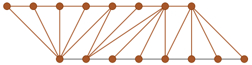
- Each cactus graph is a subgraph of an augmented fan-tree

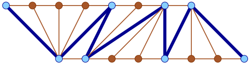
⇒ Each cactus graph admits a balanced packing

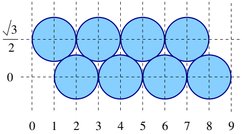
Balanced Packing for Outerpaths



Outerplanar graph whose weak dual is a path

Balanced Packing for Outerpaths

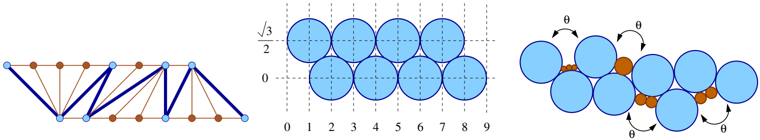




Outerplanar graph whose weak dual is a path

- Draw Circles for spine vertices

Balanced Packing for Outerpaths



Outerplanar graph whose weak dual is a path

- Draw Circles for spine vertices
- Rotate to create space for other vertices

Our Result

Balanced circle packing

- √ trees.
- √ cactus graphs.
- √ outerpaths.

- √ bounded degree and $O(\log n)$ outerplanarity.
- × bounded degree but linear outerplanarity.
- × bounded outerplanarity but linear degree.

- √ bounded tree-depth.

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Balanced Packing for Maximal Planar Graphs

[Malitz and Papakostas, 1994]

- G: maximal planar graph
- Δ: maximum vertex-degree in G

Balanced Packing for Maximal Planar Graphs

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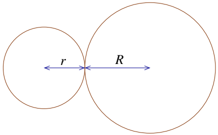
$\Rightarrow$ G admits circle packing where ratio of radii of adjacent circles $\frac{r}{R} \geq \alpha^{\Delta-2}$, $\alpha \approx 0.15$.



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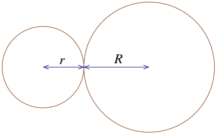


Corollary:

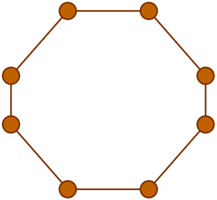
A maximal planar graph with bounded degree and logarithmic diameter has a balanced packing.

Packing for Bounded Degree and Outerplanarity

Claim:

A k-outerplanar graph with maximum degree Δ has packing with ratio of radii $\leq f(\Delta)^{k \log n}$.

- Idea: Triangulate with $O(\Delta)$ degree and $k \log n$ diameter.



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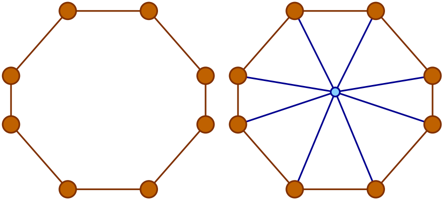

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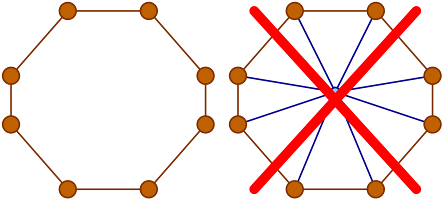

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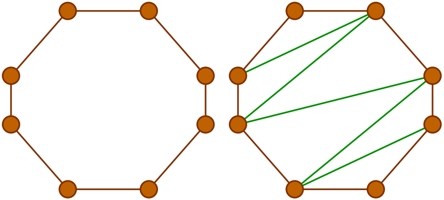

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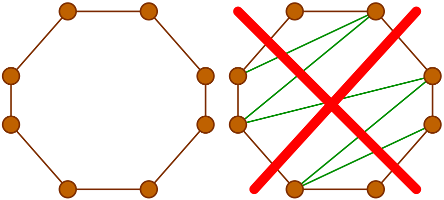

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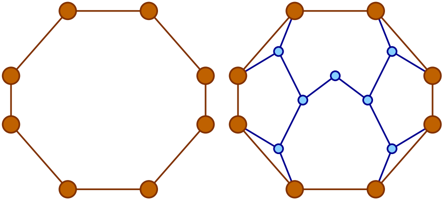

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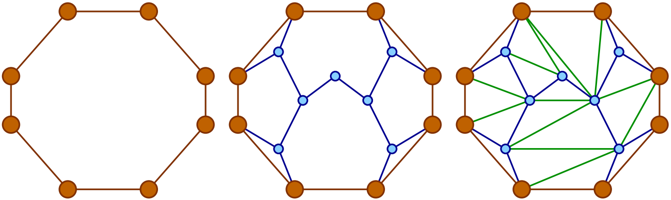
- Idea: Triangulate with $O(\Delta)$ degree and $k \log n$ diameter.

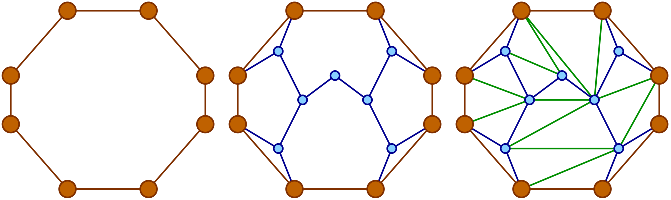


Packing for Bounded Degree and Outerplanarity

Claim:

A k-outerplanar graph with maximum degree Δ has packing with ratio of radii $\leq f(\Delta)^{k \log n}$.

- Idea: Triangulate with $O(\Delta)$ degree and $k \log n$ diameter.



Theorem:

A planar graph with bounded degree and outerplanarity has a balanced packing.

Bounded Degree and Logarithmic Outerplanarity

Claim:

> A k-outerplanar graph with maximum degree Δ has packing with ratio of radii $\leq f(\Delta)^{k+logn}$.

- Idea: Triangulate with $O(\Delta)$ degree and $k + \log n$ diameter using weight-balanced tree.

Bounded Degree and Logarithmic Outerplanarity

Claim:

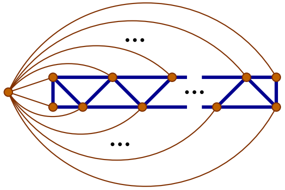
A k-outerplanar graph with maximum degree Δ has packing with ratio of radii $\leq f(\Delta)^{k+logn}$.

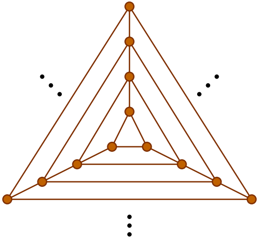
- Idea: Triangulate with $O(\Delta)$ degree and $k + \log n$ diameter using weight-balanced tree.

Theorem:

A planar graph with bounded degree and $O(\log n)$ outerplanarity has a balanced packing.

Negative Results





No balanced circle packing even with

× Bounded Outerplanarity (2-outerplanar), linear degree.

× Bounded Degree, Linear Outerplanarity.

× Bounded treewidth.

Our Result

Balanced circle packing

- √ trees.
- √ cactus graphs.
- √ outerpaths.

- √ bounded degree and $O(\log n)$ outerplanarity.
- × bounded degree but linear outerplanarity.
- × bounded outerplanarity but linear degree.

- √ bounded tree-depth.

Our Result

Balanced circle packing

- √ trees.
- √ cactus graphs.
- √ outerpaths.

- √ bounded degree and $O(\log n)$ outerplanarity.
- × bounded degree but linear outerplanarity.
- × bounded outerplanarity but linear degree.

√ bounded tree-depth.

Summary

- Balanced circle packing for graphs with
  - bounded degree and
  - $O(\log n)$ outerplanarity
  - both conditions are necessary
- Balanced circle packing for trees, cactus graphs and outerpaths
- Balanced circle packing for graphs with bounded tree-depth

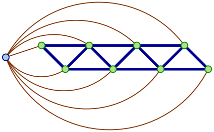
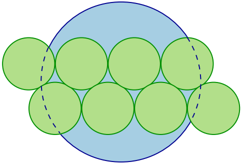
Future Work and Open Problems

- Balanced circle packing for outerplanar graphs
  - Algorithm or counter-example

Future Work and Open Problems

- Balanced circle packing for outerplanar graphs
  - Algorithm or counter-example

- Balanced intersection representation
  - 2-outerplanar graphs?
  - k-outerplanar graphs?



Thank you!