Karlsruhe Institute of Technology

A New Perspective on Clustered Planarity as a Combinatorial Embedding Problem

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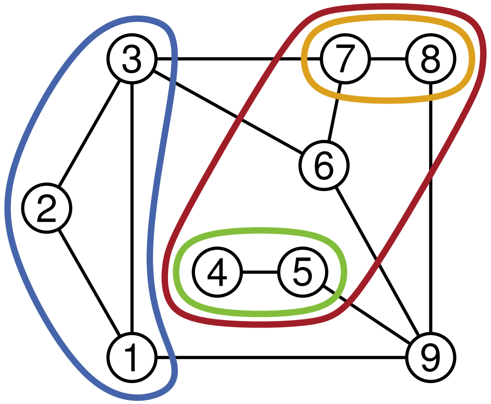
Thomas Bläsius · Ignaz Rutter

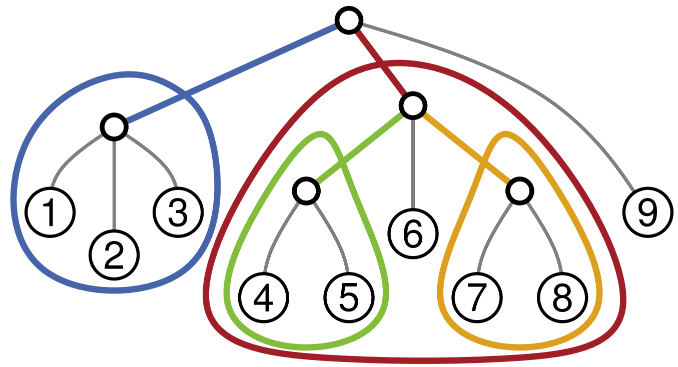
INSTITUTE OF THEORETICAL INFORMATICS · PROF. DR. DOROTHEA WAGNER

Clustered Planarity

[**Lengauer** 1989]
[**Feng, Cohen, Eades** 1995]

Input: graph $G = (V, E)$ tree T with leaves V





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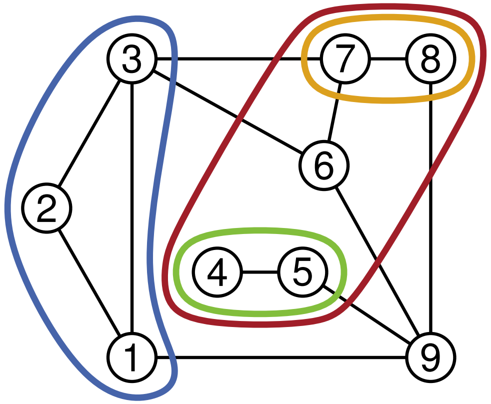
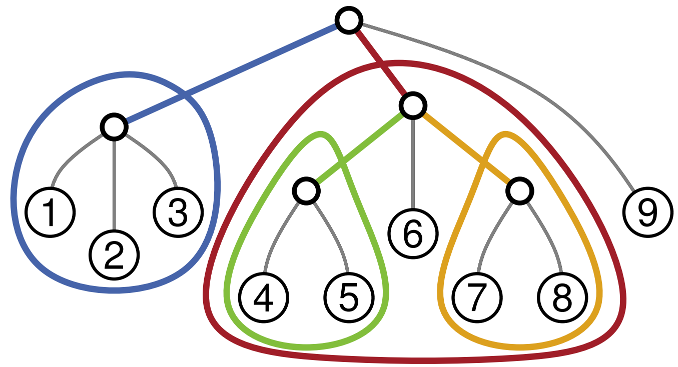
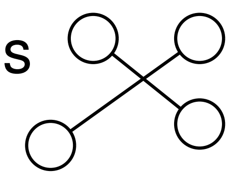




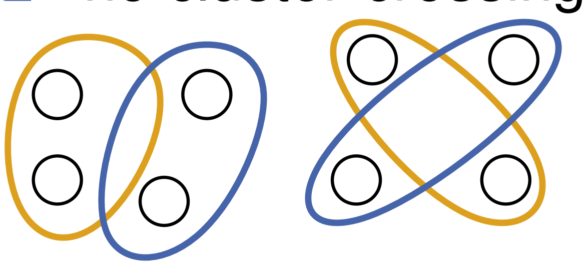
every subtree of T induces a cluster

Find: drawing of G together with regions representing the clusters

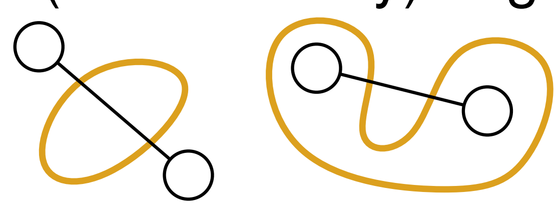
- no edge-crossings



- no cluster-crossings



- no (unnecessary) edge-cluster-crossings



My Goals for this Talk

Give you an Understanding of our Perspective on C-Planarity

- the cd-tree and a characterization
- flat clusterings and constrained planarity
- related work from the new perspective

no new c-planarity variants will be solved in this part

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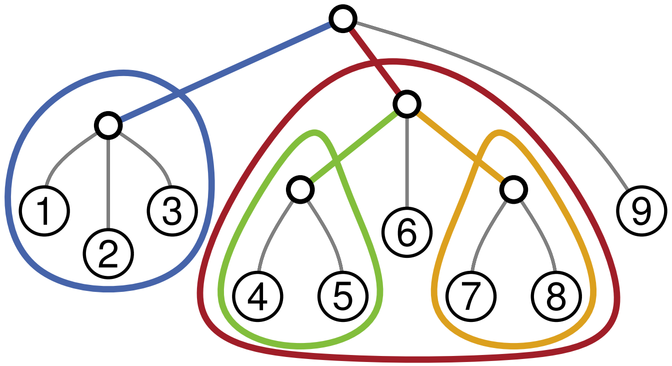
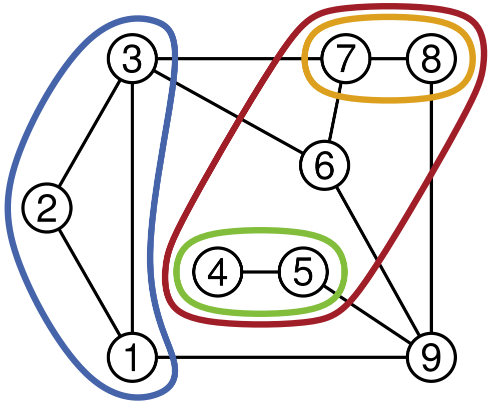
Final Remarks

- new cases we can solve

The CD-Tree – A (New) Perspective





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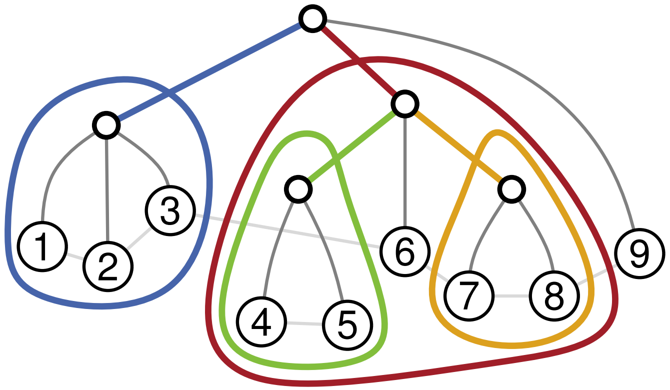

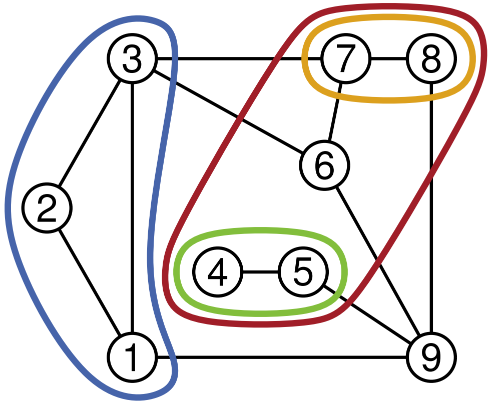

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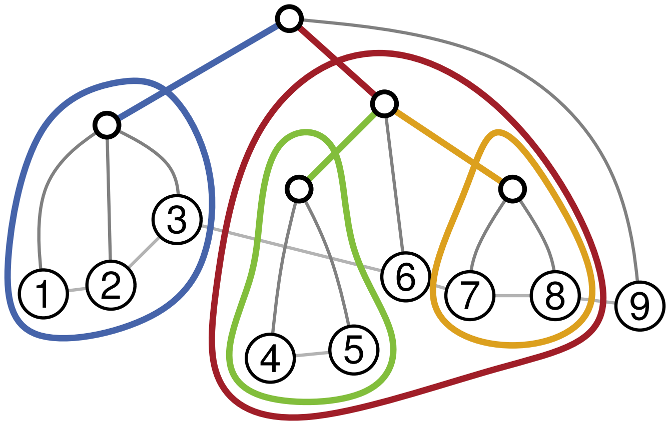

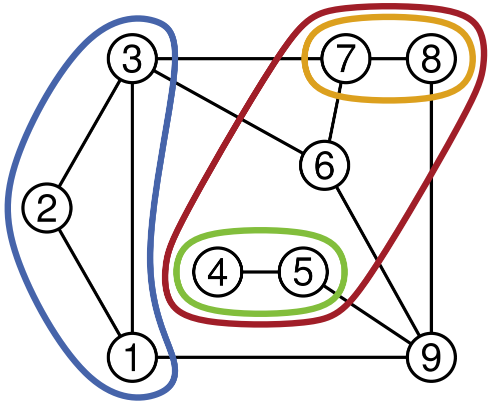

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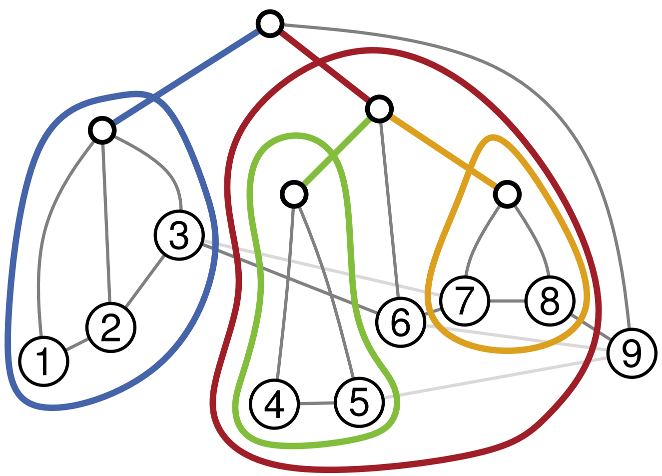

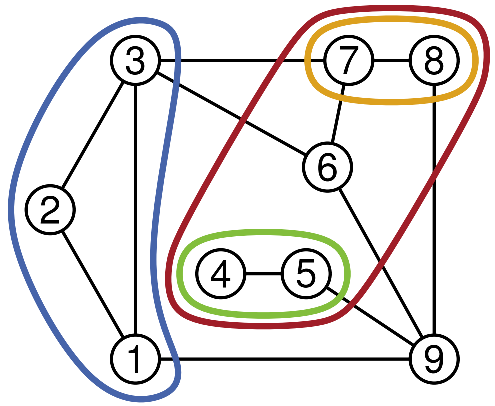

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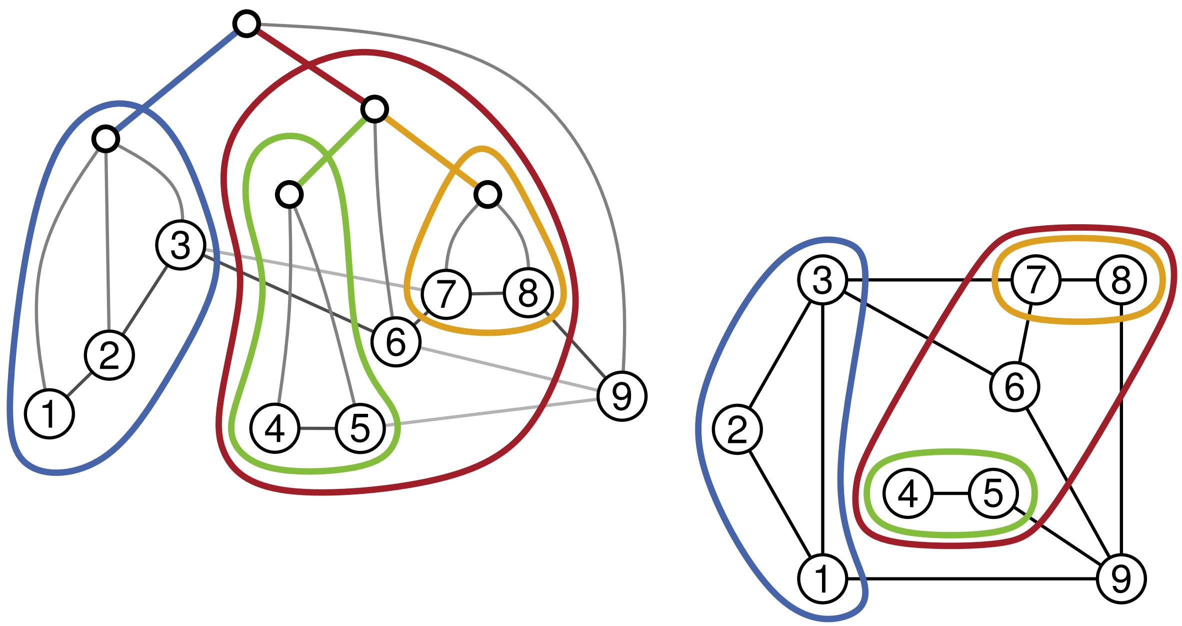

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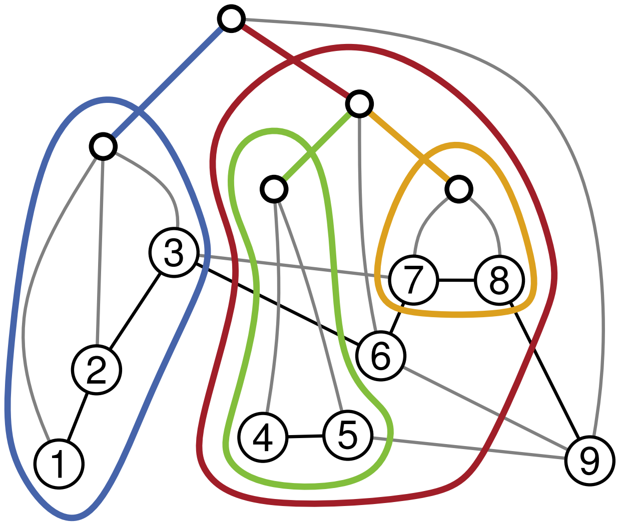

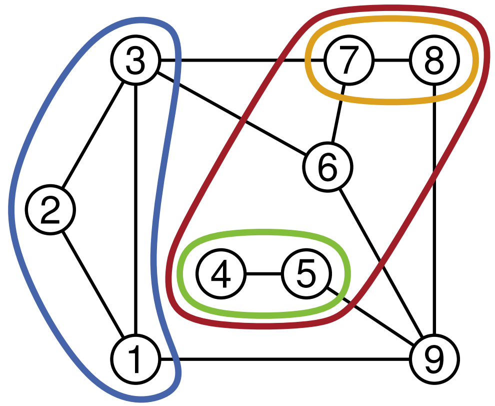

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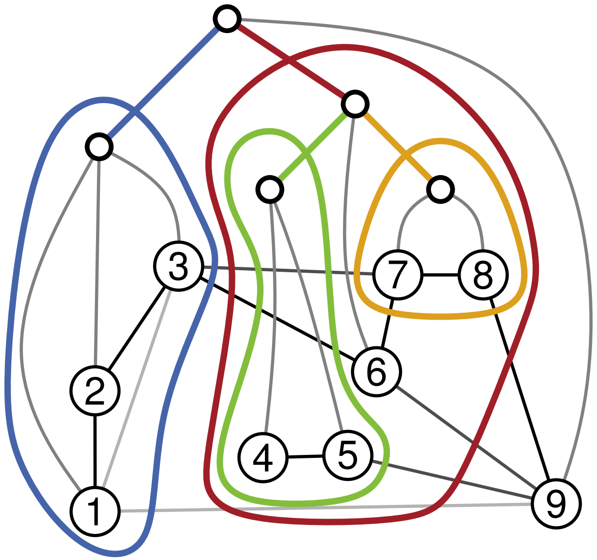

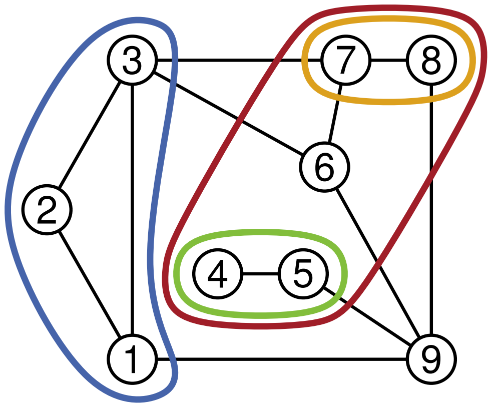

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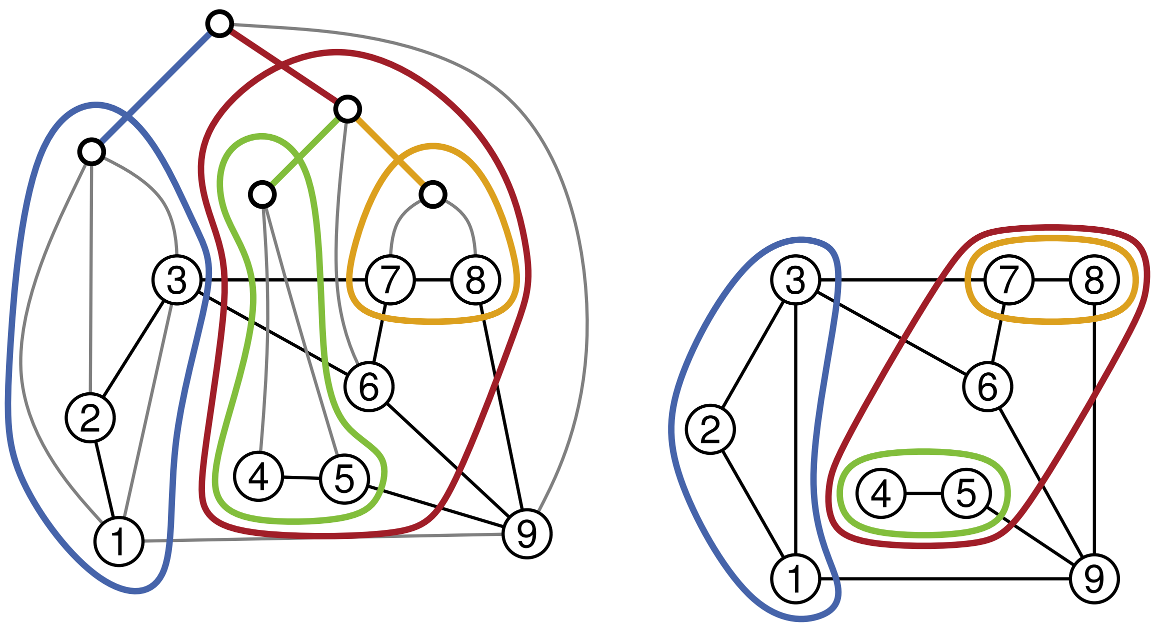

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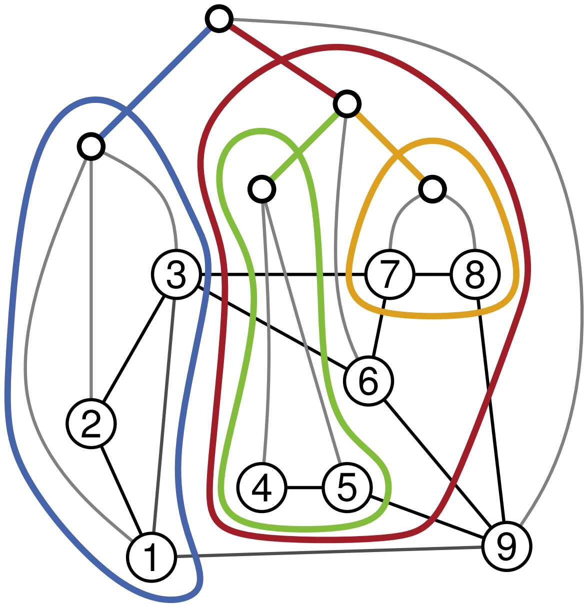

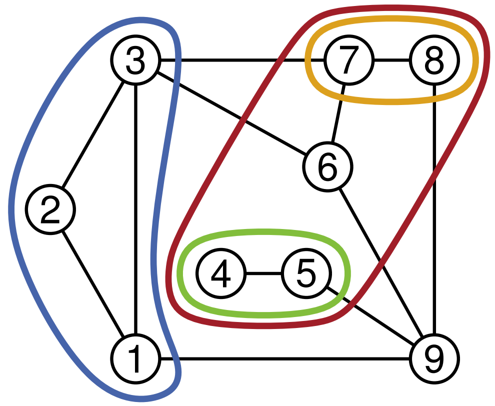

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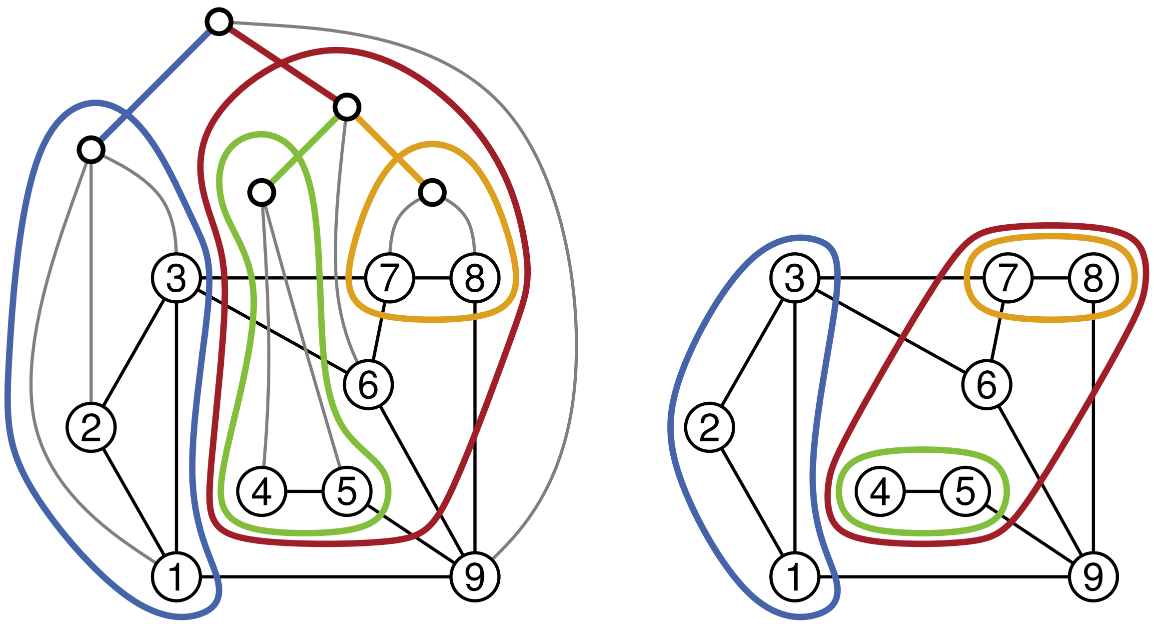

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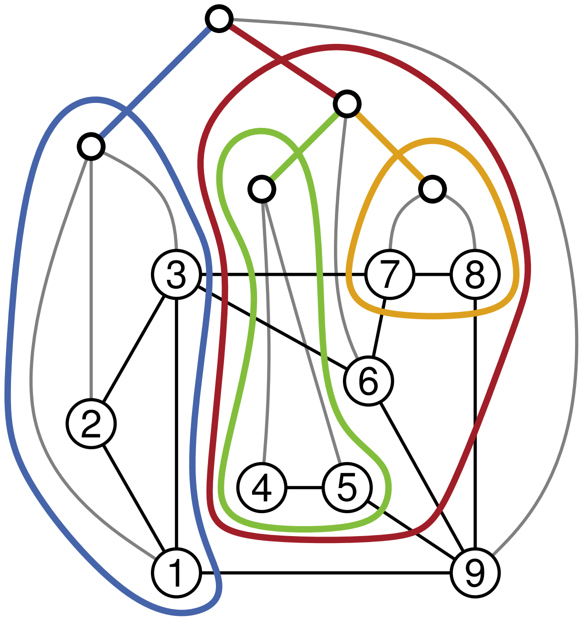

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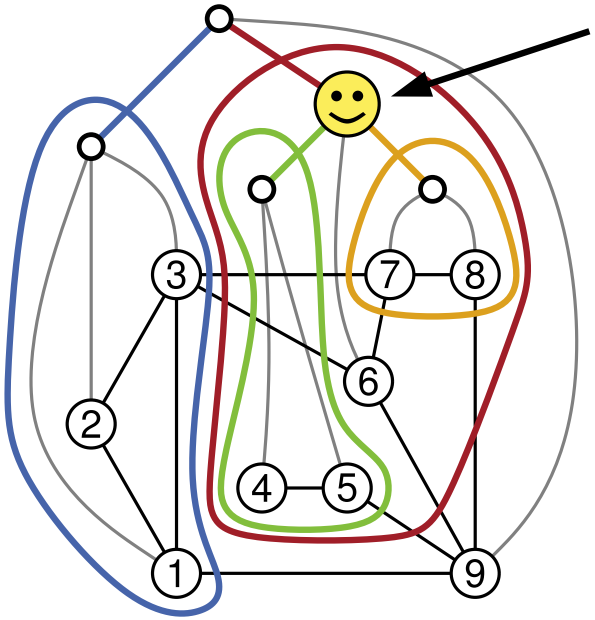

From the perspective of this node, there are

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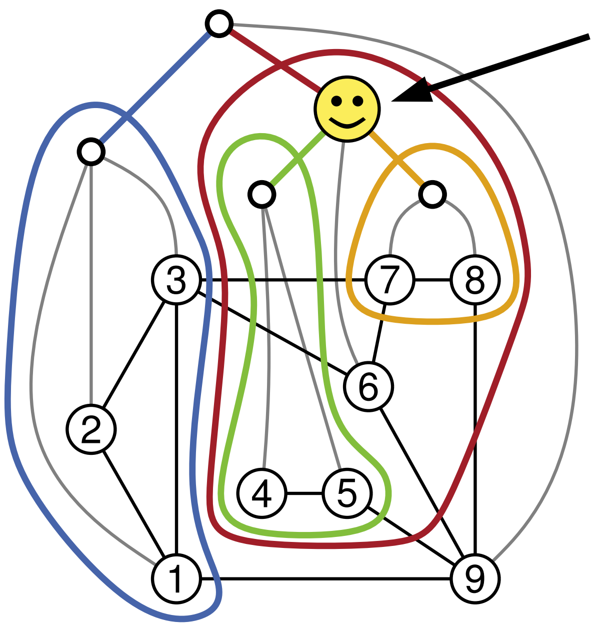

From the perspective of this node, there are

- node 6

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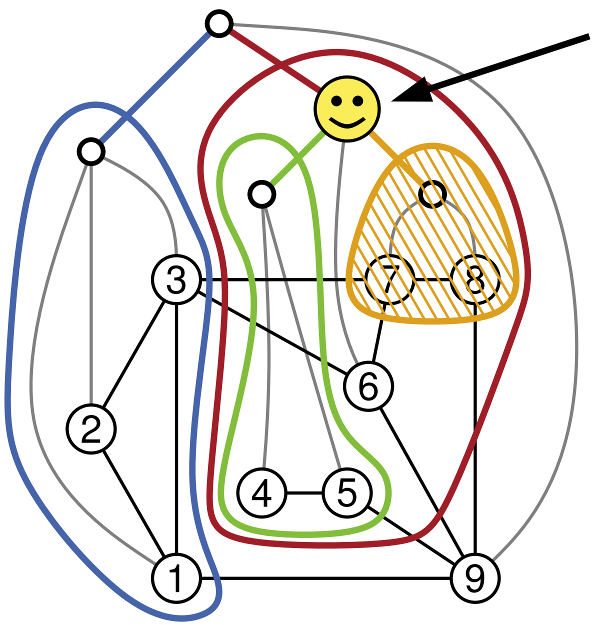
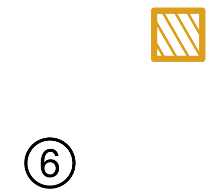


From the perspective of this node, there are

- node 6
- orange part



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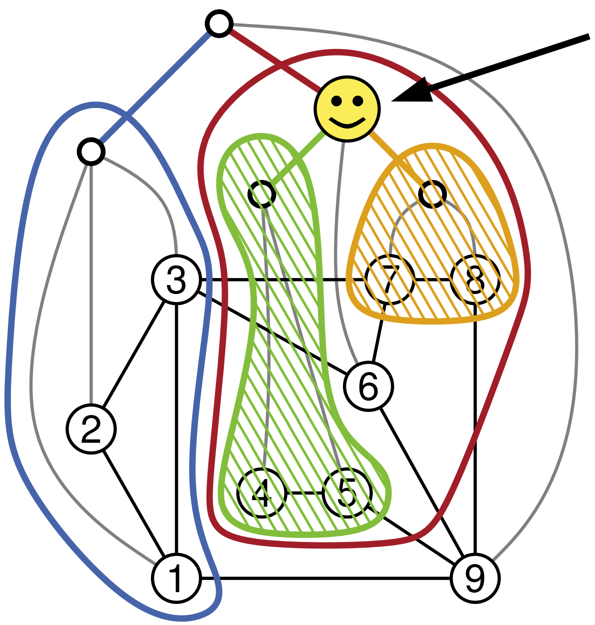
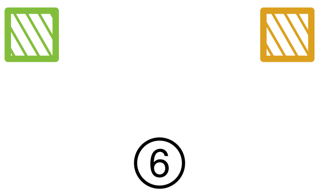


From the perspective of this node, there are

- node 6
- orange part
- green part



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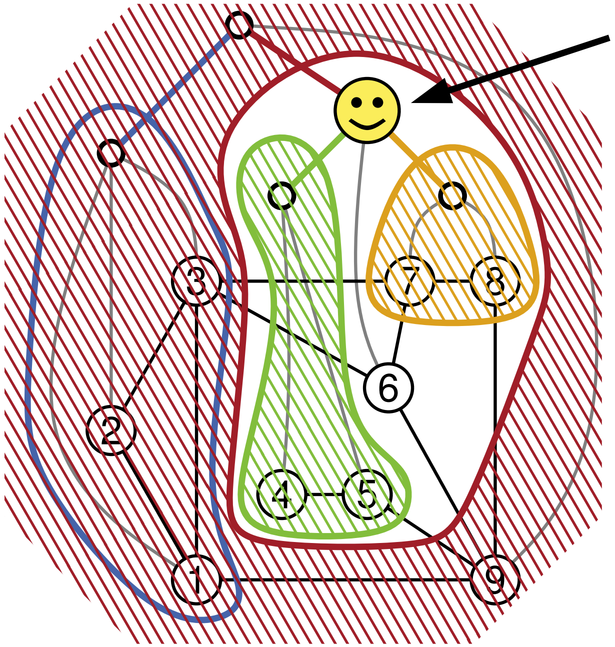

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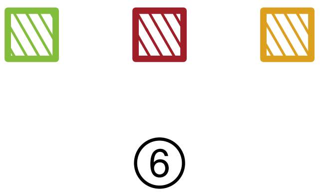
- red part



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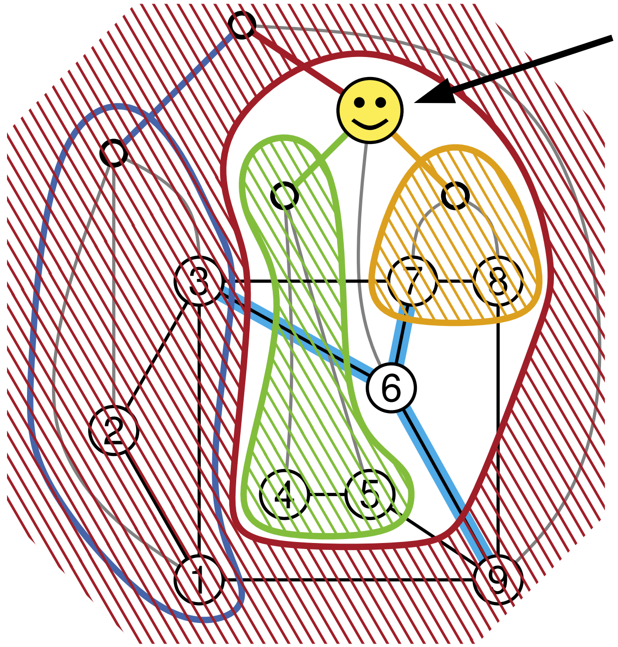

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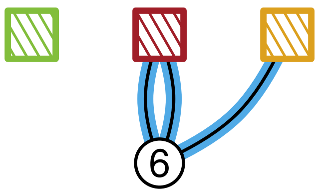
- some edges



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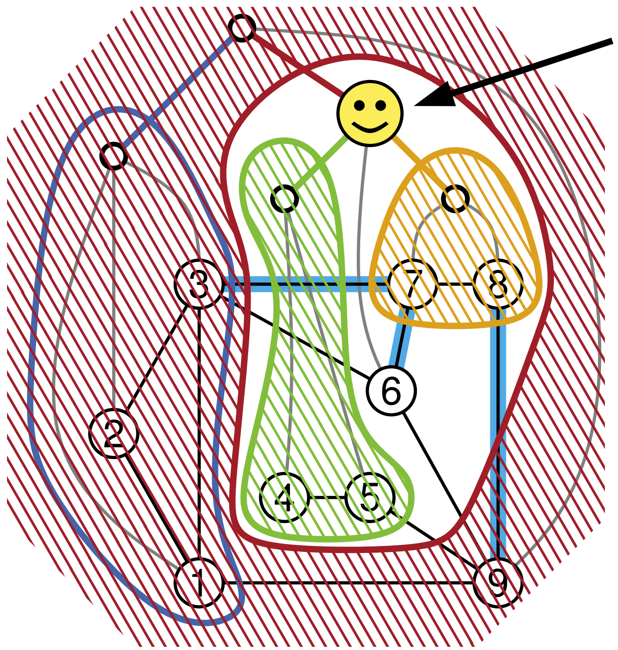

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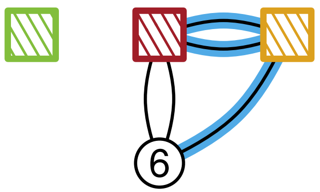

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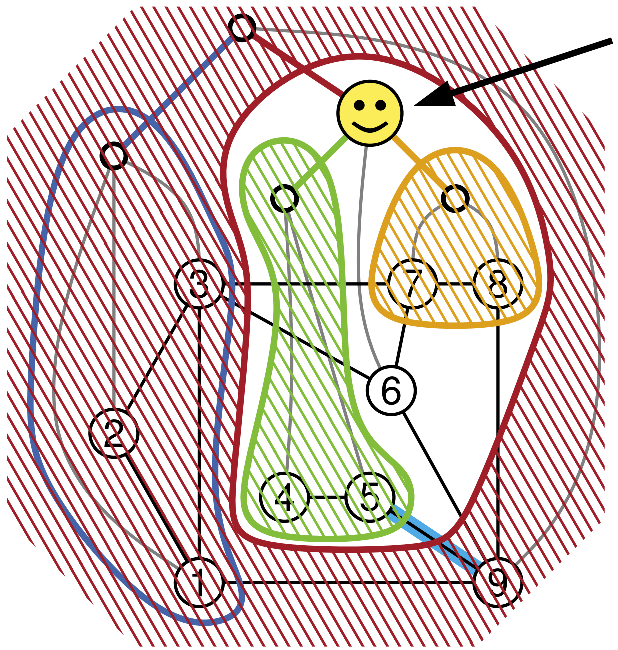

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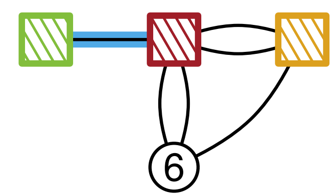

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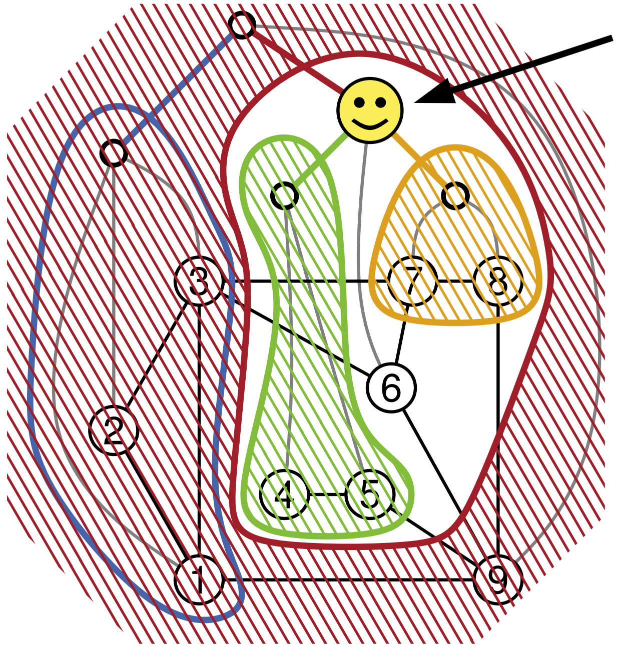

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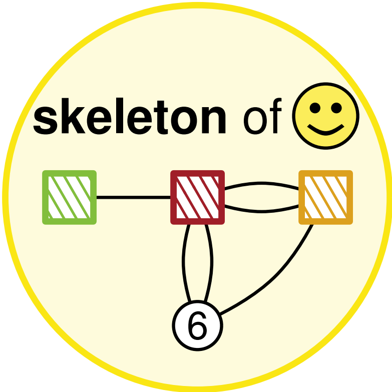

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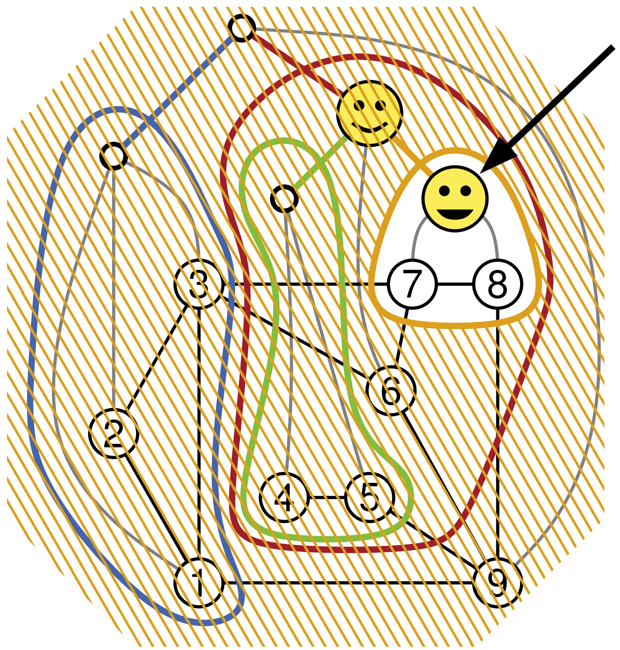

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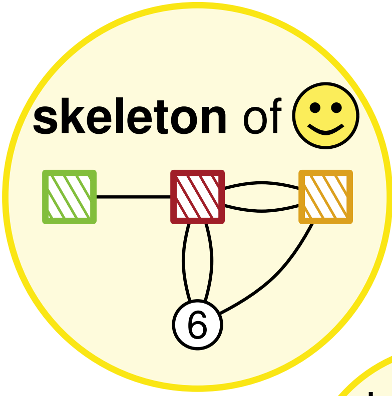

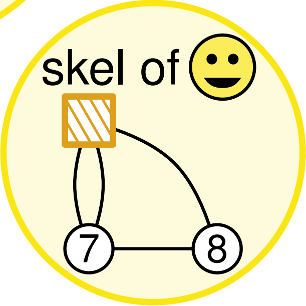

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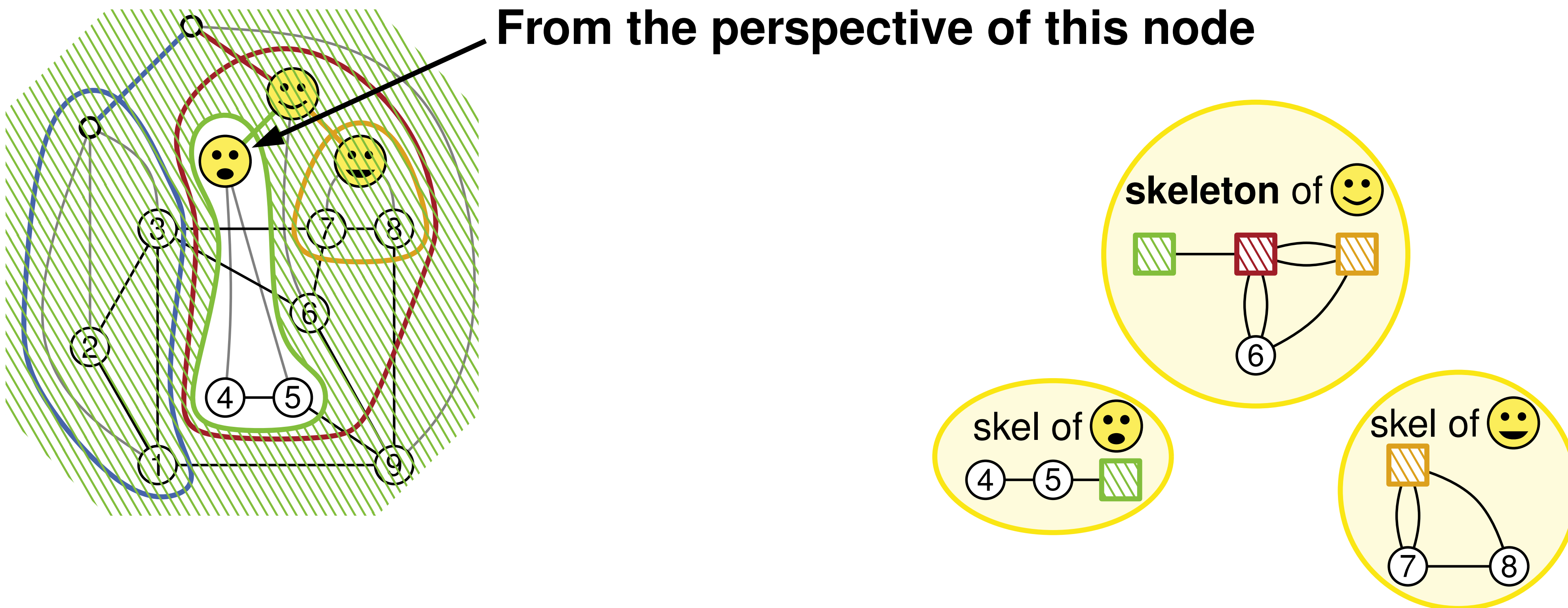

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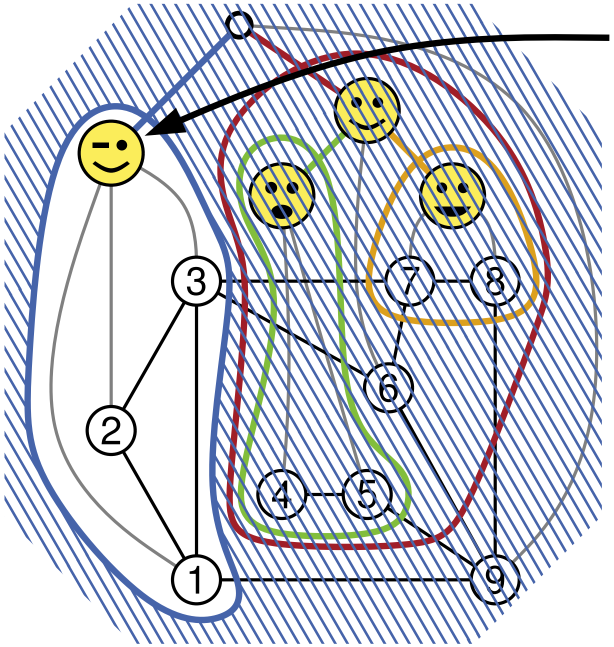

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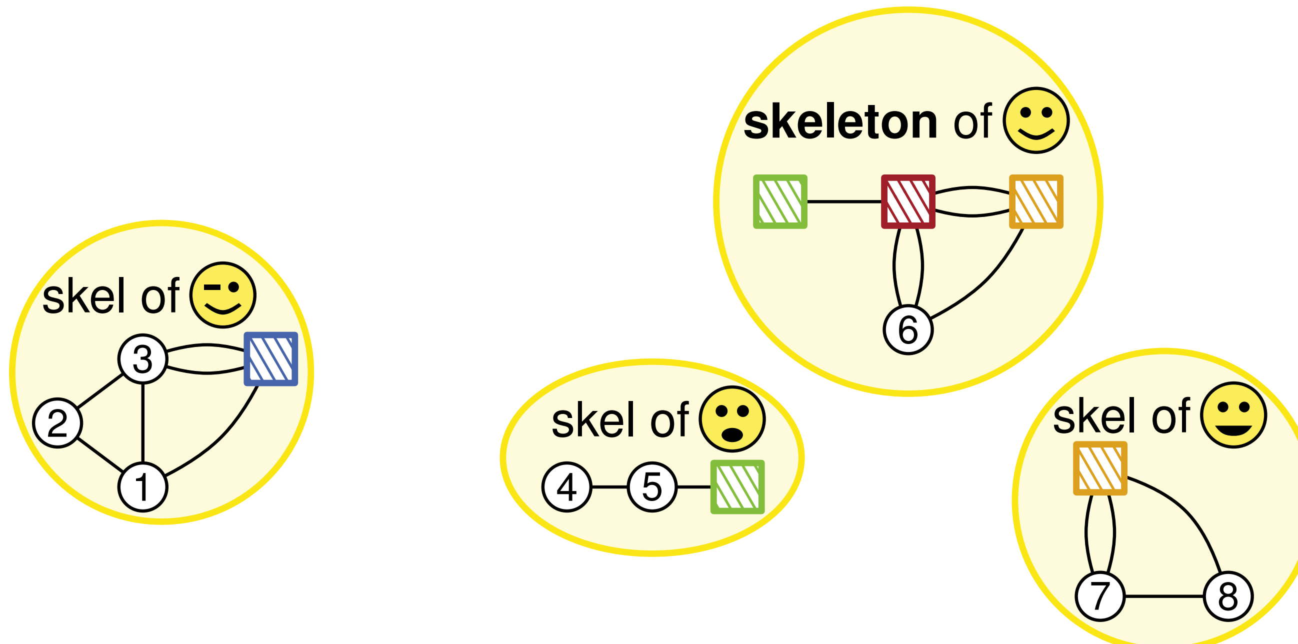

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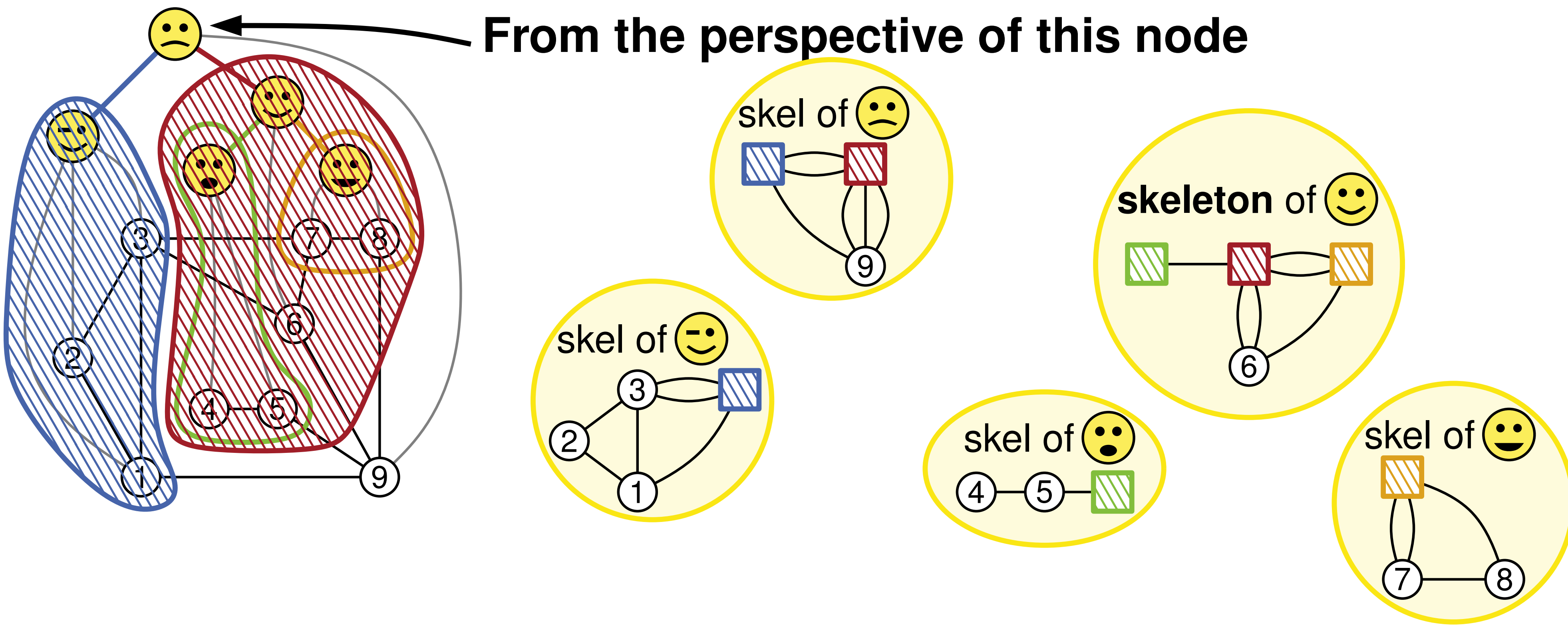

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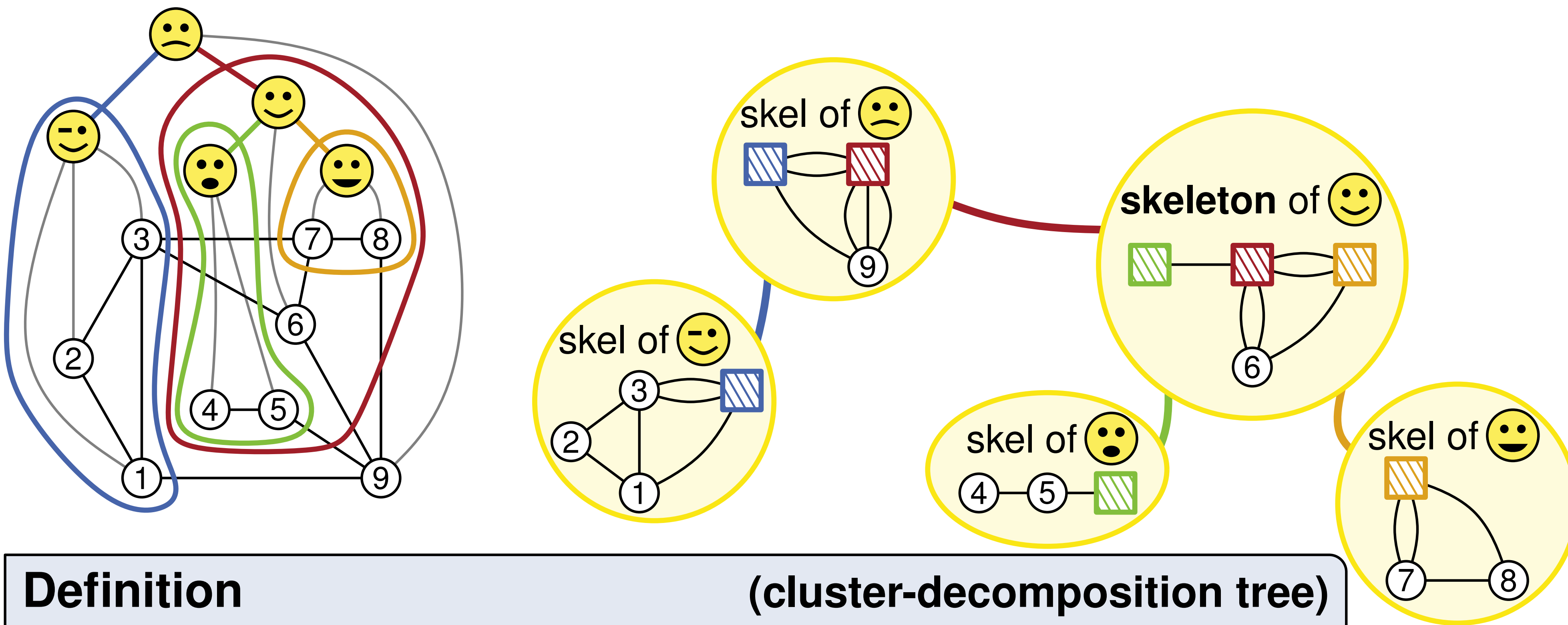
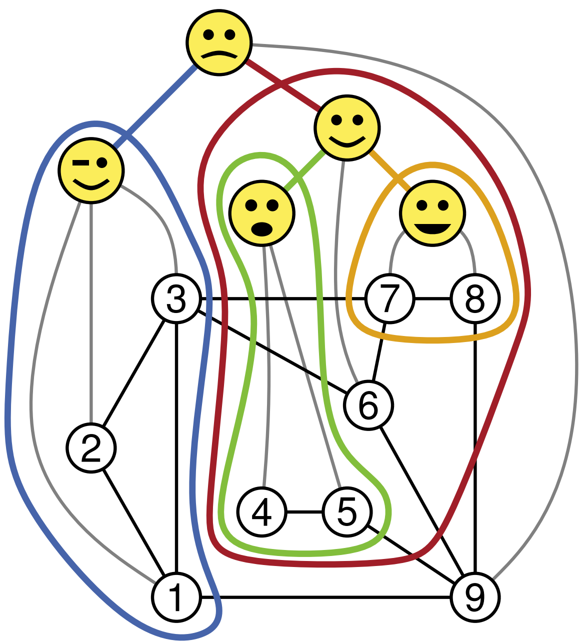
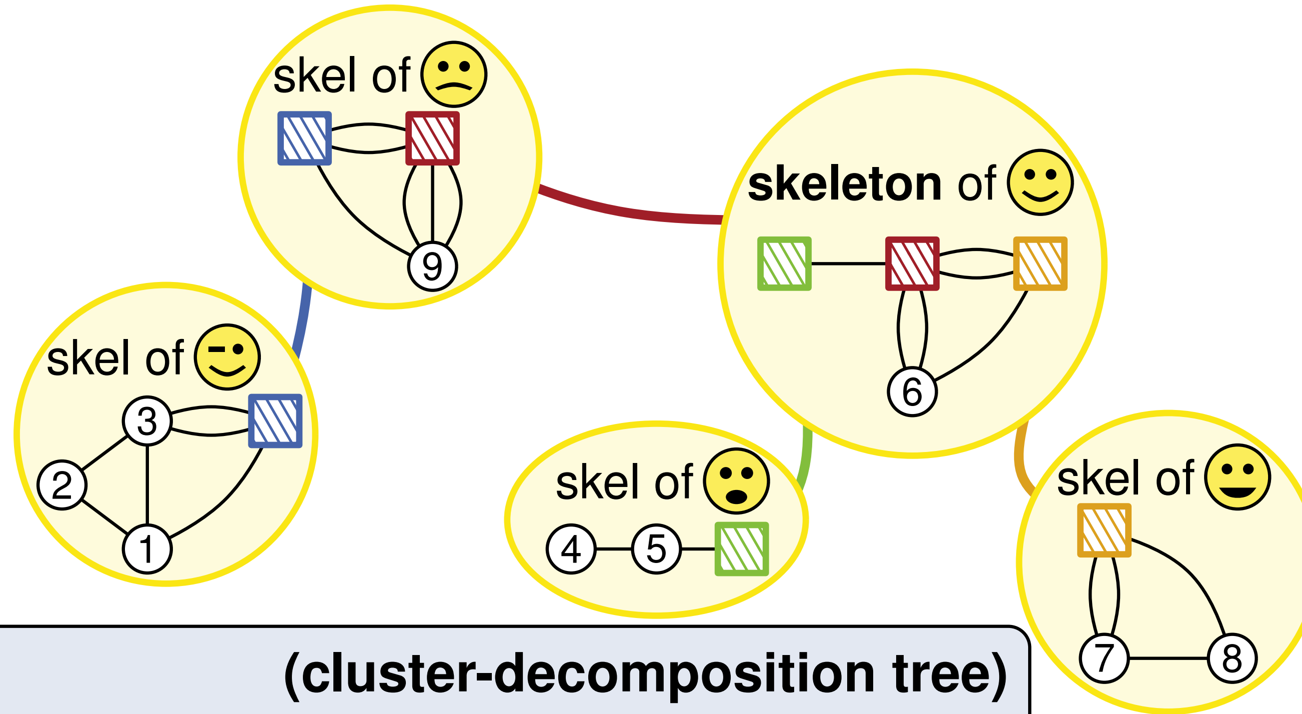


Definition **(cluster-decomposition tree)**

The **cd-tree** of a clustered graph (G, T) is T together with a bijection between T's nodes and their skeletons.

The CD-Tree – A (New) Perspective





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The **cd-tree** of a clustered graph (G, T) is T together with a bijection between T's nodes and their skeletons.

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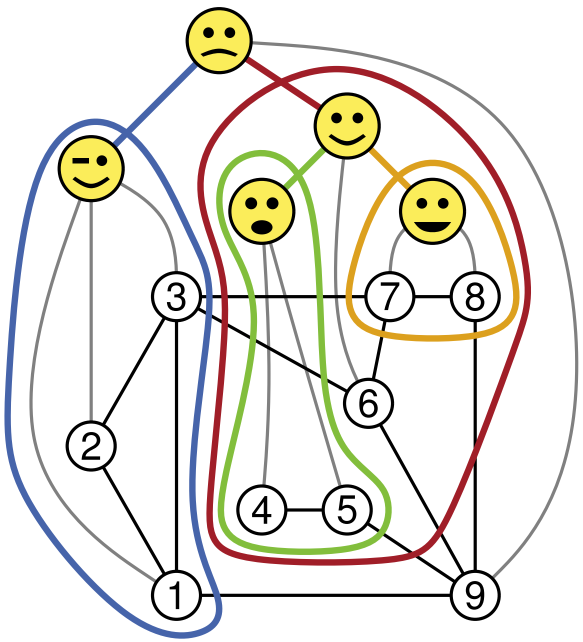

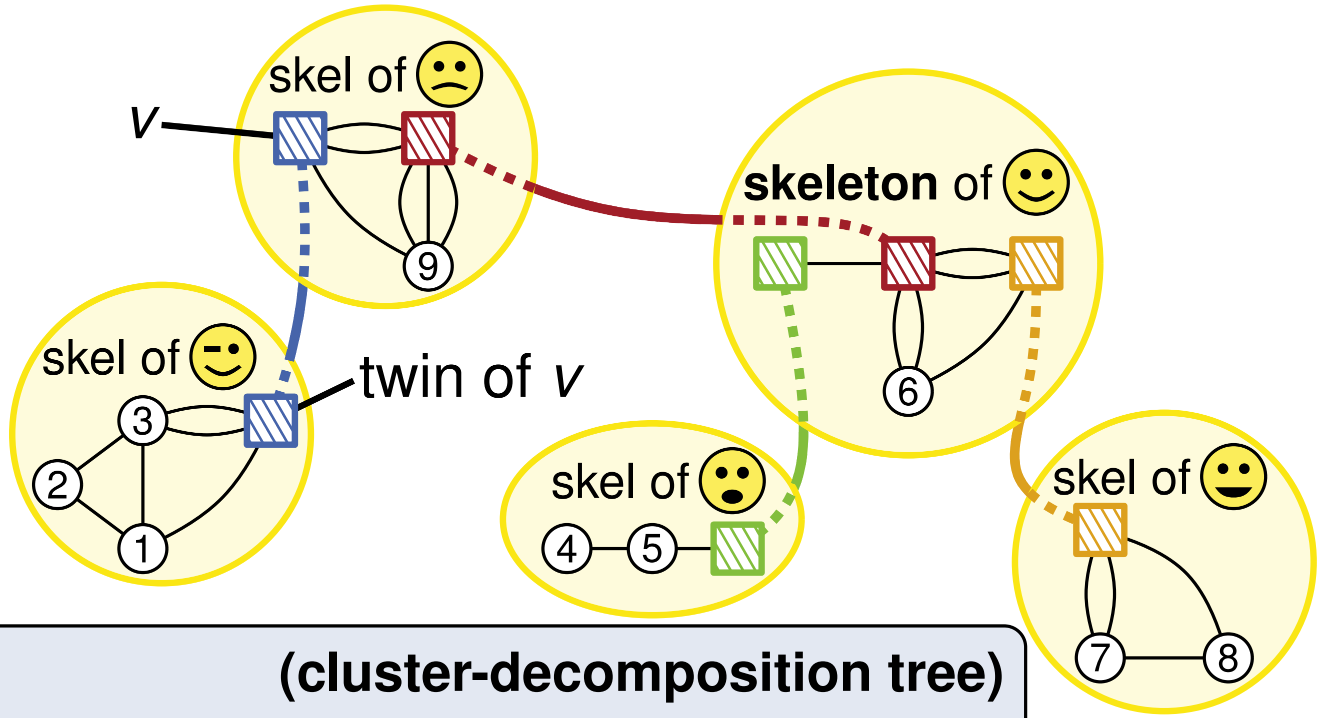

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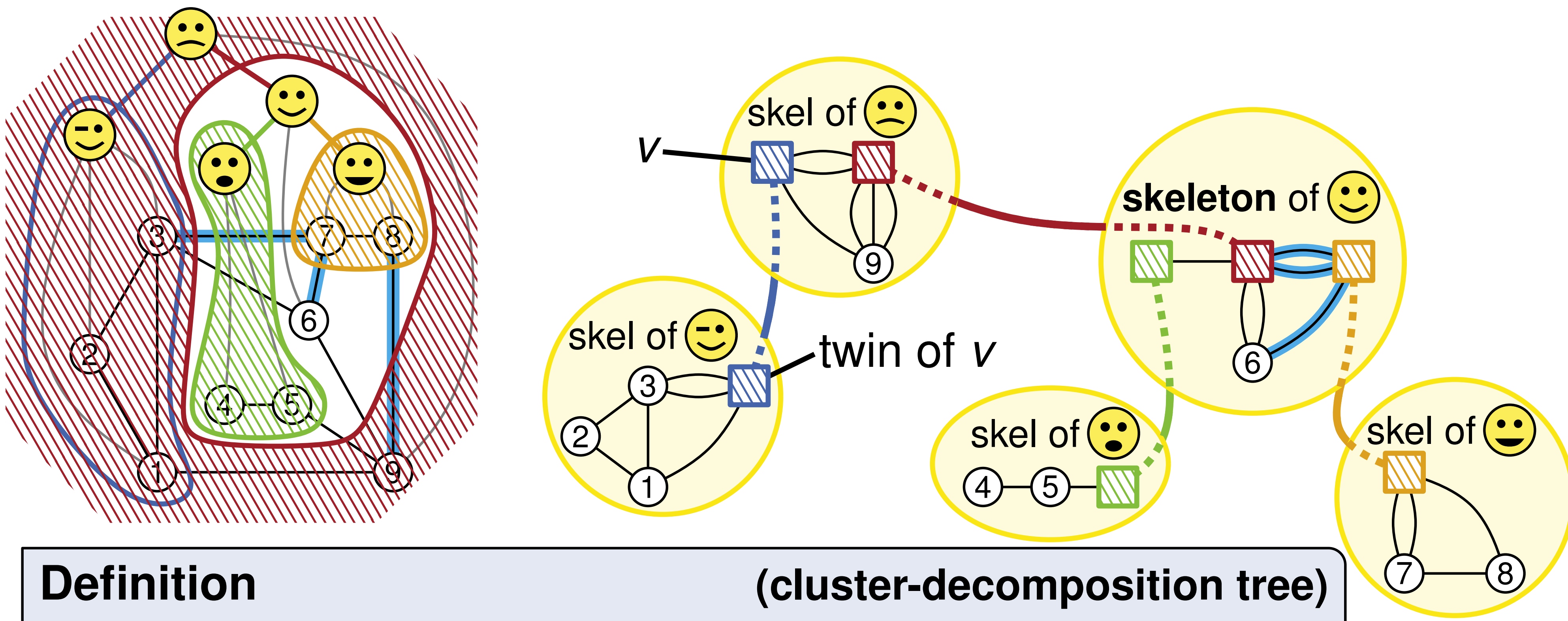

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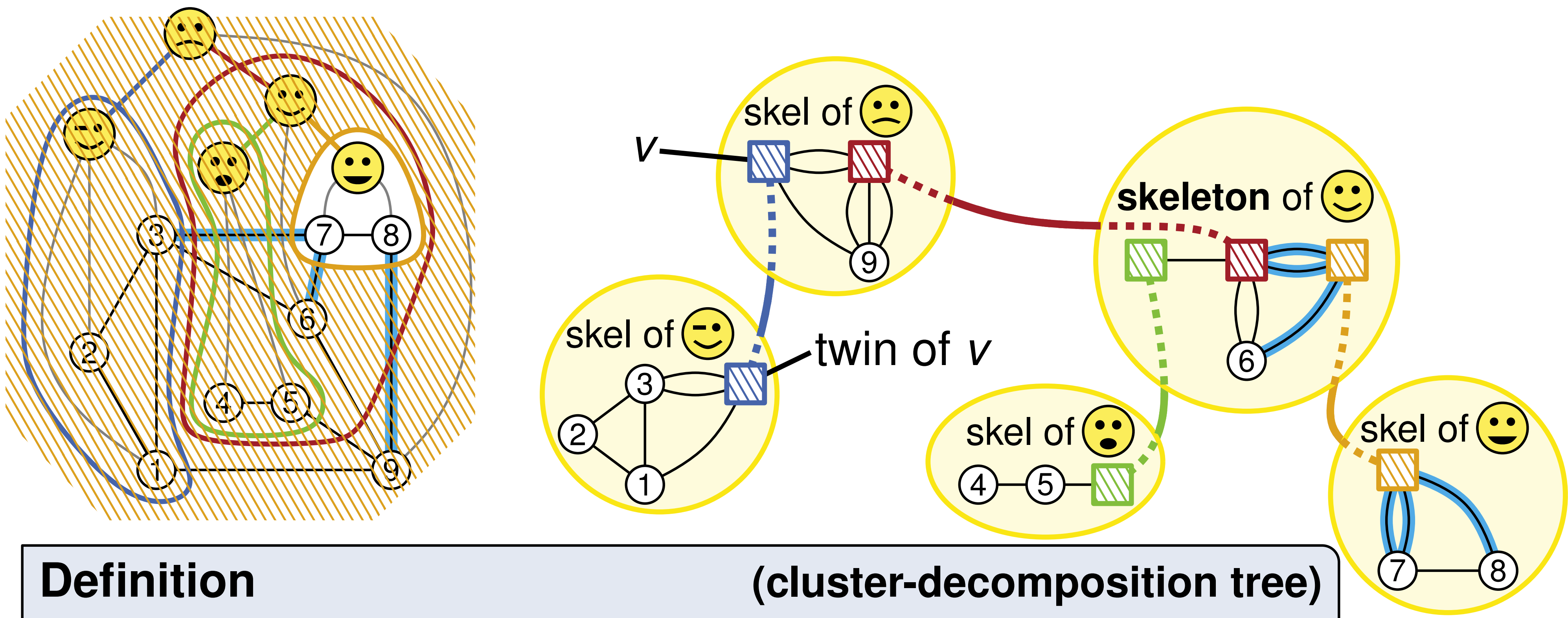

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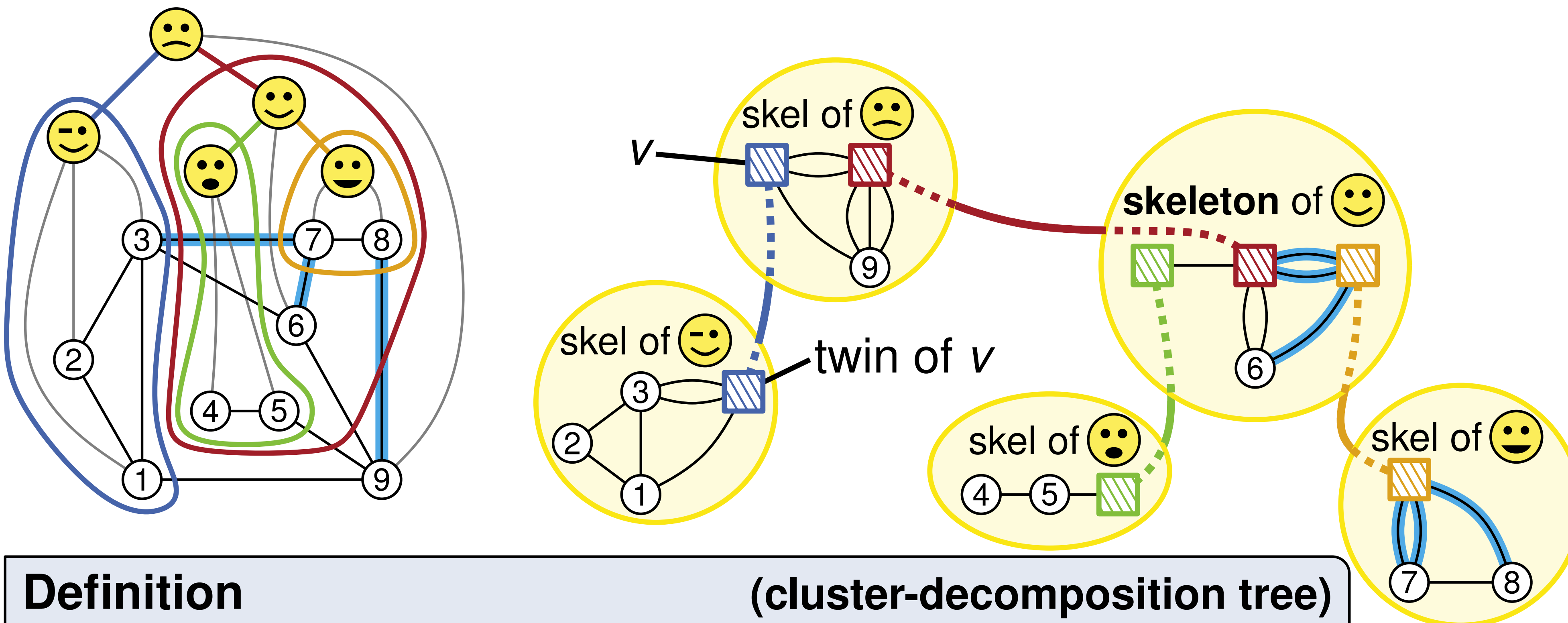

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The **cd-tree** of a clustered graph (G, T) is T together with a bijection between T's nodes and their skeletons.

Theorem **(characterization)**

(G, T) is c-planar $\Leftrightarrow$ one can embed the skeletons such that every virtual vertex and its twin have the same edge-ordering.

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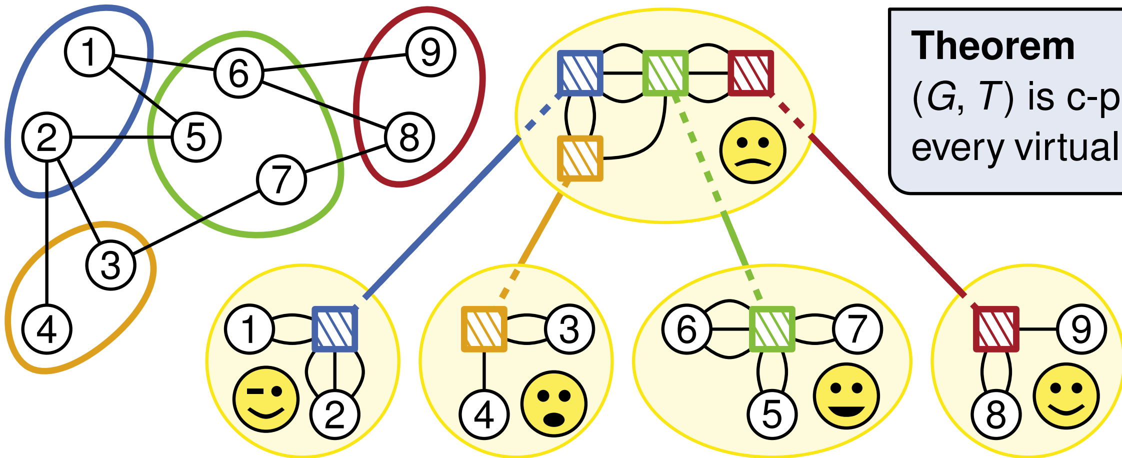
Flat-Clustered Graphs – Isolated Vertices



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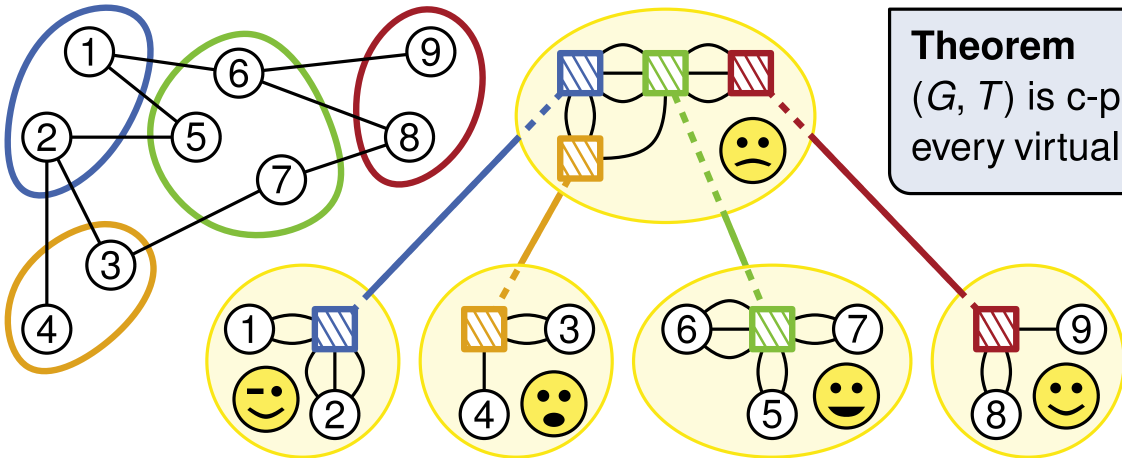

Theorem **(characterization)**
(G, T) is c-planar $\Leftrightarrow$ one can embed the skeletons such that every virtual vertex and its twin have the same edge-ordering.

Why is this instance special?

- flat hierarchy
- cluster = isolated vertices

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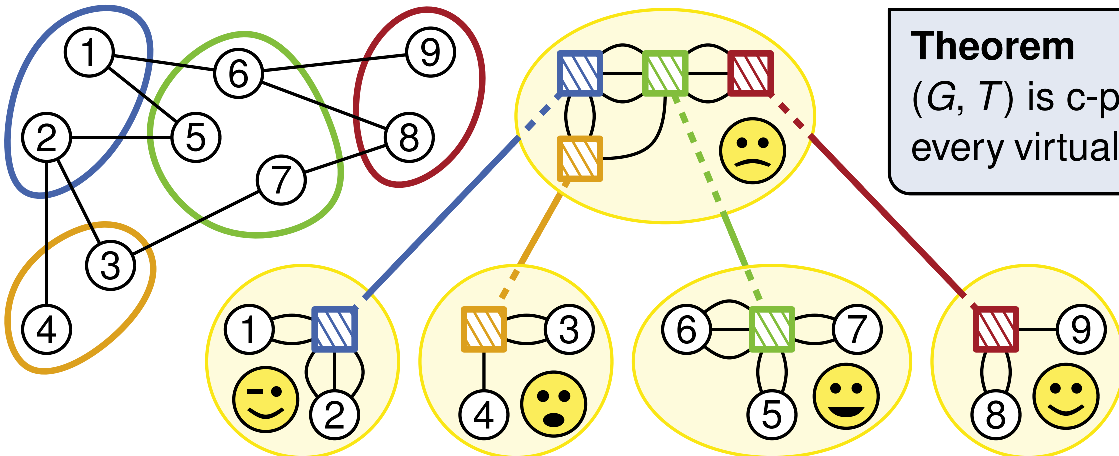
Flat-Clustered Graphs – Isolated Vertices



Theorem **(characterization)**
(G, T) is c-planar $\Leftrightarrow$ one can embed the skeletons such that every virtual vertex and its twin have the same edge-ordering.

Why is this instance special?

- flat hierarchy
- cluster = isolated vertices

Which edge-orderings around does 's skeleton allow?

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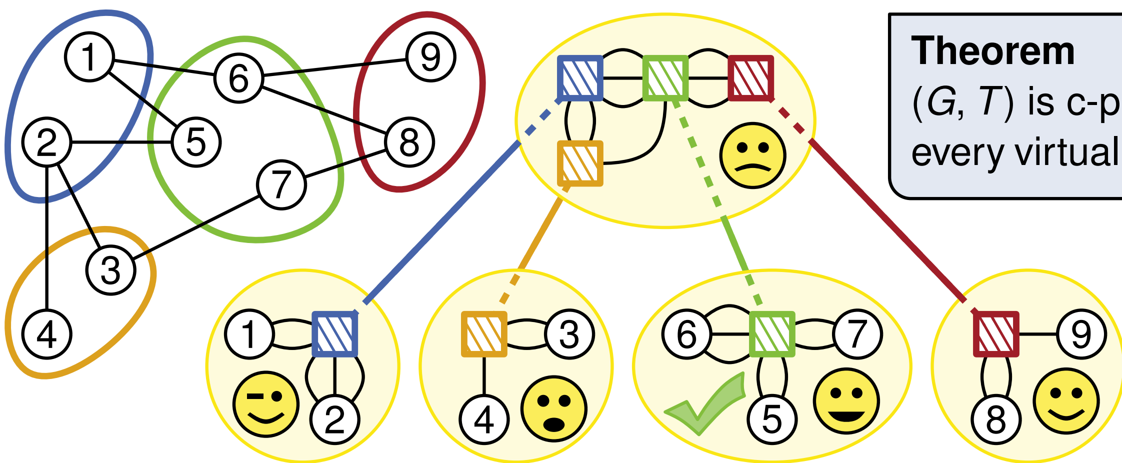
Flat-Clustered Graphs – Isolated Vertices



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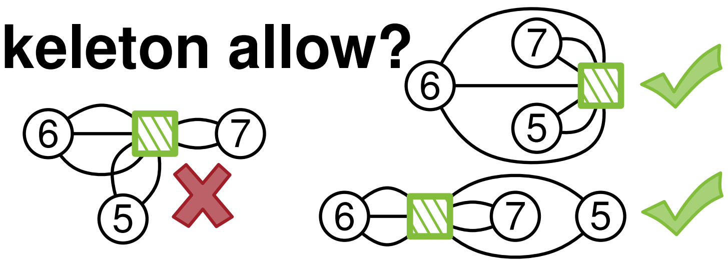
- vertices ⑤, ⑥, and ⑦ partition the edges
- partitions must not alternate



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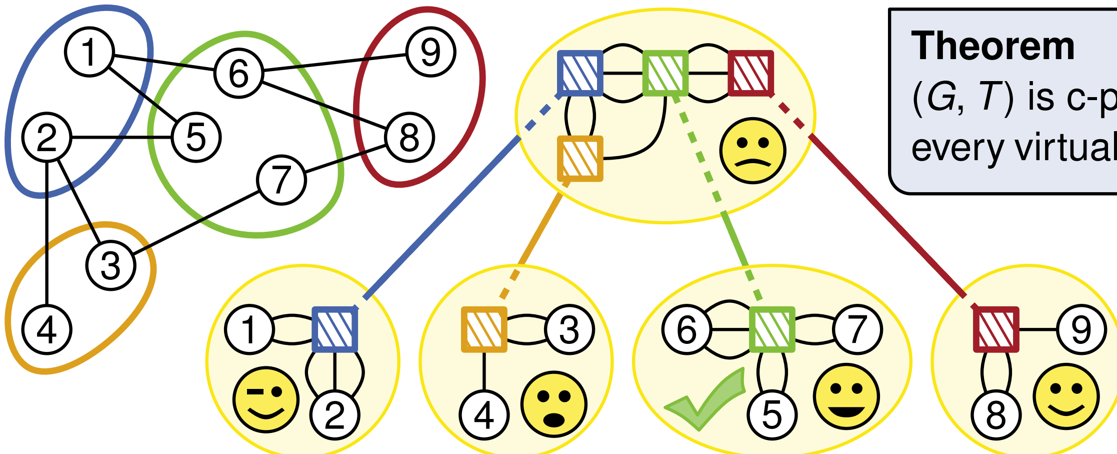
Flat-Clustered Graphs – Isolated Vertices



Theorem **(characterization)**
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Why is this instance special?

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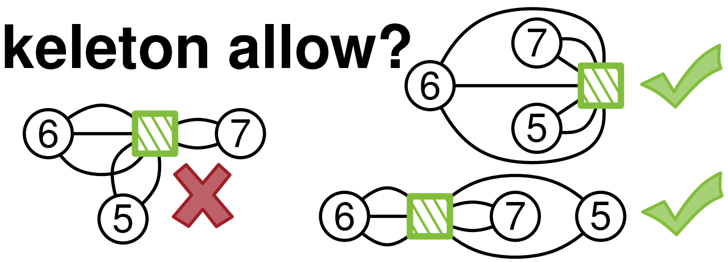
- cluster = isolated vertices

Which edge-orderings around does 's skeleton allow?

- vertices 5, 6, and 7 partition the edges
- partitions must not alternate



Definition **(partition-constraint)**
We call such a restriction of edge-orderings a **partition-constraint**.

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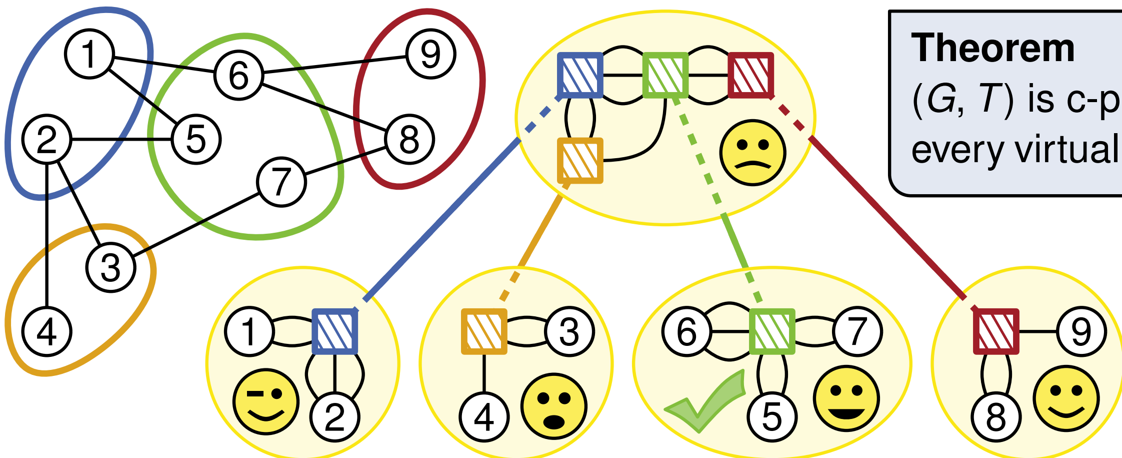
Flat-Clustered Graphs – Isolated Vertices



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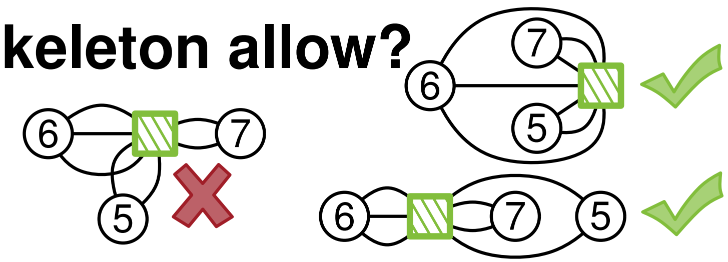
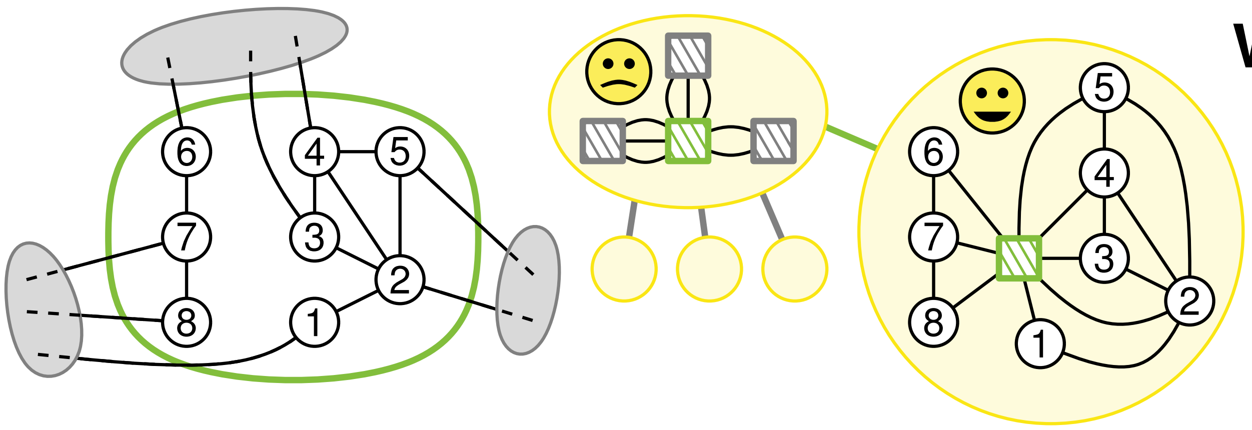


Definition **(partition-constraint)**
We call such a restriction of edge-orderings a **partition-constraint**.

Theorem
C-planarity for flat-clustered graphs where every cluster is a set of isolated vertices is equivalent to **planarity with partition-constraints**.

Flat-Clustered Graphs – Fixed Embedding



What is fixed?

- embedding of G

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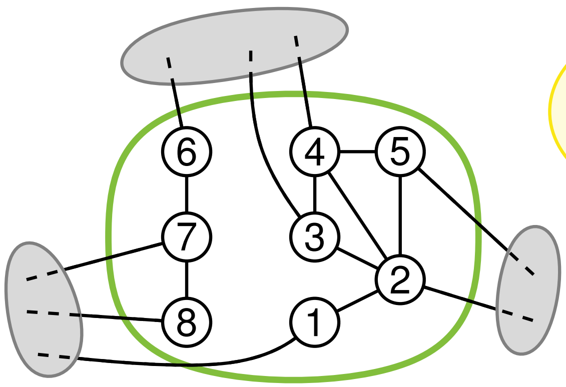
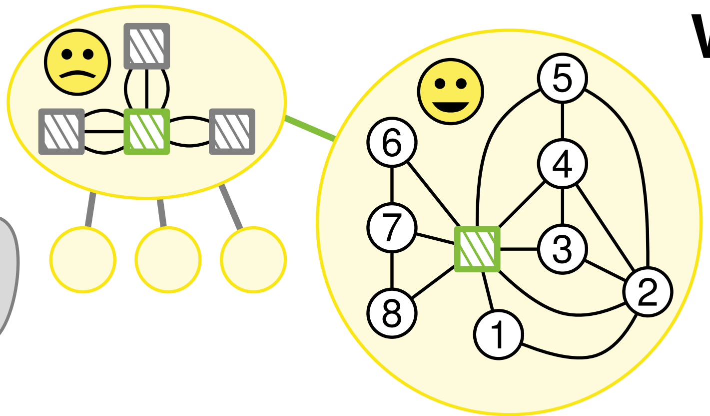
Flat-Clustered Graphs – Fixed Embedding





What is fixed?

- embedding of G
- edge-orderings of non-virtual vertices

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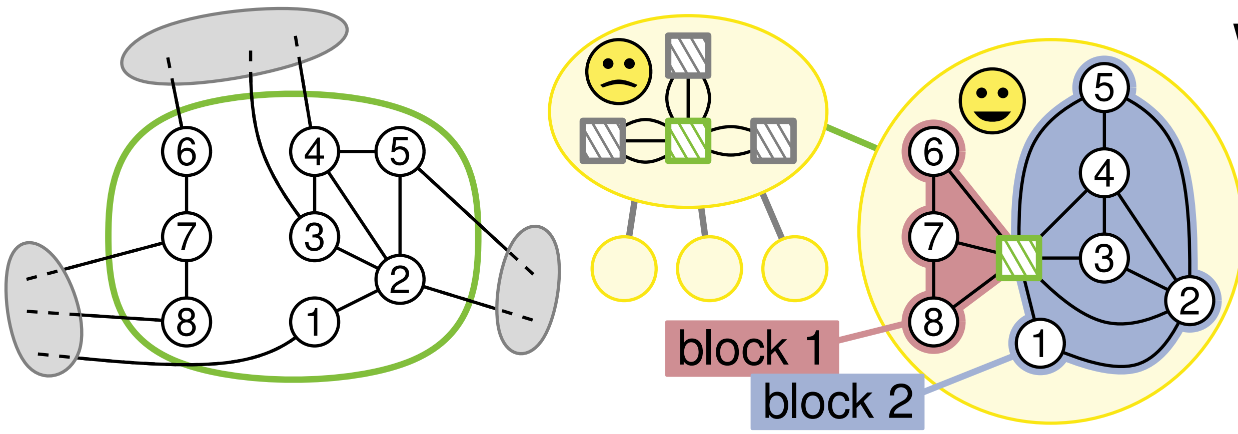
Flat-Clustered Graphs – Fixed Embedding



What is fixed?

- embedding of G
- edge-orderings of non-virtual vertices
- embeddings of blocks

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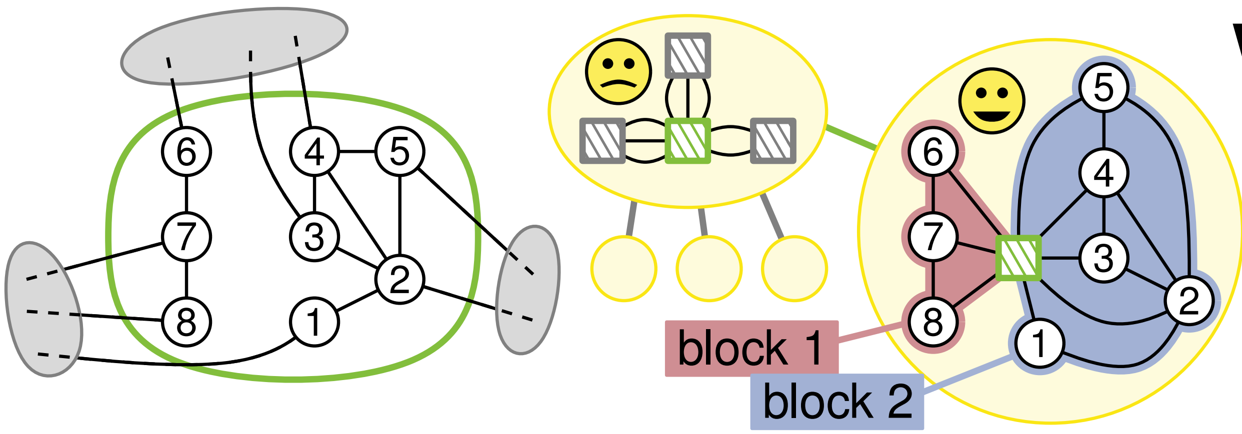
Flat-Clustered Graphs – Fixed Embedding



What is fixed?

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Which edge-orderings around  does 's skeleton allow?

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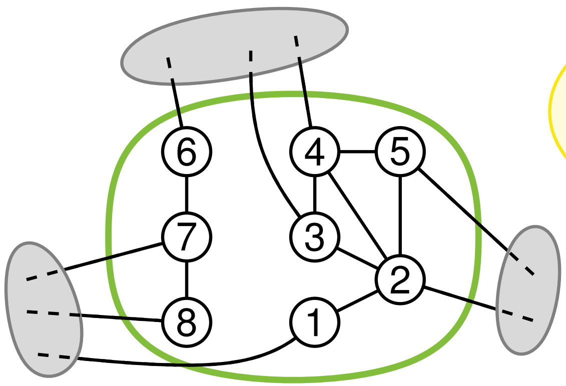
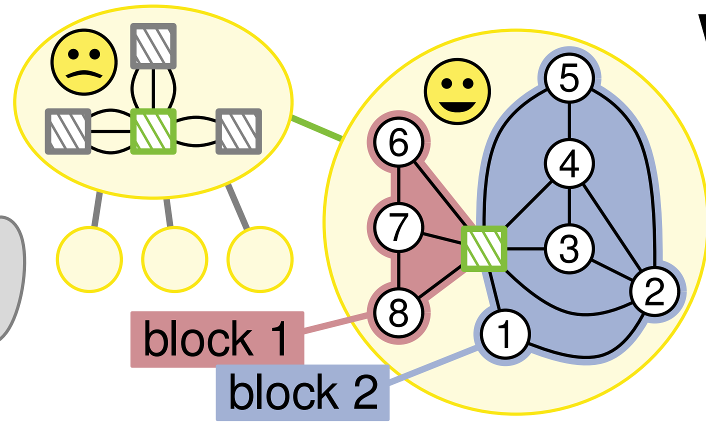
Flat-Clustered Graphs – Fixed Embedding





What is fixed?

- embedding of G
- edge-orderings of non-virtual vertices
- embeddings of blocks

Which edge-orderings around does 's skeleton allow?

- blocks in 's skeleton partition the edges

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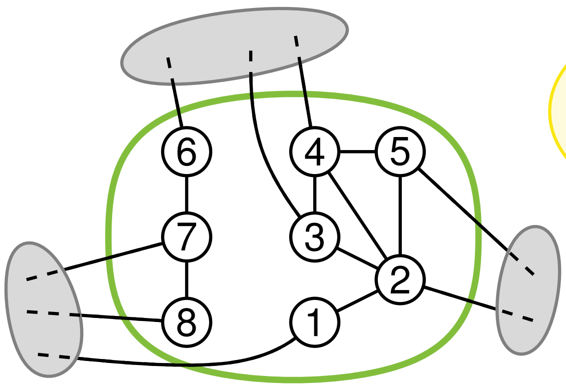
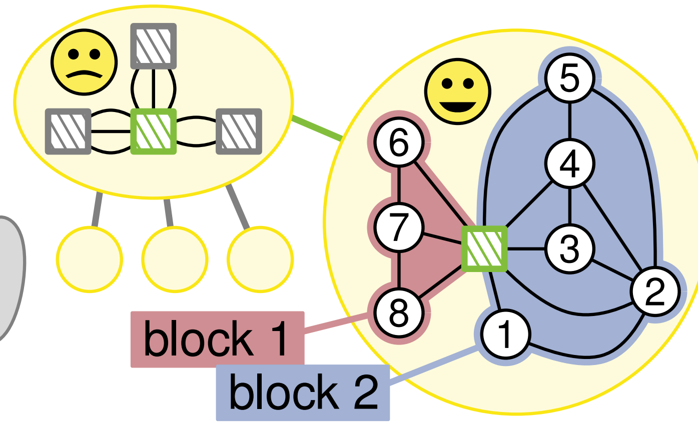
Flat-Clustered Graphs – Fixed Embedding





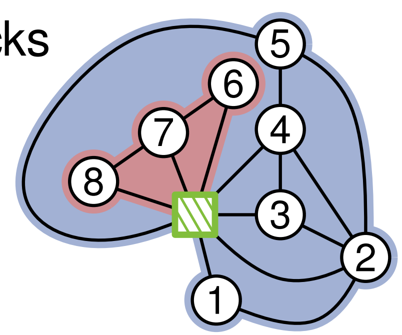
What is fixed?

- embedding of *G*
- edge-orderings of non-virtual vertices
- embeddings of blocks

Which edge-orderings around does 's skeleton allow?

- blocks in 's skeleton partition the edges
- partitions must not alternate → **partition-constraint**

Nesting the blocks is possible:



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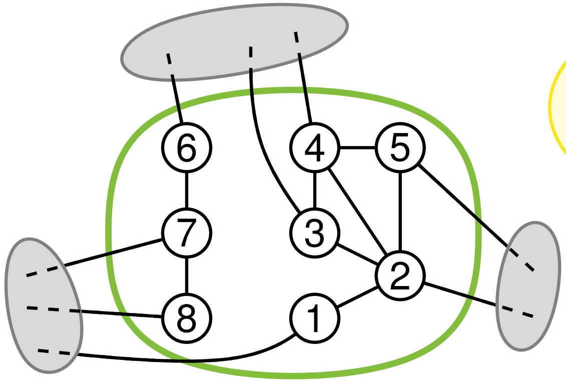
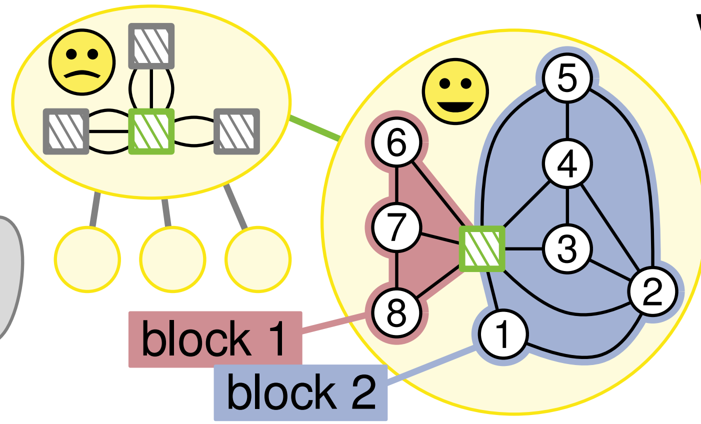
Flat-Clustered Graphs – Fixed Embedding





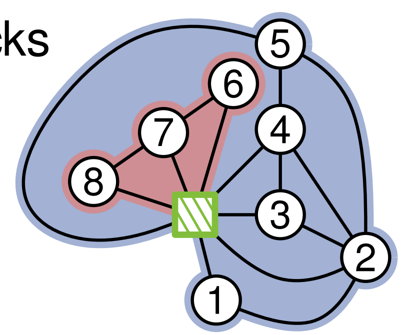
What is fixed?

- embedding of *G*
- edge-orderings of non-virtual vertices
- embeddings of blocks

Which edge-orderings around ▧ does 🙂's skeleton allow?

- blocks in 🙂's skeleton partition the edges
- partitions must not alternate → **partition-constraint**
- order in each partition is fixed → **full-constraint**

Nesting the blocks is possible:



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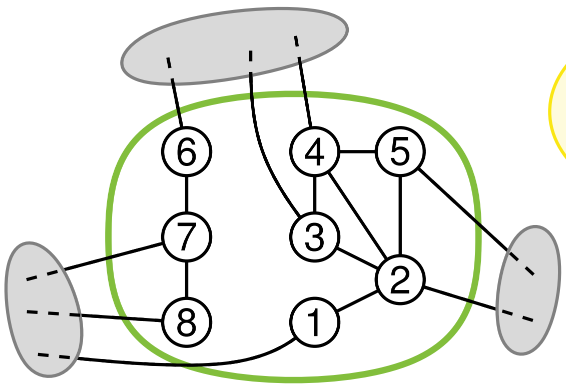
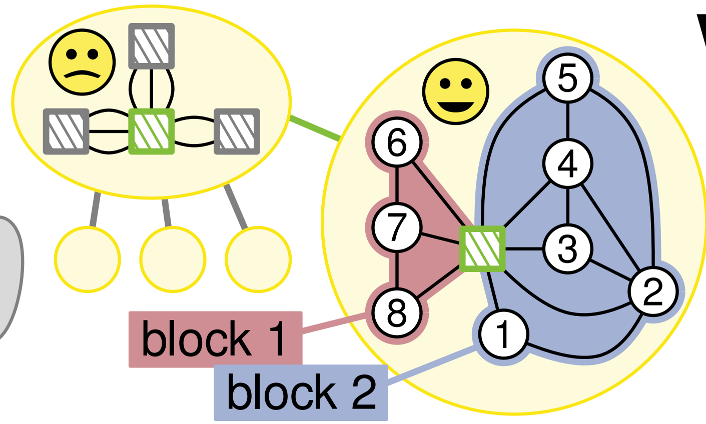
Flat-Clustered Graphs – Fixed Embedding





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partitioned full-constraint

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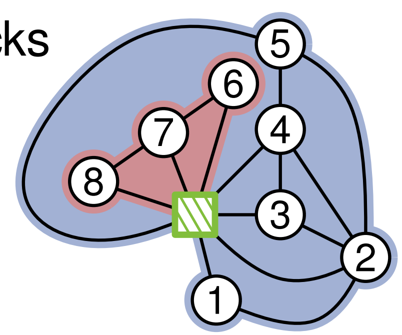

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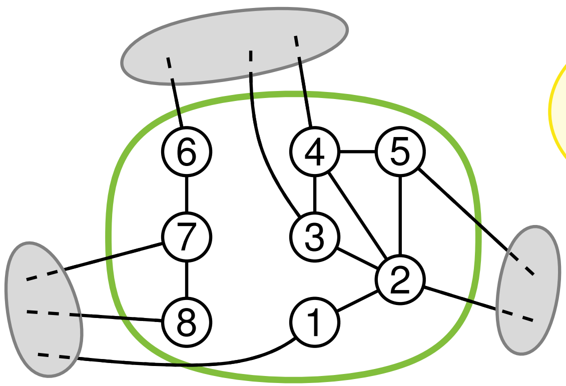
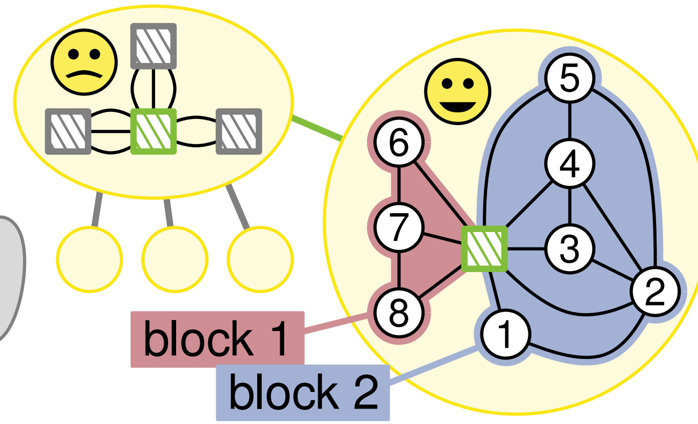
Flat-Clustered Graphs – Fixed Embedding





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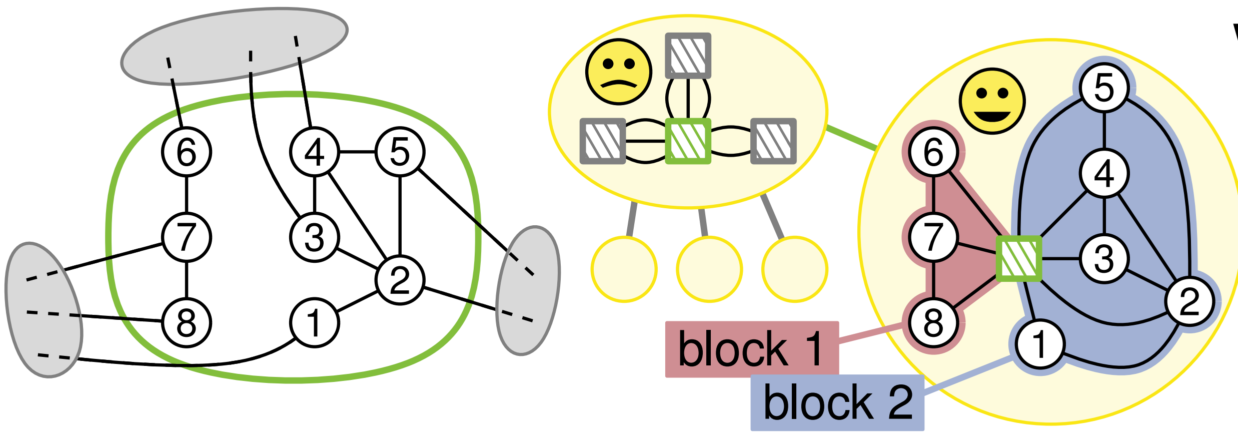
partitioned full-constraint

Theorem
C-planarity for flat-clustered graphs with fixed planar embedding is equivalent to **planarity with partitioned full-constraints**.

Flat-Clustered Graphs – ~~Fixed Embedding~~



What is fixed?

- ~~embedding of *G*~~
- ~~edge-orderings of non-virtual vertices~~
- ~~embeddings of blocks~~

Which edge-orderings around does 's skeleton allow?

- blocks in 's skeleton partition the edges
- partitions must not alternate → **partition-constraint**
- order in each partition **??** → **??-constraint**

partitioned ??-constraint

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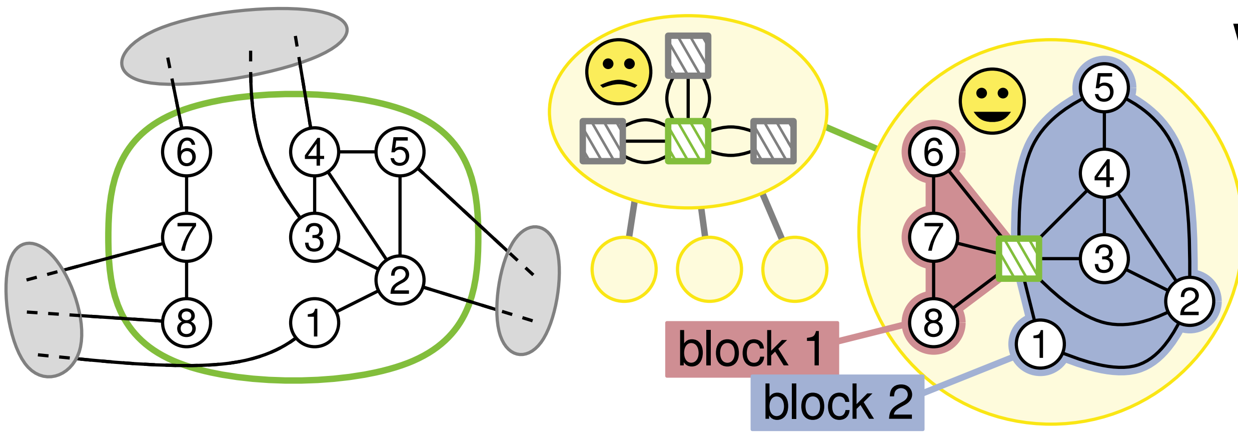
Flat-Clustered Graphs – ~~Fixed Embedding~~



What is fixed?

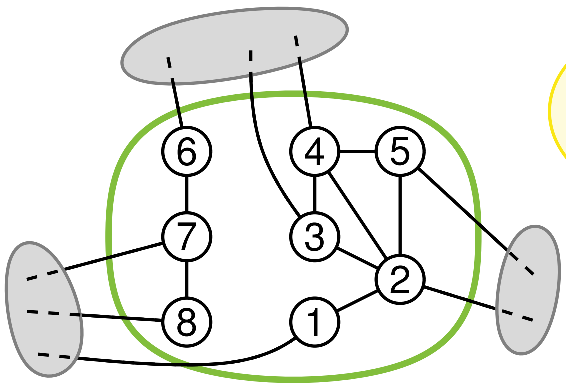
- ~~embedding of *G*~~
- ~~edge-orderings of non-virtual vertices~~
- ~~embeddings of blocks~~

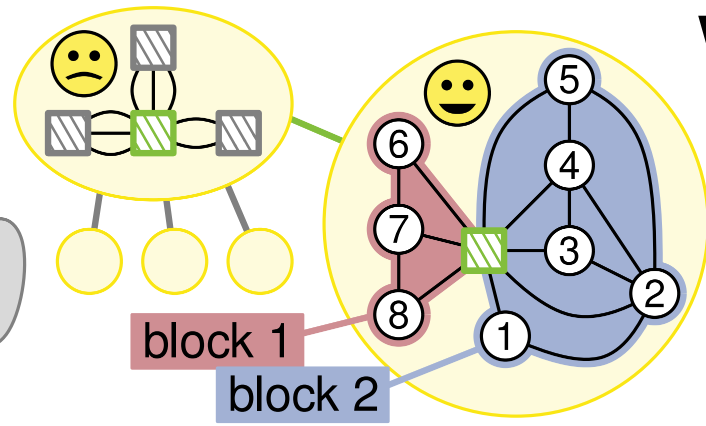
Which edge-orderings around ▧ does 🙂's skeleton allow?

- blocks in 🙂's skeleton partition the edges
- partitions must not alternate → **partition-constraint**
- order in each partition **is restricted by a PQ-tree** → **PQ-constraint**

partitioned PQ-constraint

Flat-Clustered Graphs – ~~Fixed Embedding~~





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partitioned PQ-constraint

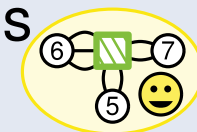
> **Theorem**
> C-planarity for flat-clustered graphs is equivalent to **planarity with partitioned PQ-constraints**.

Conclusion

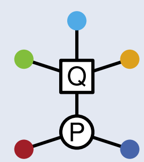
Theorem **(flat-clustered graphs)**

These variants of c-planarity and constrained embedding are equivalent:

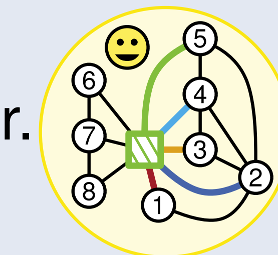
- flat, isolated vertices
  ⟷ partition-constr.



- flat
  ⟷ partitioned PQ-constr.



- flat, fixed embedding
  ⟷ partitioned full-constr.



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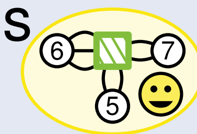

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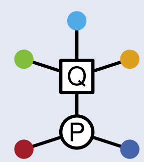

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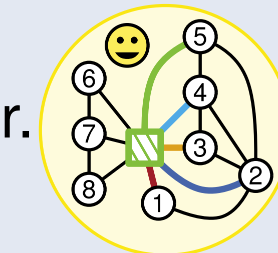
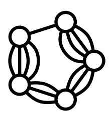


solved variants

graph class	constraints	
multi-cycle 	partition constraints partitions of size 2 	[Cortese, Di Battista, Patrignani, Pizzonia 2005]

Conclusion

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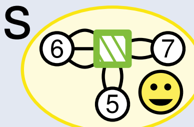
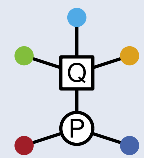
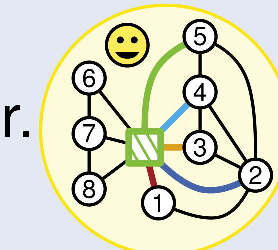
- flat, isolated vertices ⟷ partition-constr. 

- flat ⟷ partitioned PQ-constr. 

- flat, fixed embedding ⟷ partitioned full-constr. 

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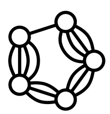
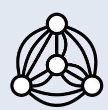
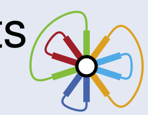
multi-cycle 	partition constraints partitions of size 2 	**[Cortese, Di Battista, Patrignani, Pizzonia** 2005]
fixed embedding (up to reordering multi-edges) 	partition constraints partitions of size 2 	**[Cortese, Di Battista, Patrignani, Pizzonia** 2009]

Conclusion

Karlsruhe Institute of Technology

Theorem **(flat-clustered graphs)**

These variants of c-planarity and constrained embedding are equivalent:

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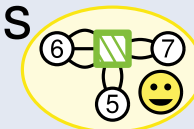
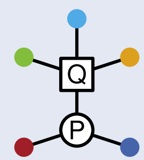
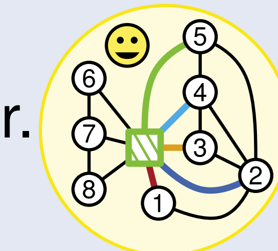


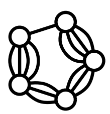
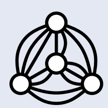
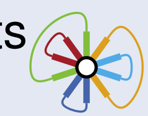
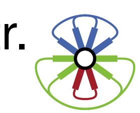
- flat ⟷ partitioned PQ-constr.



- flat, fixed embedding ⟷ partitioned full-constr.



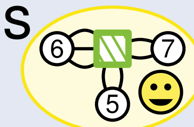
solved variants

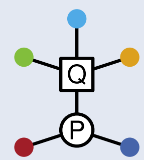
graph class	**constraints**	
multi-cycle 	partition constraints partitions of size 2 	[Cortese, Di Battista, Patrignani, Pizzonia 2005]
fixed embedding (up to reordering multi-edges) 	partition constraints partitions of size 2 	[Cortese, Di Battista, Patrignani, Pizzonia 2009]
complicated restriction (not very strong)	partitioned-full constr. at most 3 partitions 	[Jelínková, Kára, Kratochvíl, Pergel, Suchý, Vyskočil 2009]

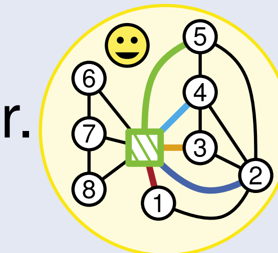
Conclusion

Theorem **(flat-clustered graphs)**

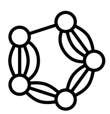
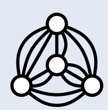
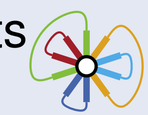
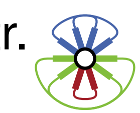
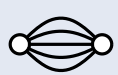
These variants of c-planarity and constrained embedding are equivalent:

- flat, isolated vertices ⟷ partition-constr. 

- flat ⟷ partitioned PQ-constr. 

- flat, fixed embedding ⟷ partitioned full-constr. 

solved variants

graph class	constraints	
multi-cycle 	partition constraints partitions of size 2 	**[Cortese, Di Battista, Patrignani, Pizzonia** 2005]
fixed embedding (up to reordering multi-edges) 	partition constraints partitions of size 2 	**[Cortese, Di Battista, Patrignani, Pizzonia** 2009]
complicated restriction (not very strong)	partitioned-full constr. at most 3 partitions 	**[Jelínková, Kára, Kratochvíl, Pergel, Suchý, Vyskočil** 2009]
only two vertices 	partitioned PQ-constraints	**[Biedl, Kaufmann, Mutzel** 1998] **[Hong, Nagamochi** 2014]

Conclusion

Theorem **(flat-clustered graphs)**

These variants of c-planarity and constrained embedding are equivalent:

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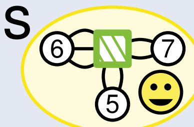

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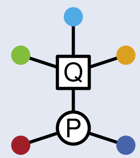

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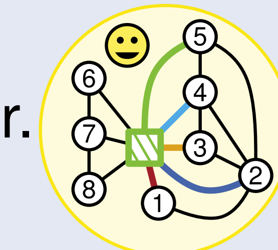

solved variants

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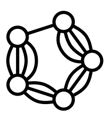
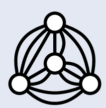
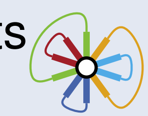
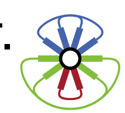
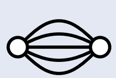
multi-cycle 	partition constraints partitions of size 2 	**[Cortese, Di Battista, Patrignani, Pizzonia** 2005]
fixed embedding (up to reordering multi-edges) 	partition constraints partitions of size 2 	**[Cortese, Di Battista, Patrignani, Pizzonia** 2009]
complicated restriction (not very strong)	partitioned-full constr. at most 3 partitions 	**[Jelínková, Kára, Kratochvíl, Pergel, Suchý, Vyskočil** 2009]
only two vertices 	partitioned PQ-constraints	**[Biedl, Kaufmann, Mutzel** 1998] **[Hong, Nagamochi** 2014]

open problem: extend this table

Final Remarks

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- lowest level: same constraints as in the flat-clustered case
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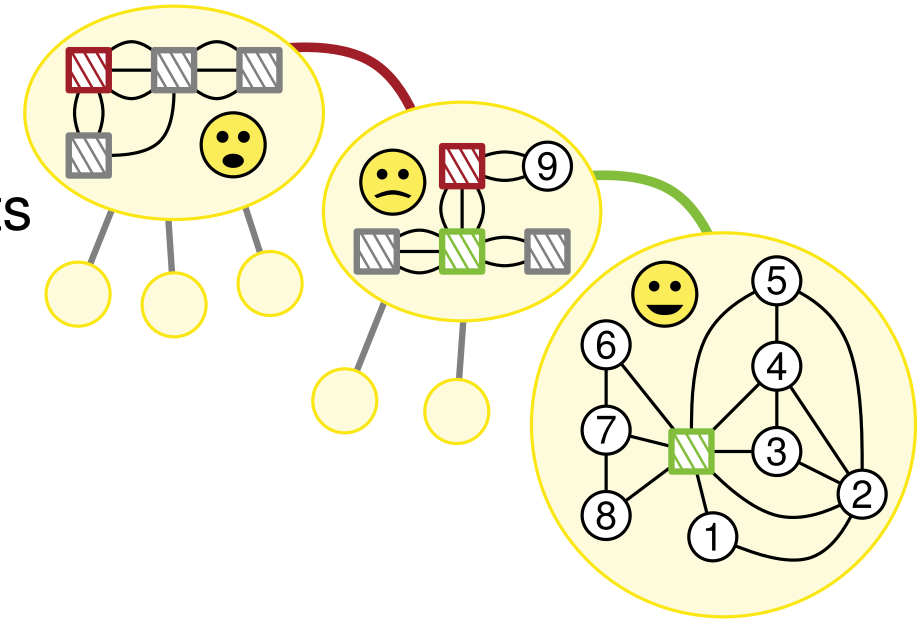

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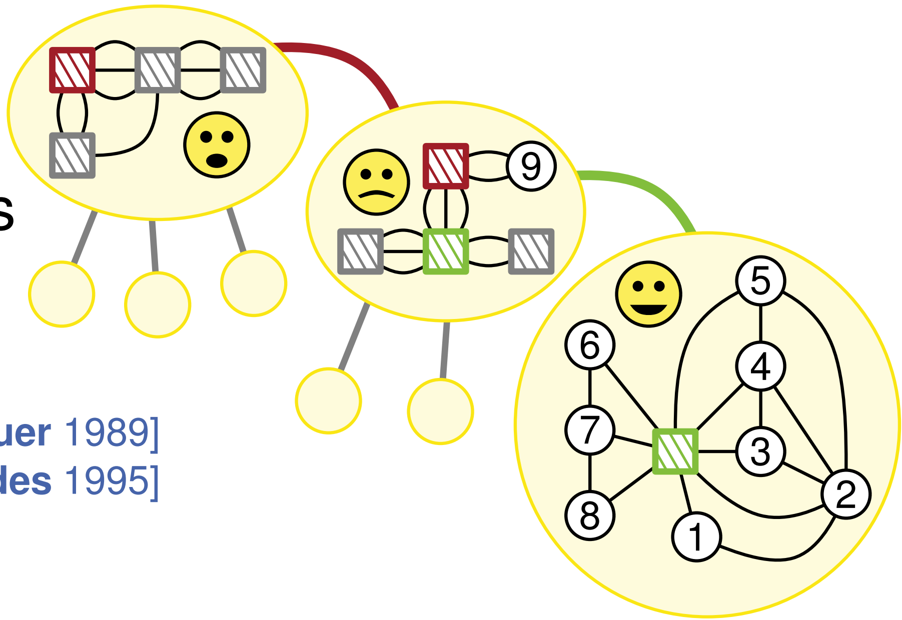

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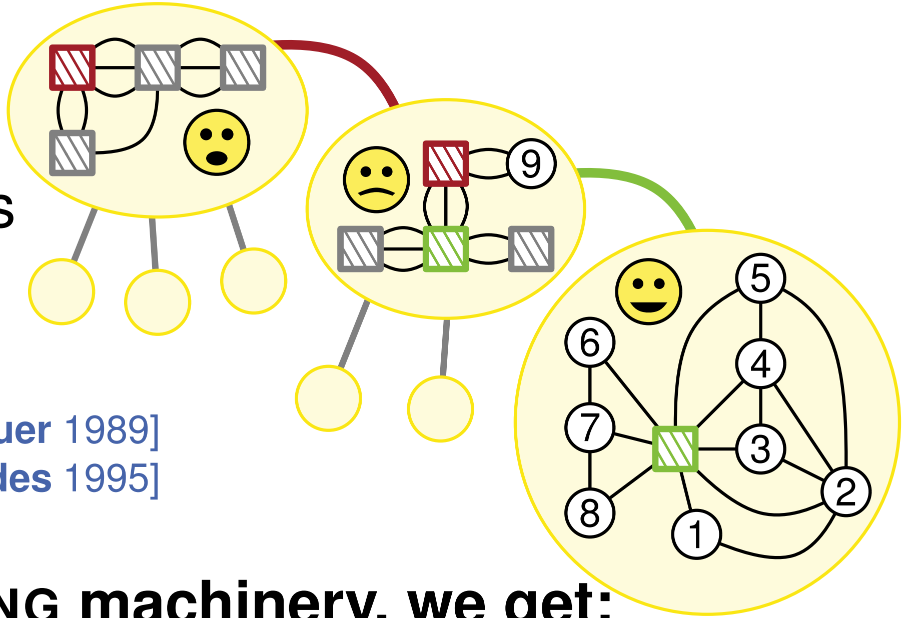

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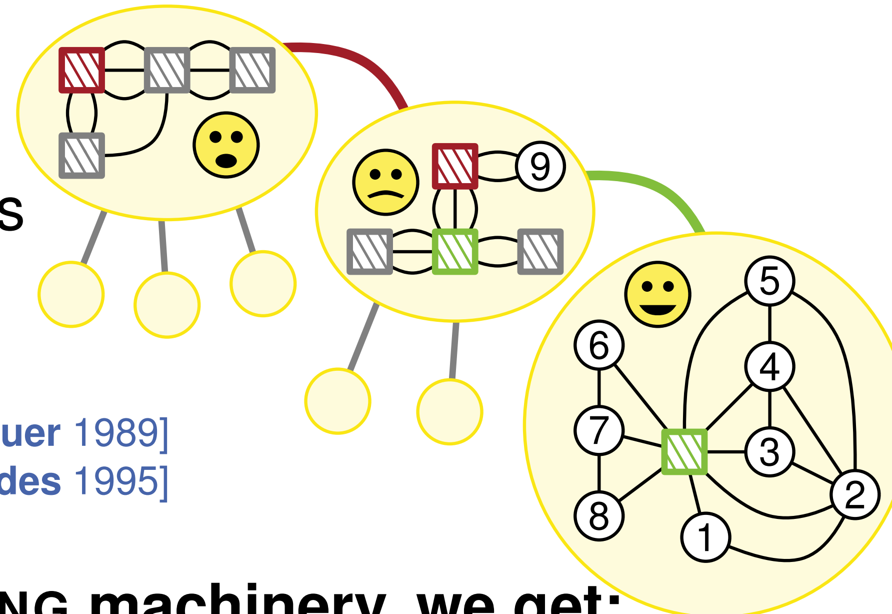

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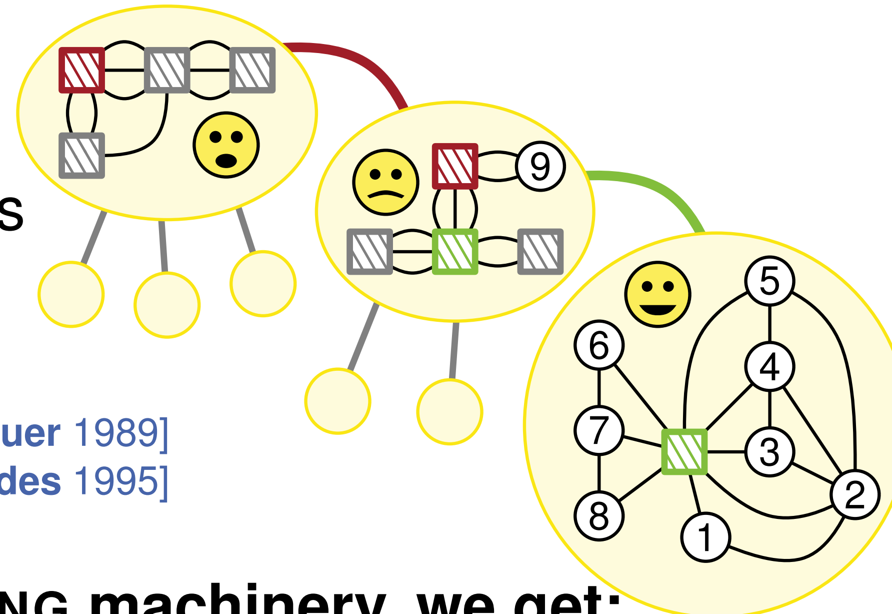

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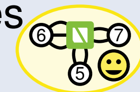

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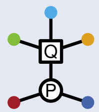

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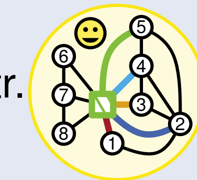

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Thank you!

Questions?

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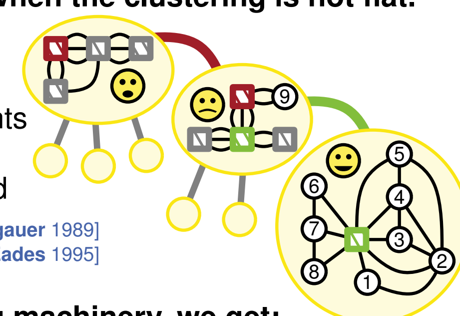

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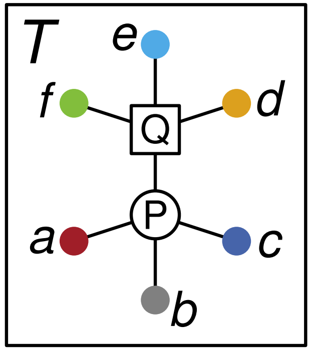

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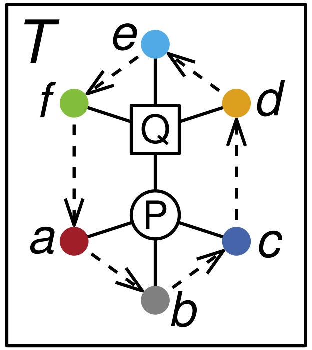

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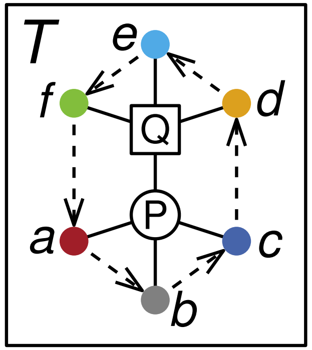

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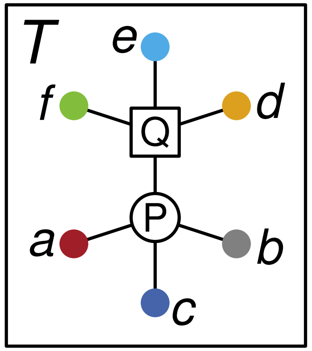

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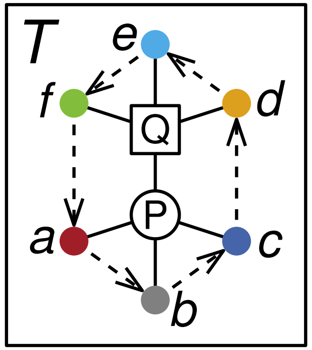

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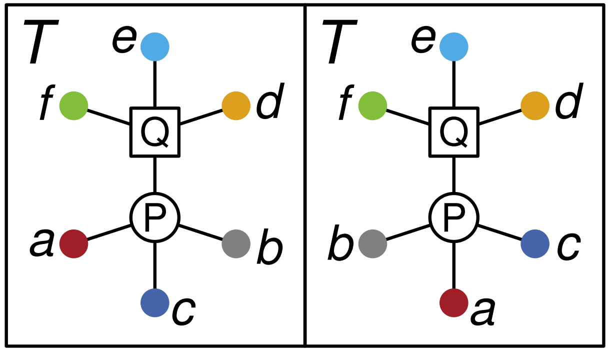

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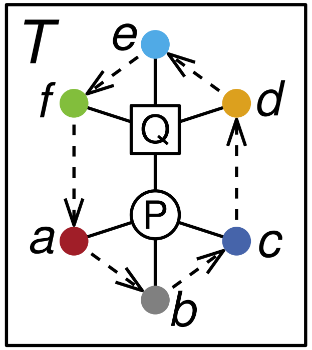

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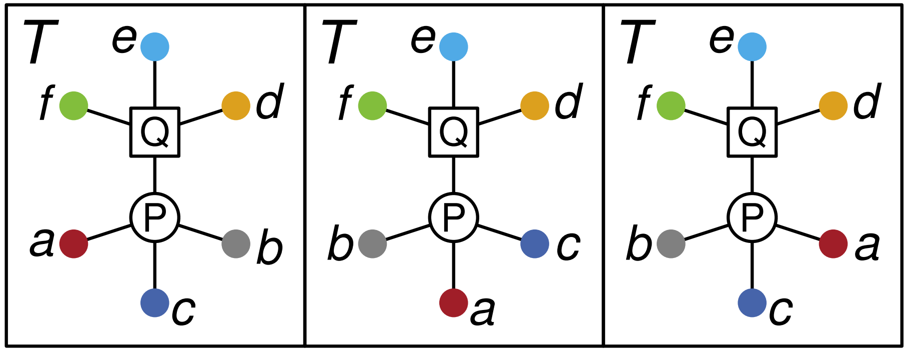

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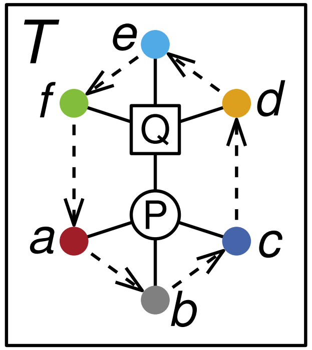

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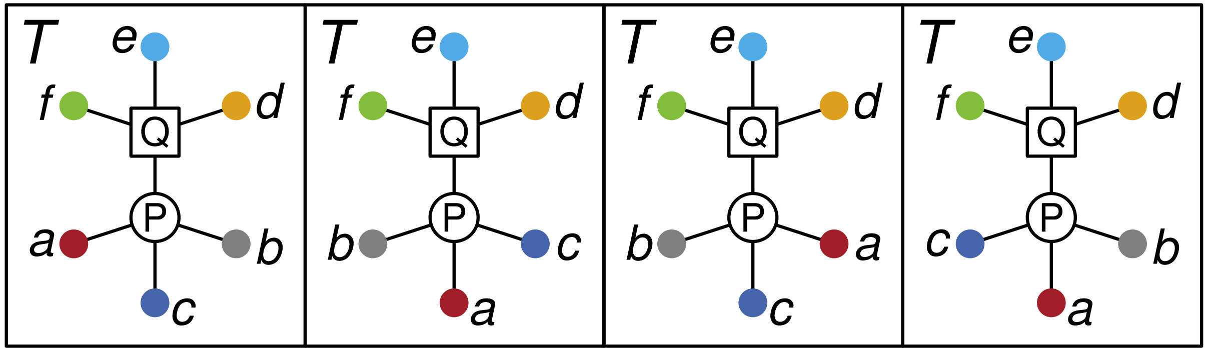

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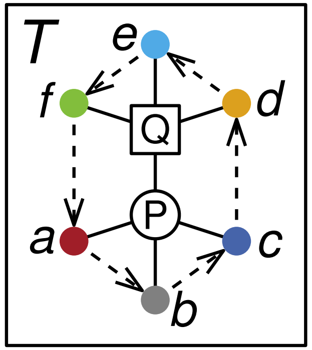

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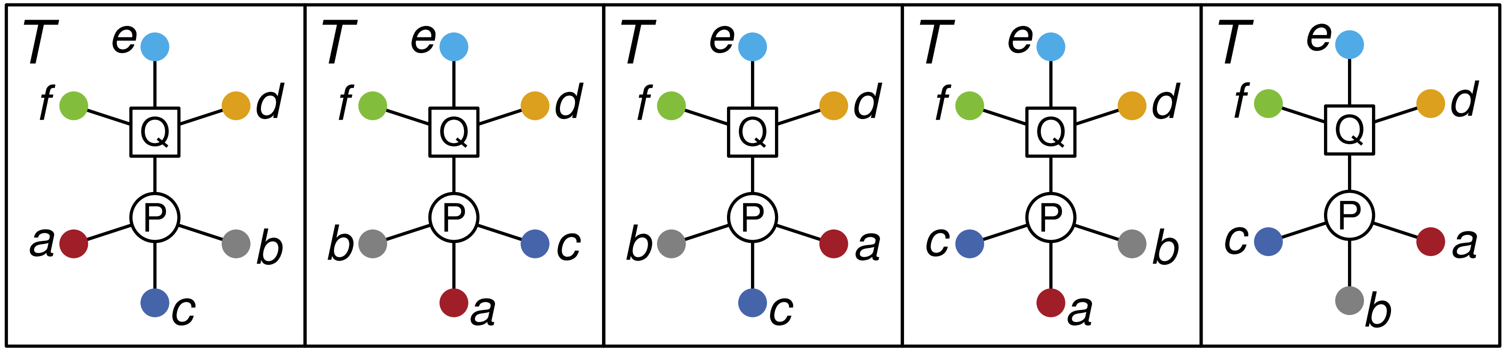

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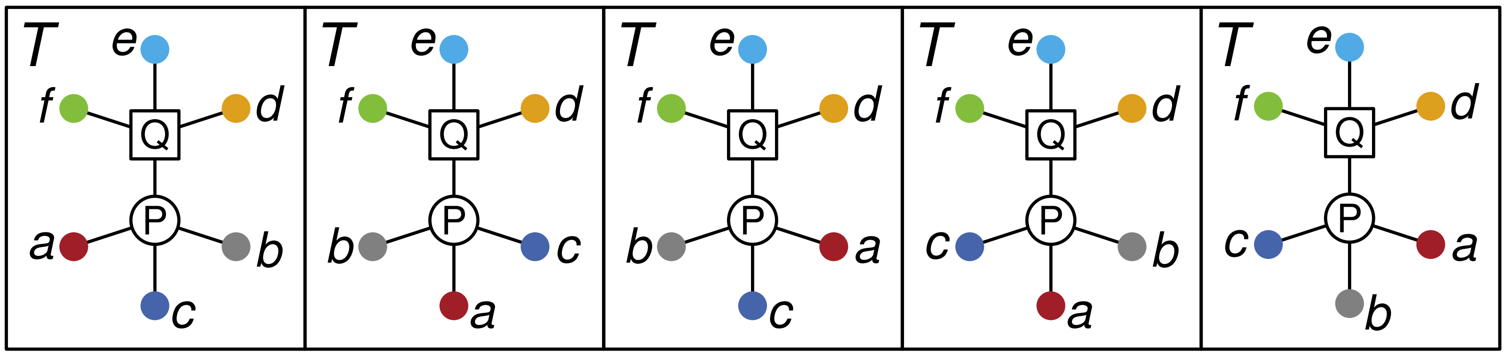

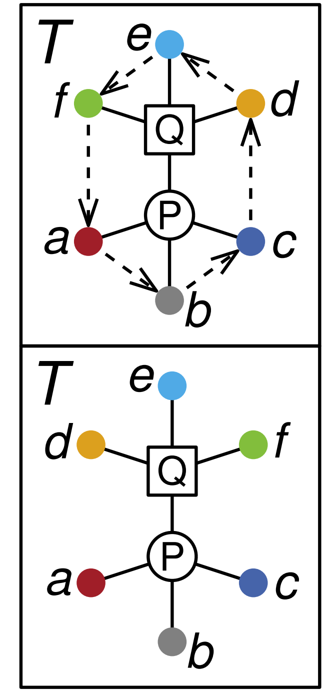

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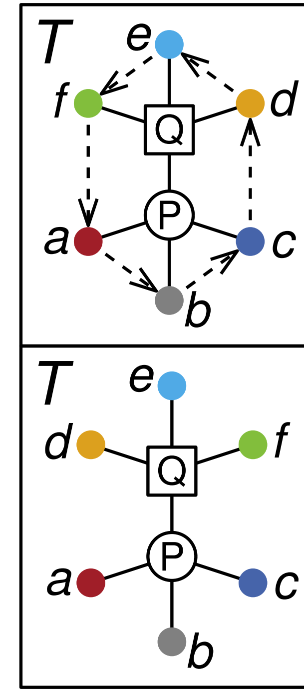

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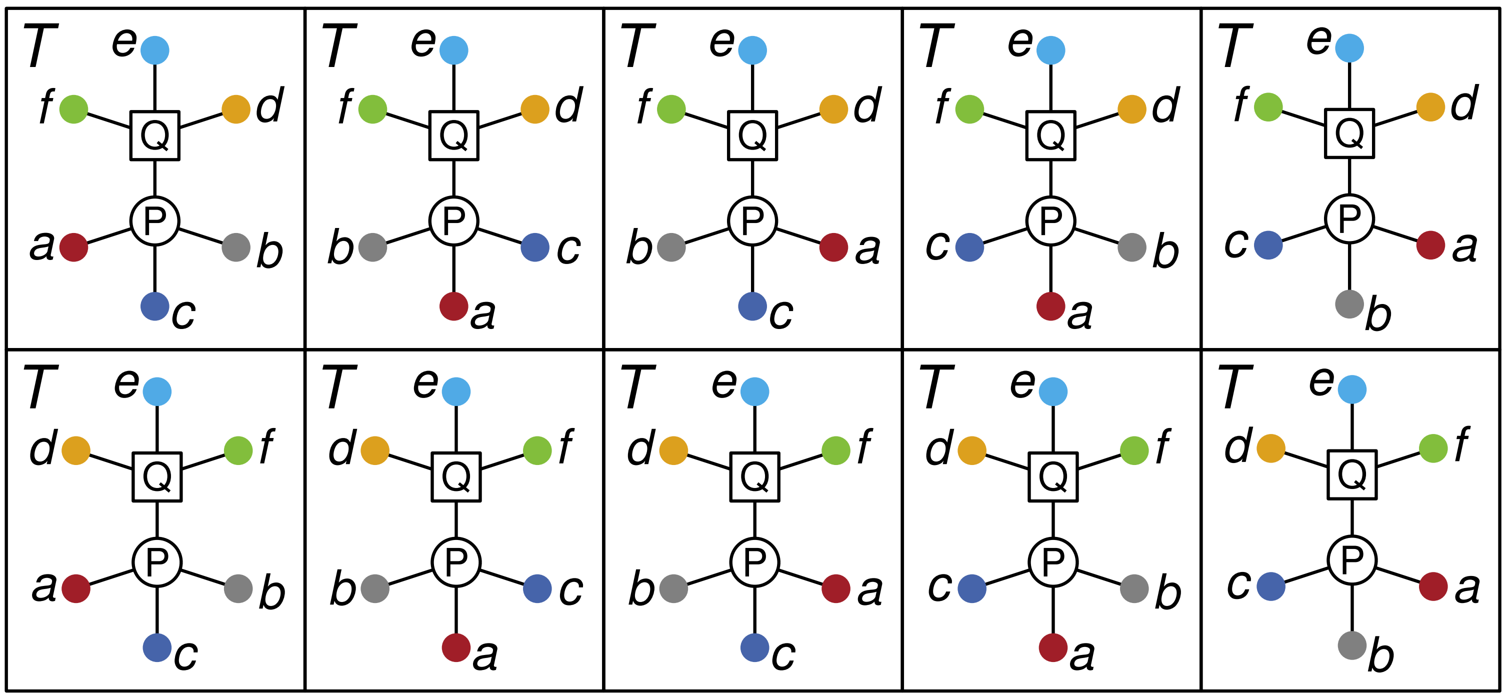

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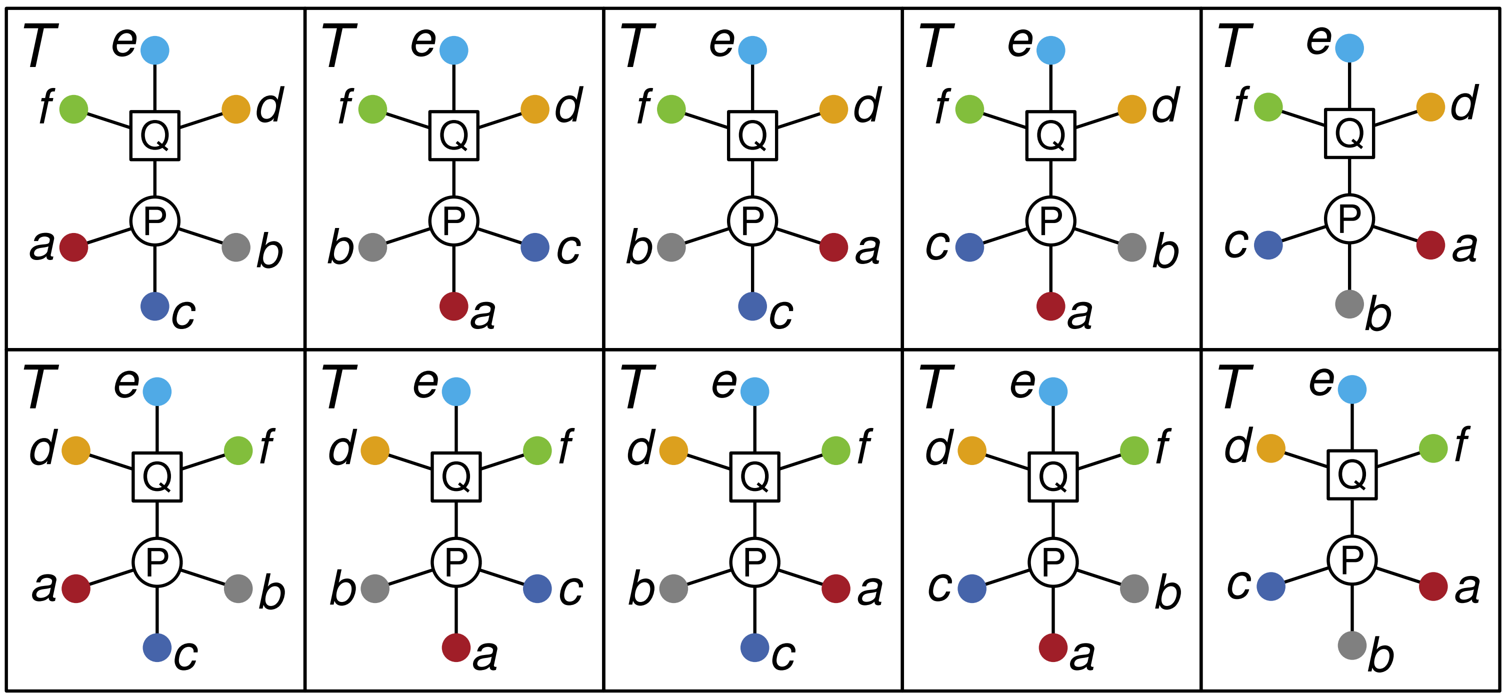

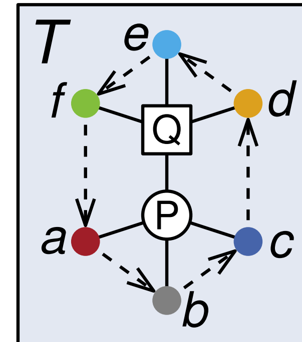

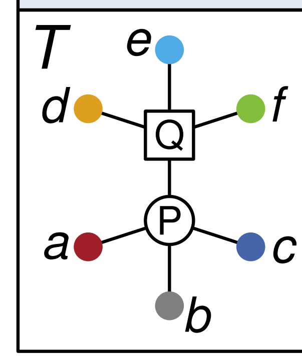

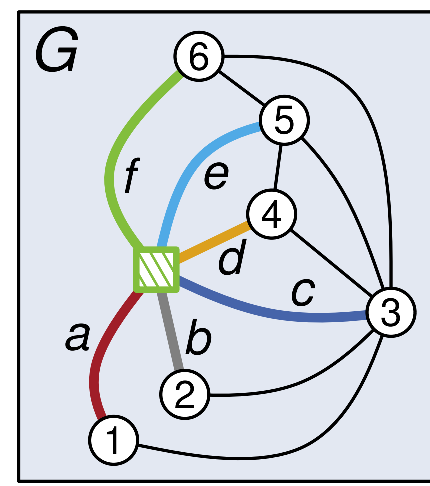

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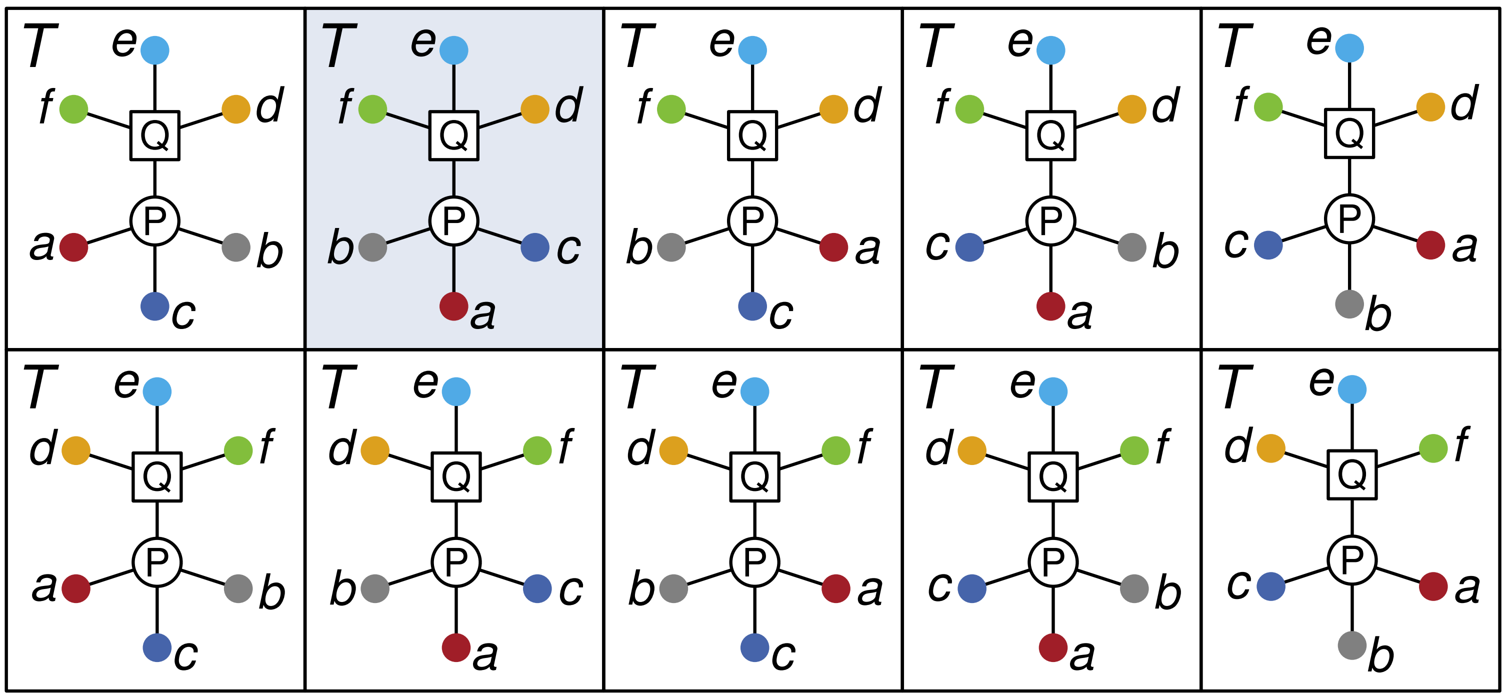

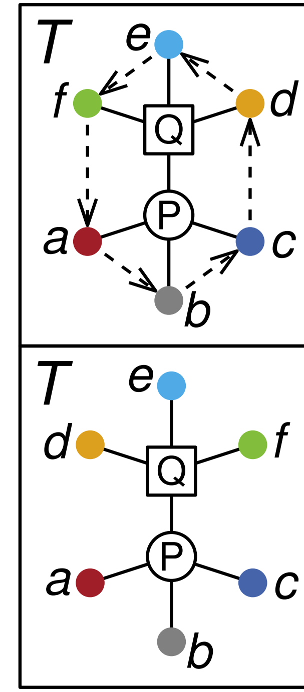

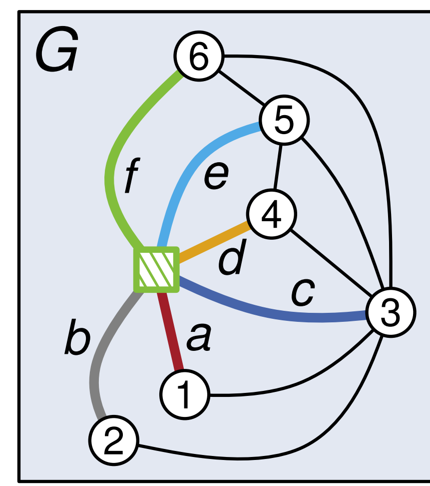

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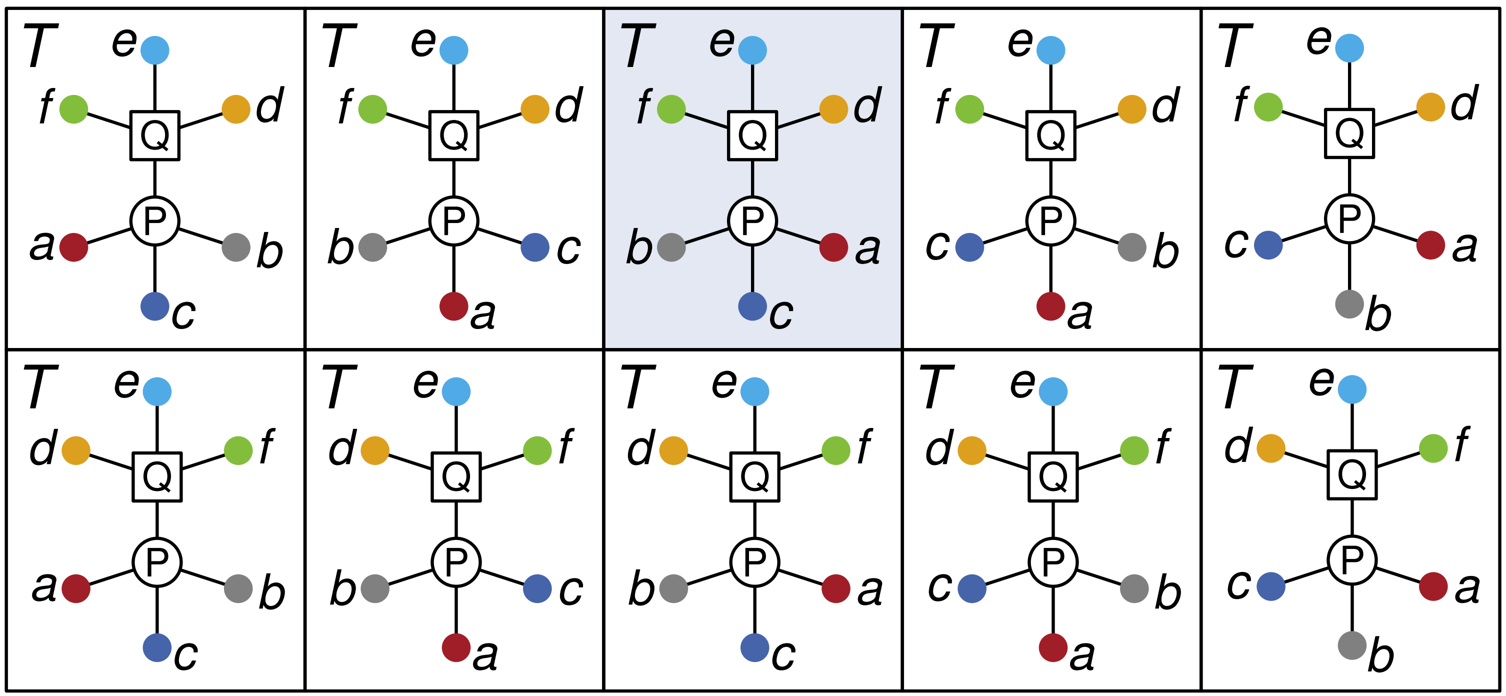

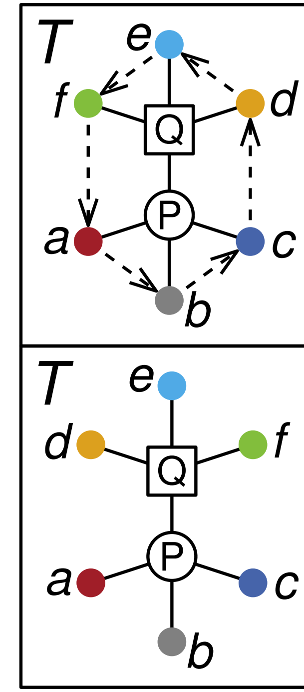

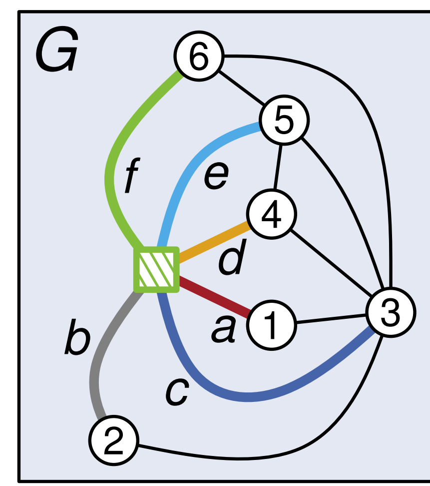

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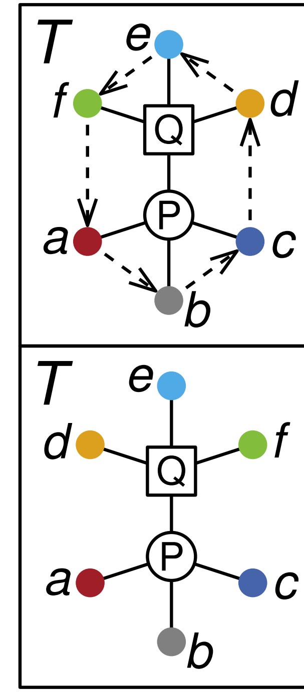

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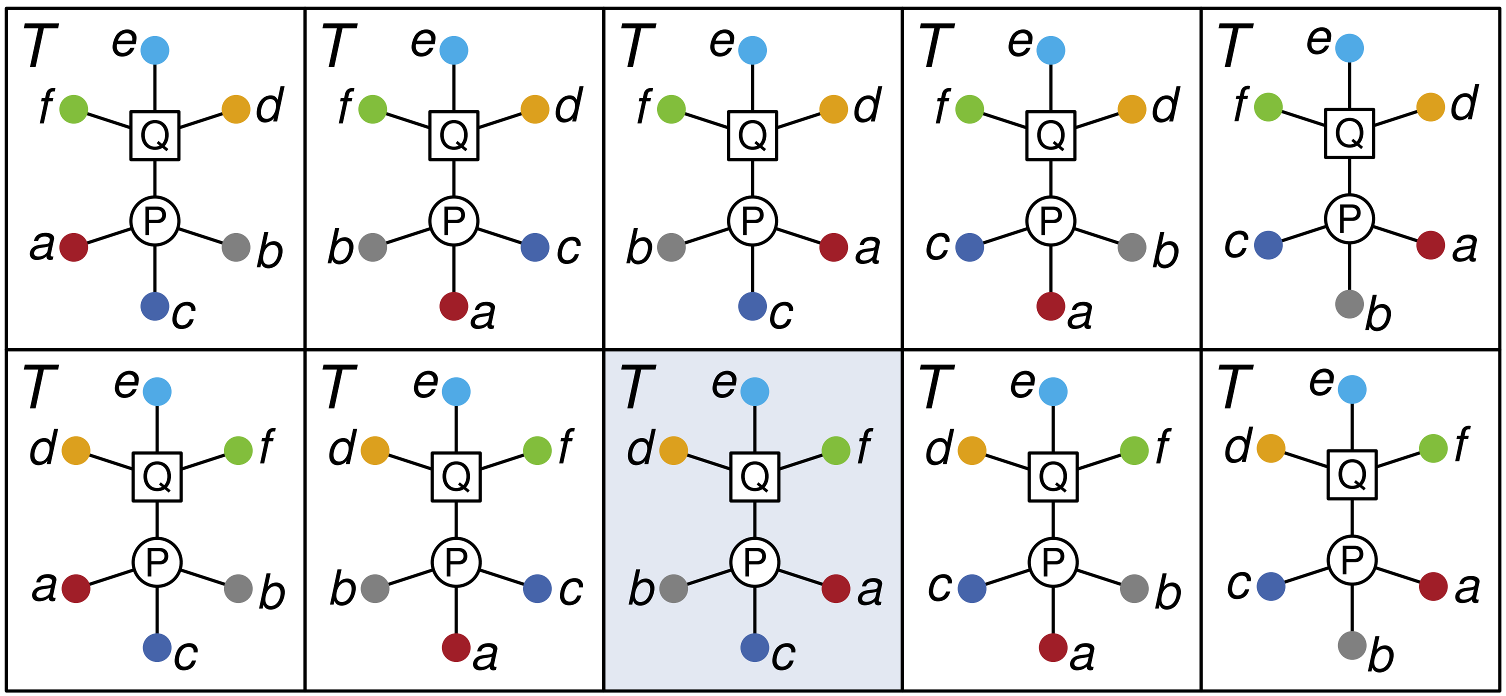

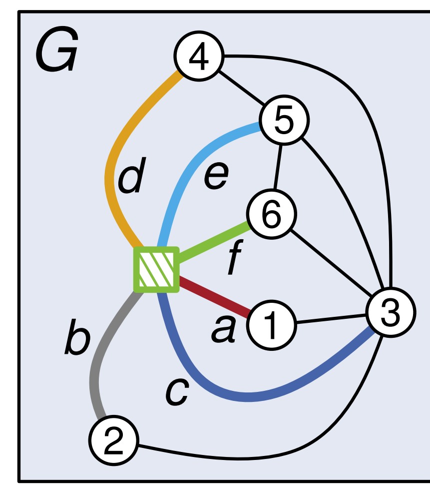

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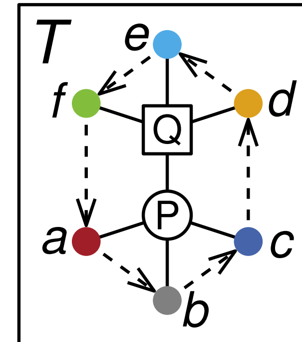
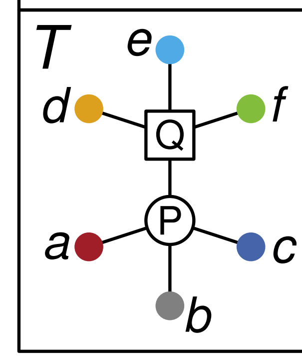
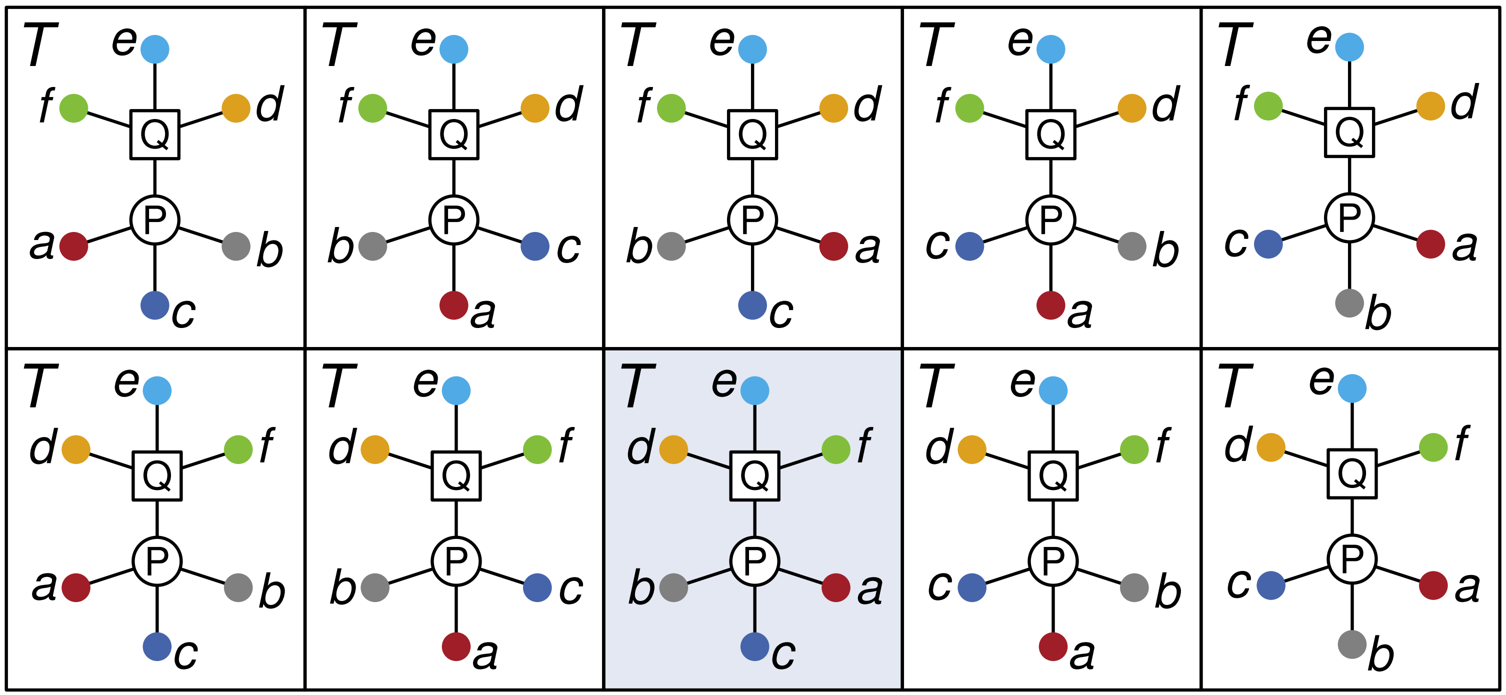
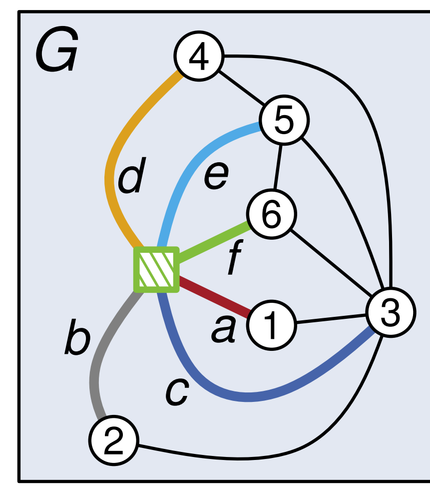




Every embedding induces a (cyclic) ordering on the leaves.

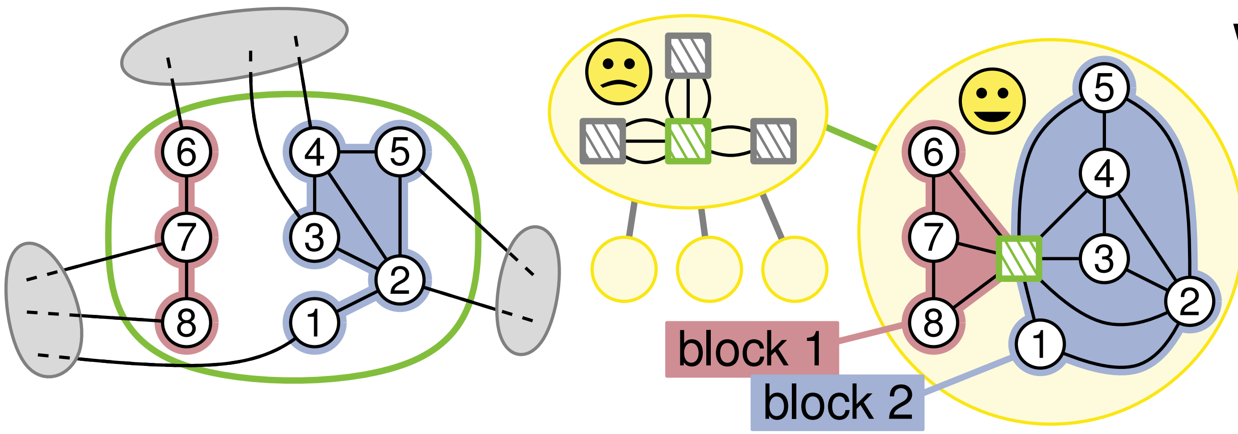




T represents the possible edge-orderings around ▨ in *G*.

Institute of Theoretical Informatics
Prof. Dr. Dorothea Wagner

Flat-Clustered Graphs – ~~Fixed Embedding~~



What is fixed?

- ~~embedding of *G*~~
- ~~edge-orderings of non-virtual vertices~~
- ~~embeddings of blocks~~

Which edge-orderings around does 's skeleton allow?

- blocks in 's skeleton partition the edges
- partitions must not alternate → **partition-constraint**
- order in each partition **is restricted by a PQ-tree** → **PQ-constraint**

partitioned PQ-constraint

> **Theorem**
> C-planarity for flat-clustered graphs is equivalent to **planarity with partitioned PQ-constraints**.