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On Self-Approaching and Increasing-Chord Drawings of 3-Connected Planar Graphs

Martin Nöllenburg, **Roman Prutkin**, and Ignaz Rutter

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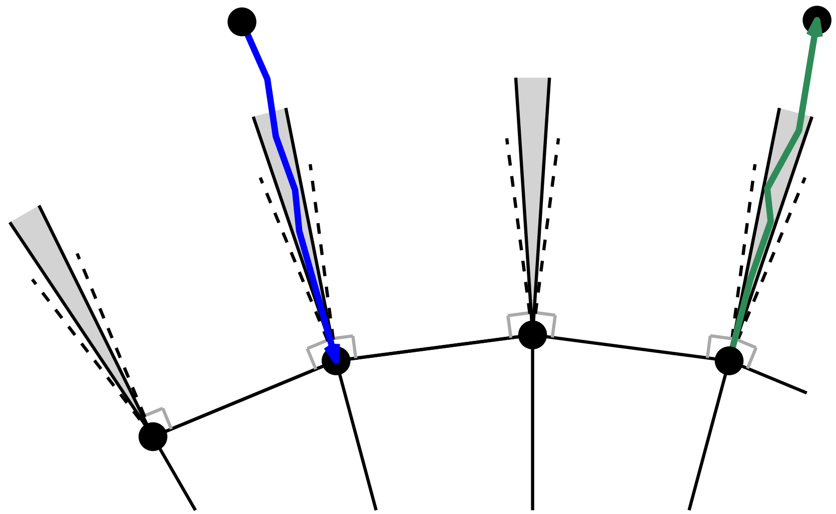
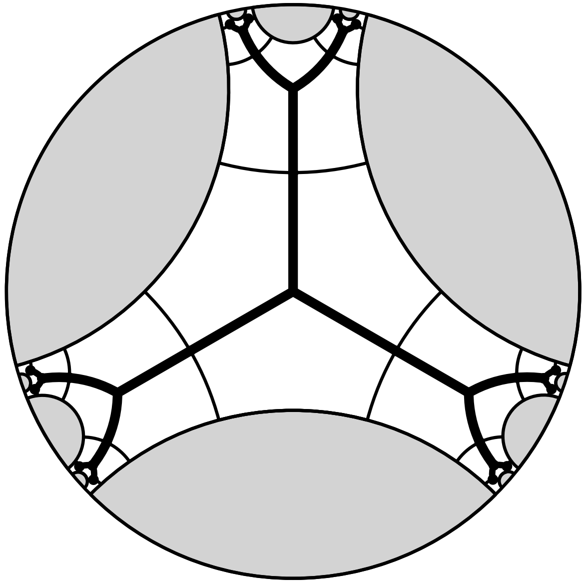
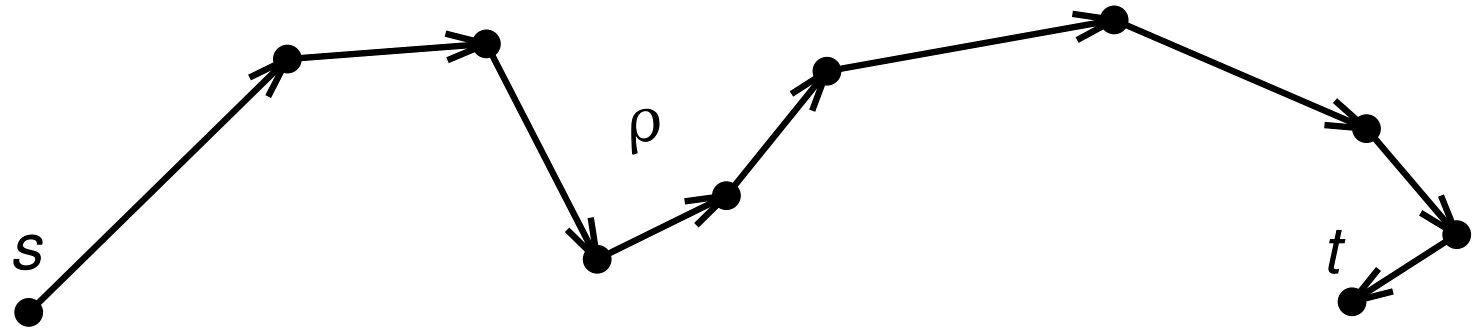




Drawings with Geodesic-Path Tendency

straight-line drawings of $G = (V, E)$; for each pair $s, t \in V$ exists st path ρ, along which we get closer to t



Empirical findings

such drawings make path-finding tasks easier

[Huang et al. 2009], [Purchase et al. 2013]

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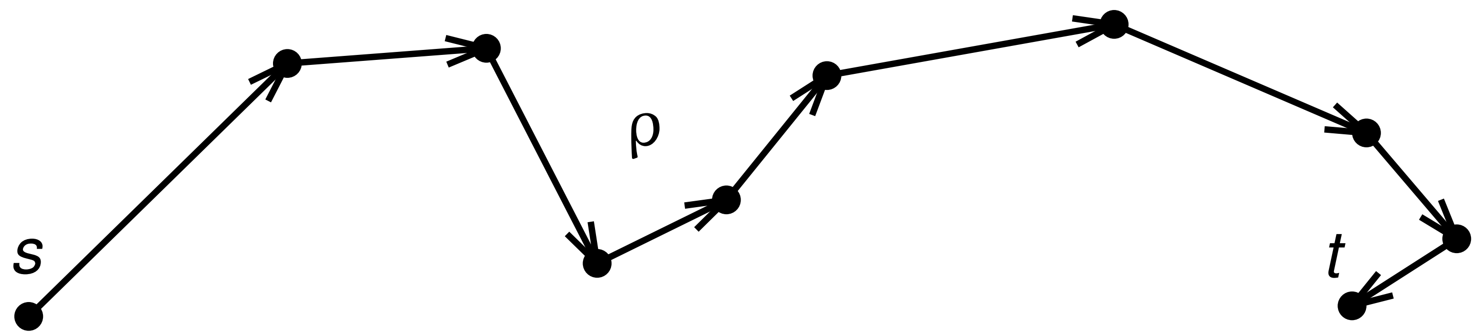


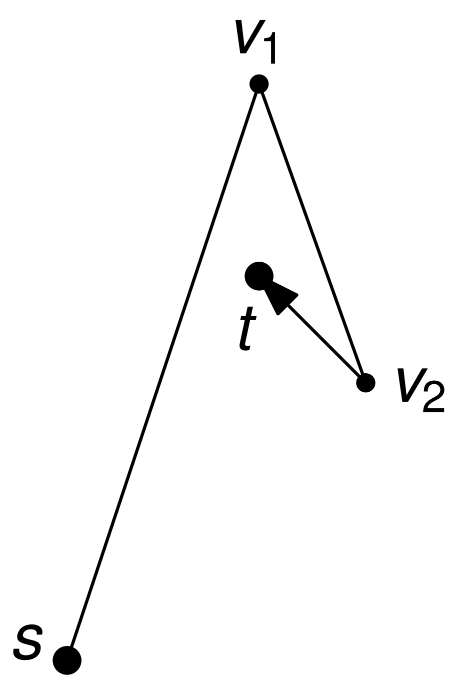
possible interpretations of closer

- **greedy:** get closer on vertices
- **self-approaching:** . . . on all intermediate points
- **increasing chords:** self-approaching in both directions
- **monotone:** closer regarding projection on some line
- **strongly monotone:** . . . regarding projection on line st

Greedy Embeddings (GE) [Rao et al. 2003]

greedy path exists between each pair $s, t \in V$

- path $\rho = (v_1, v_2, \ldots, t)$ greedy if $|v_{i+1}t| < |v_i t|$ for all i
- motivated by local routing in wireless sensor networks



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Related Work

3-conn. planar graphs have GE in $\mathbb{R}^2$
[Papadimitriou, Ratajczak 2005], [Leighton, Moitra 2010], [Angelini et al. 2010]

virtual coordinates with $O(\log n)$ bits in $\mathbb{H}^2$ and $\mathbb{R}^2$
[Eppstein, Goodrich 2008], [Goodrich, Strash 2009]

every tree has GE in hyperbolic plane $\mathbb{H}^2$
[Kleinberg, 2007]

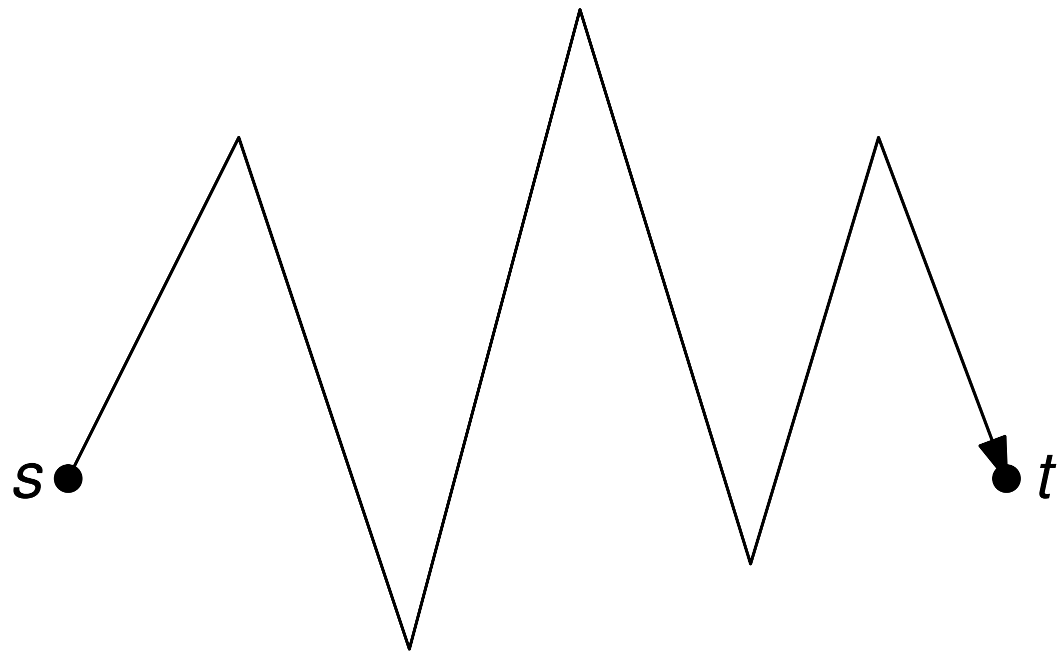
characterization of trees with GE in $\mathbb{R}^2$
[Nöllenburg, Prutkin 2013]

open: planar GE of 3-conn. graphs?

Monotone Drawings [Angelini et al. 2012]

monotone path exists between each pair $s, t \in V$

- path monotone if its curve monotone
- strongly monotone: monotonicity direction $\vec{st}$



strongly monotone path

Monotone Drawings [Angelini et al. 2012]

monotone path exists between each pair $s, t \in V$

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- strongly monotone: monotonicity direction $\vec{st}$

biconnected planar graphs admit monotone drawings

plane graphs admit monotone drawings with few bends

open: strongly monotone planar drawings of triangulations

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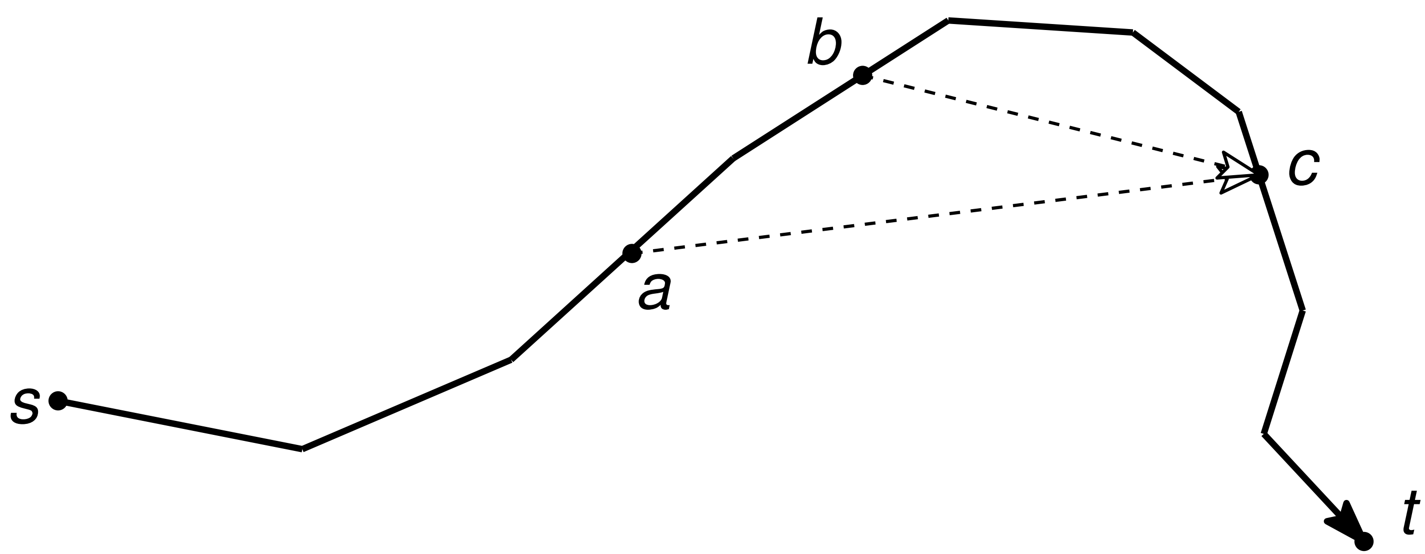
Self-Approaching Drawings

self-approaching curve: for any a, b, c along the curve, $|bc| \leq |ac|$
equivalent: no normal crosses the curve later on



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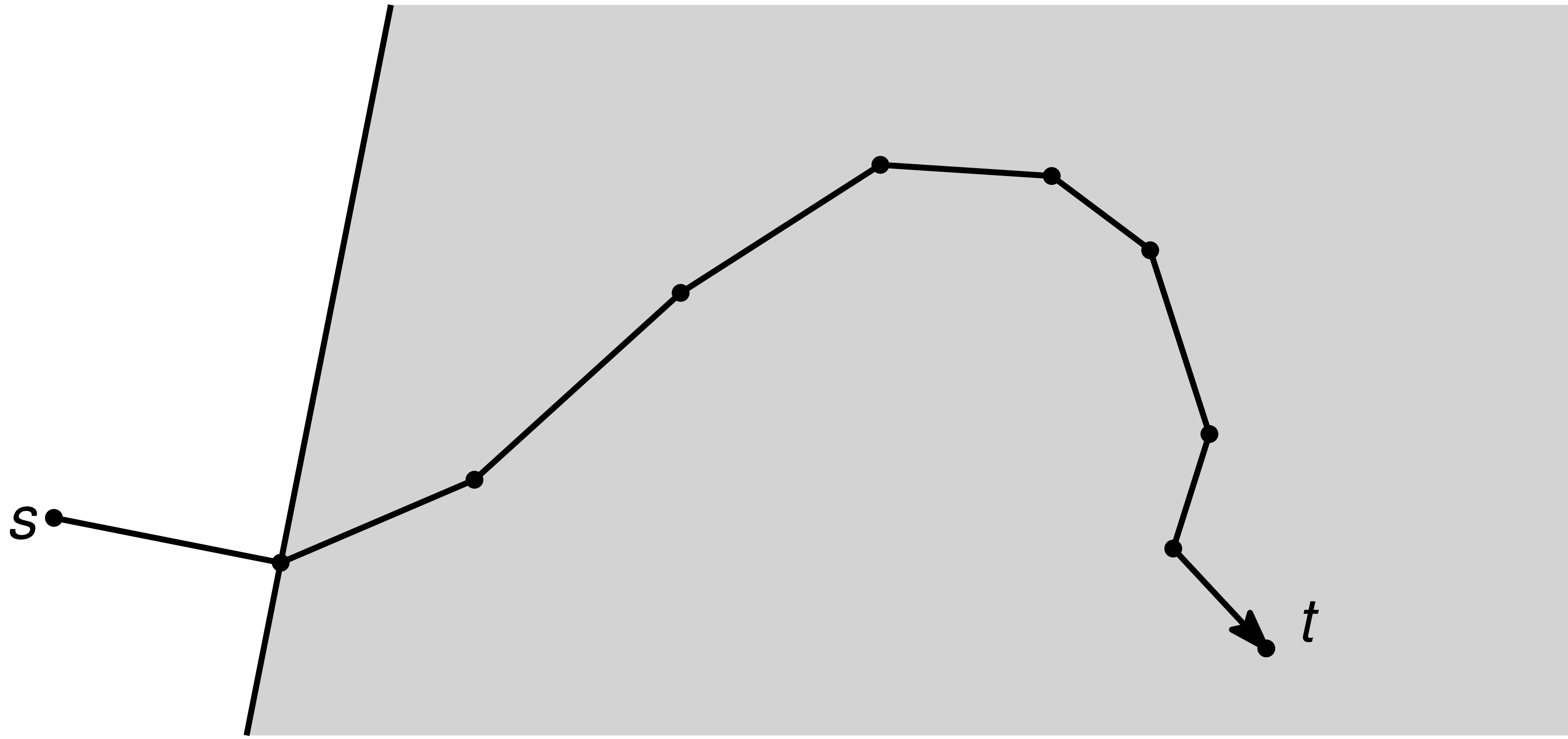

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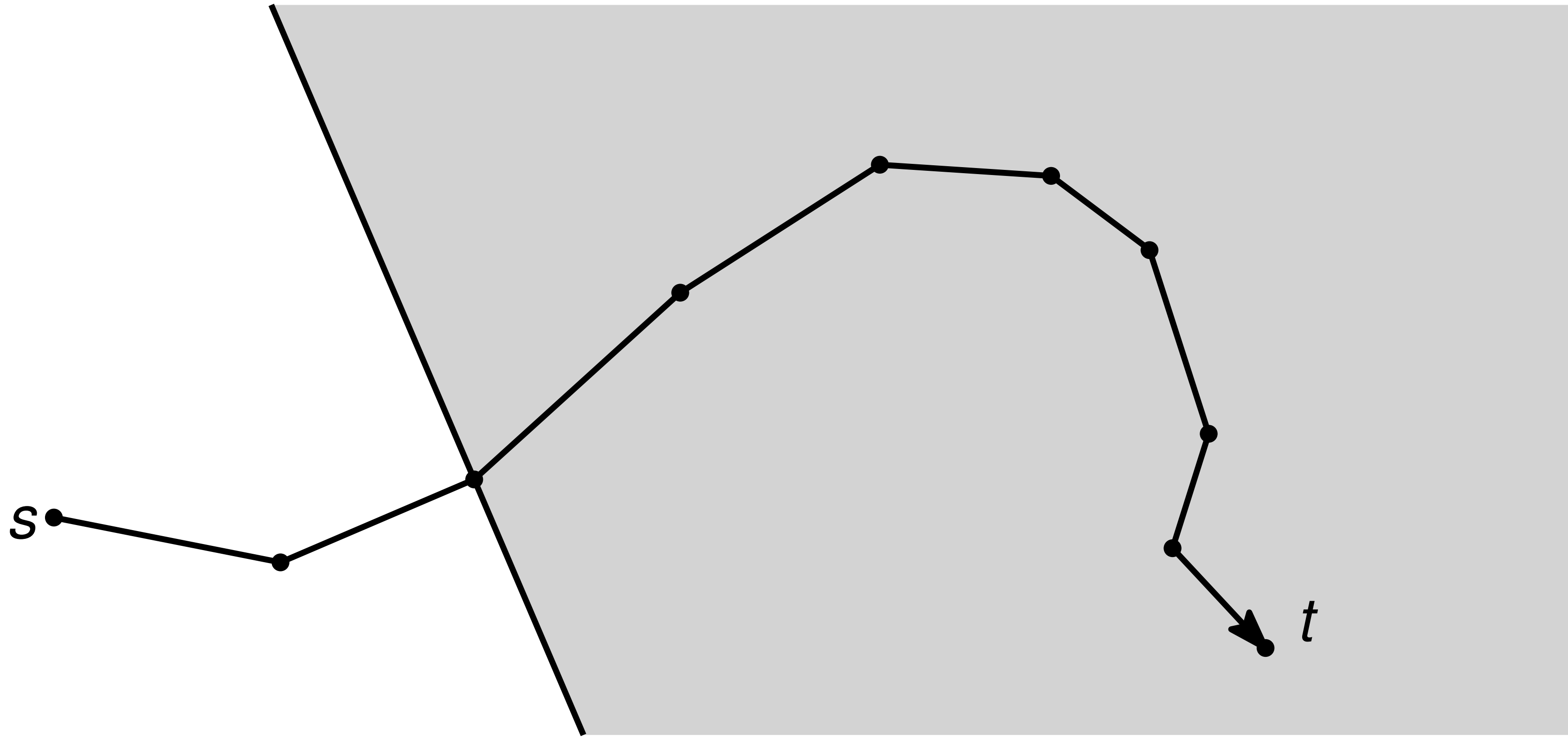

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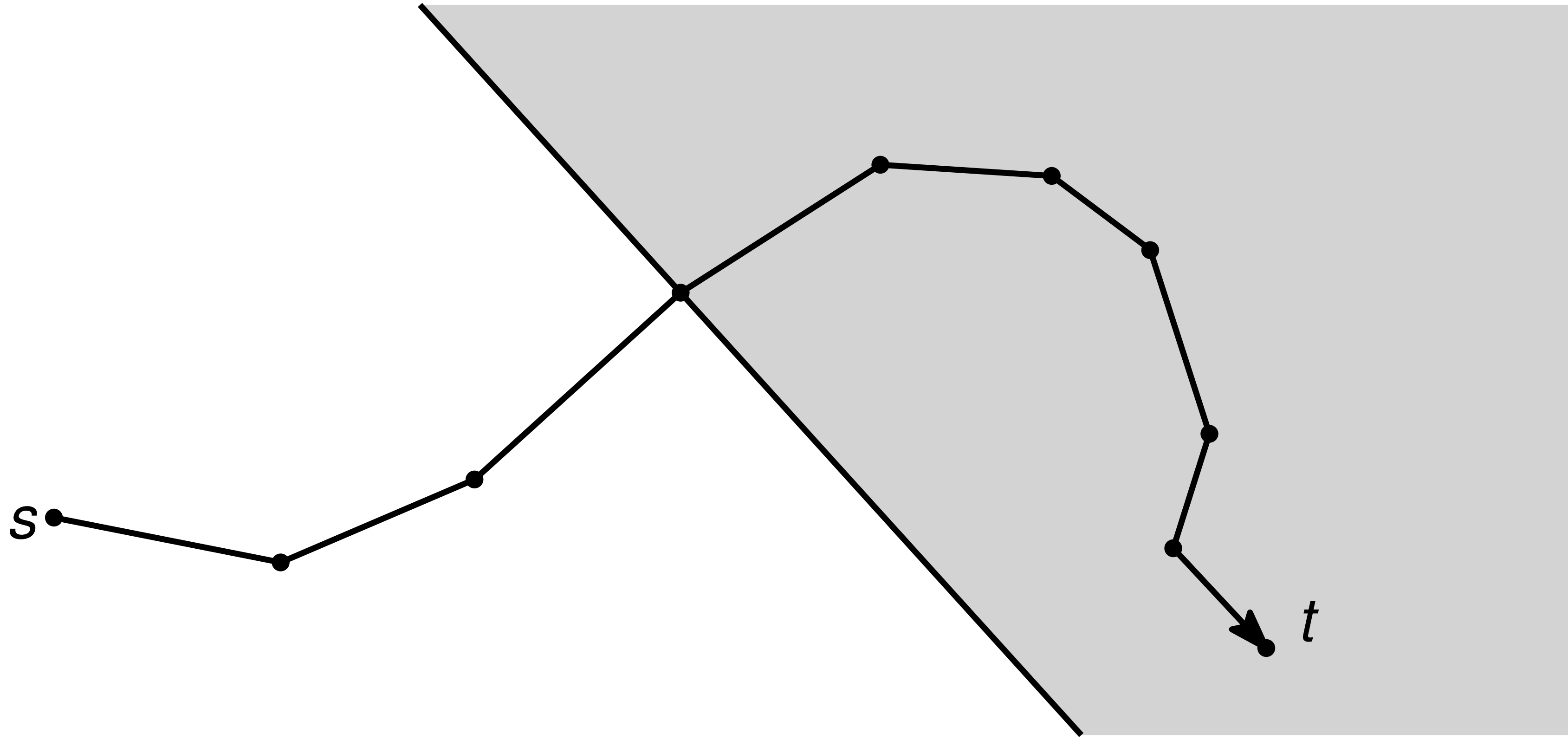

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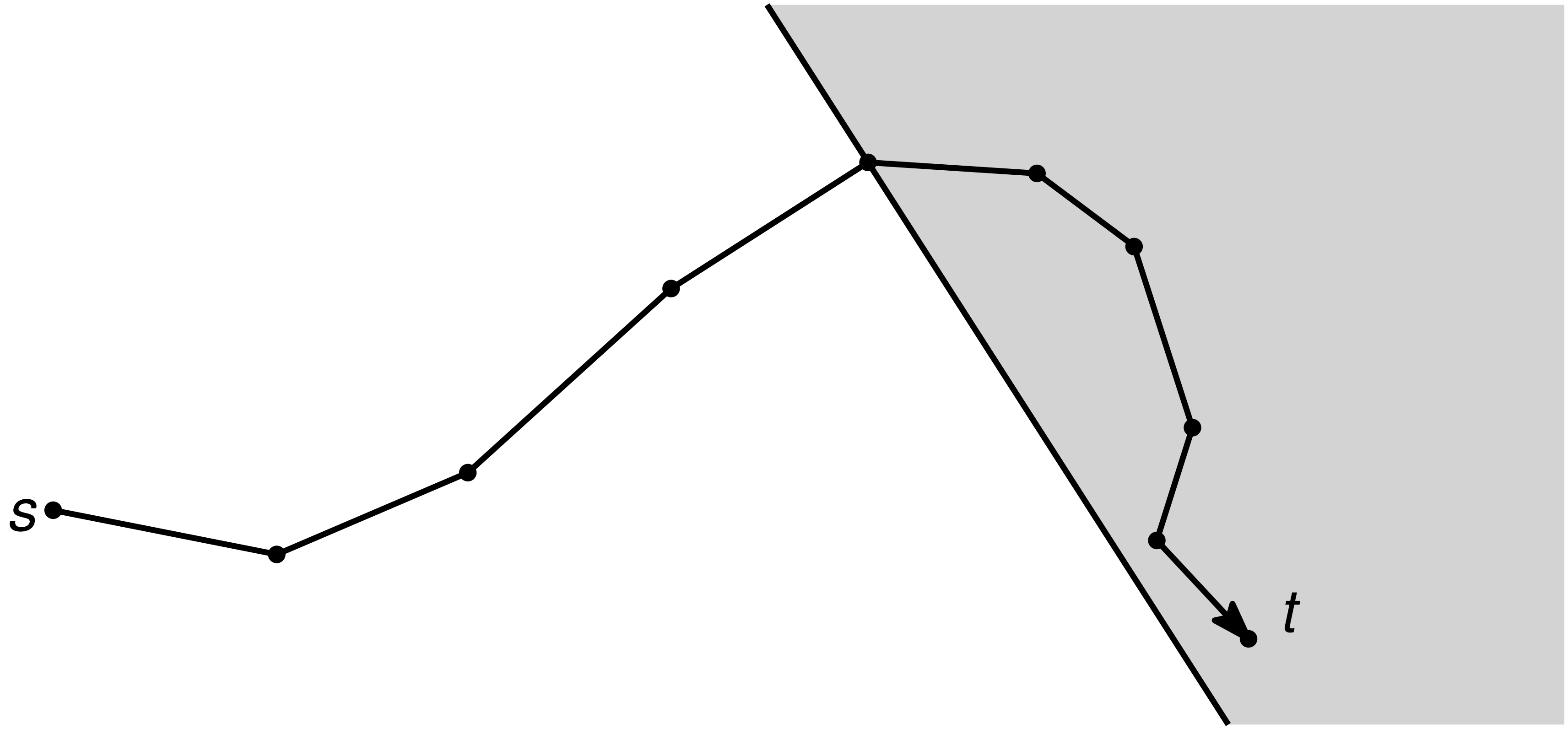

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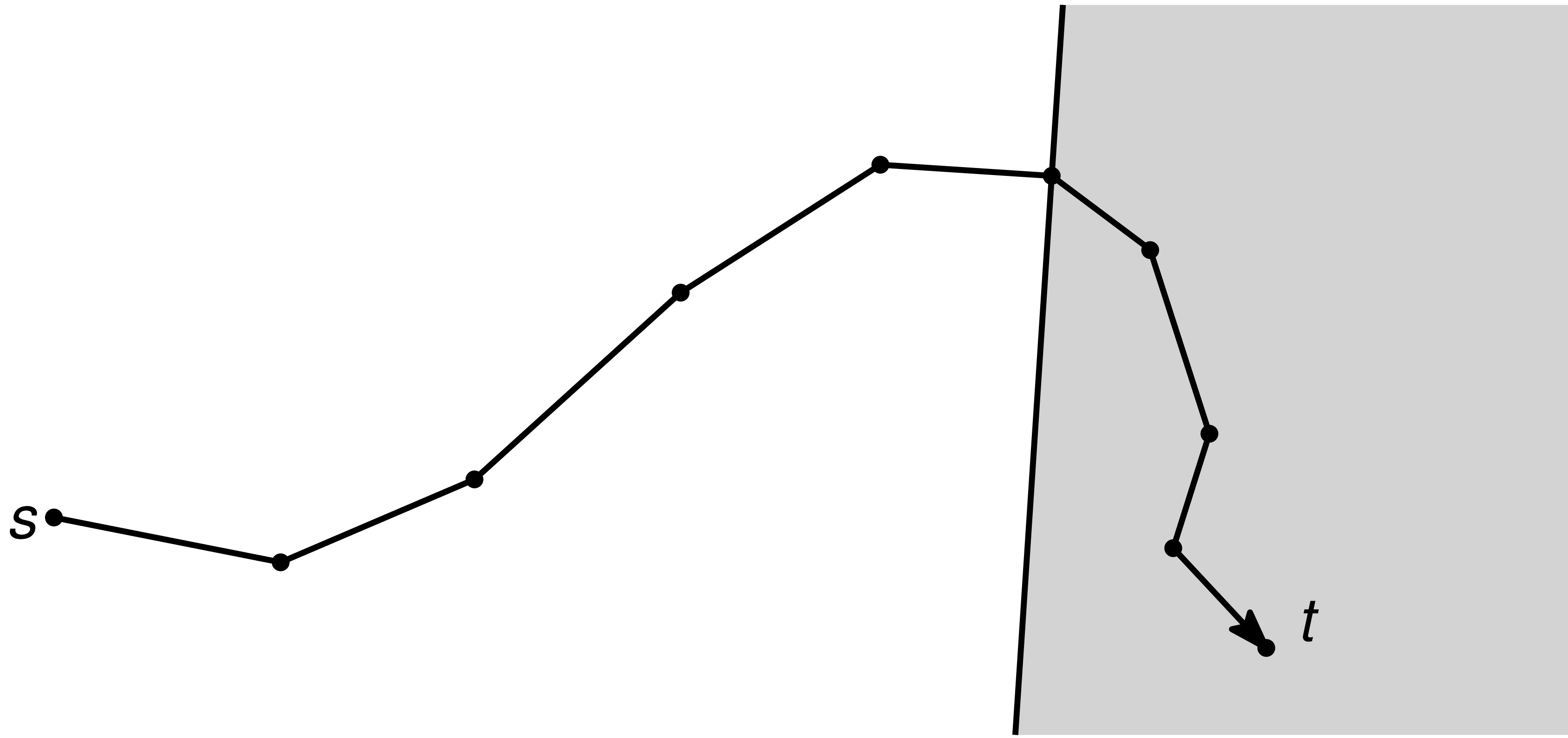

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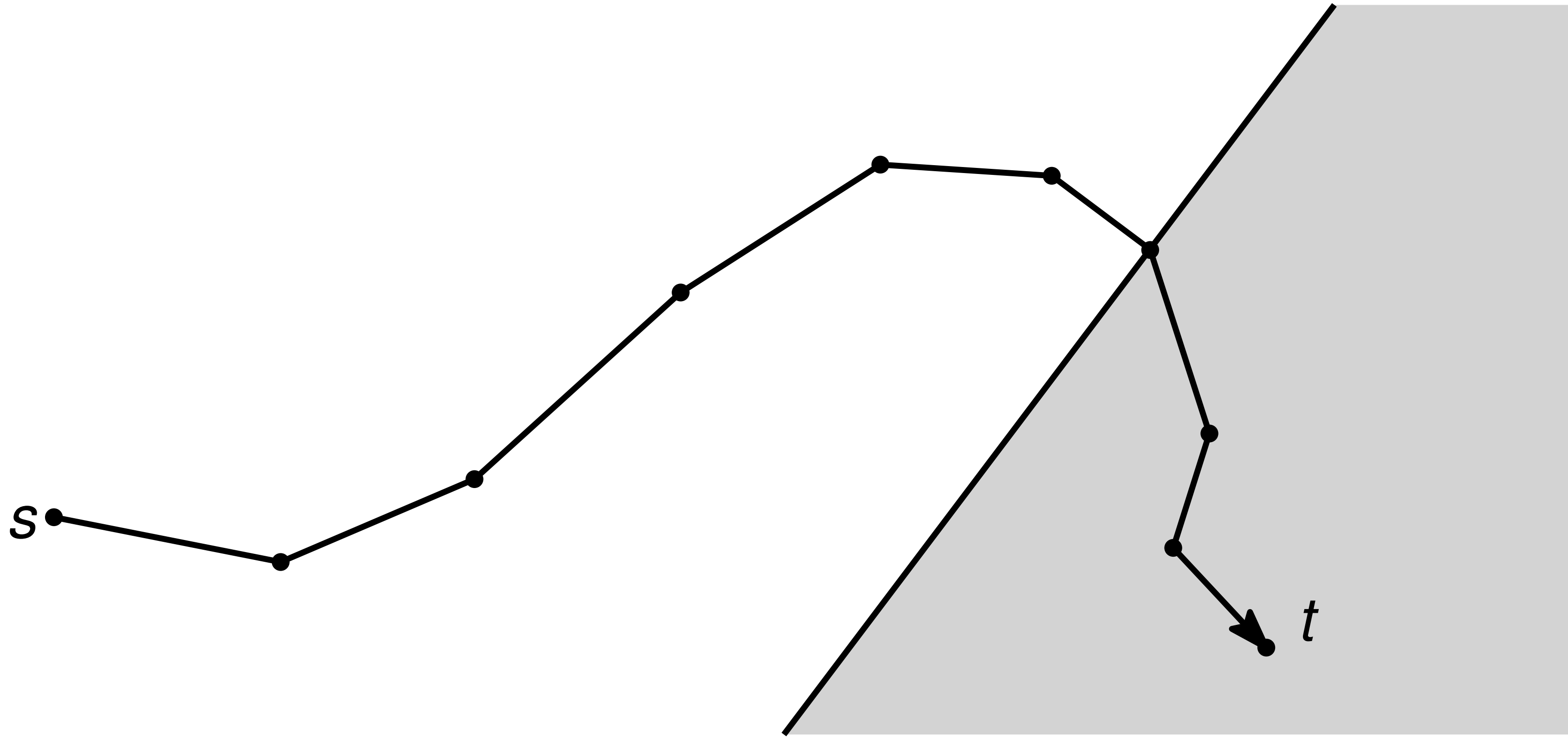

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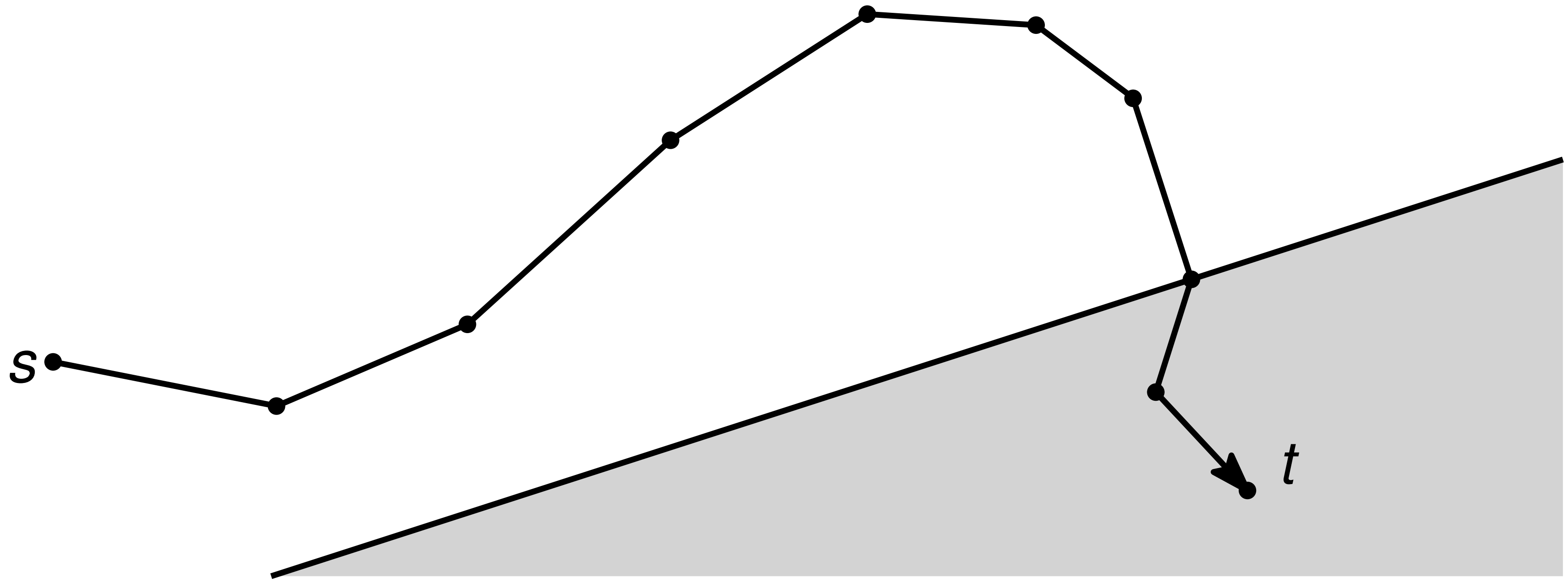

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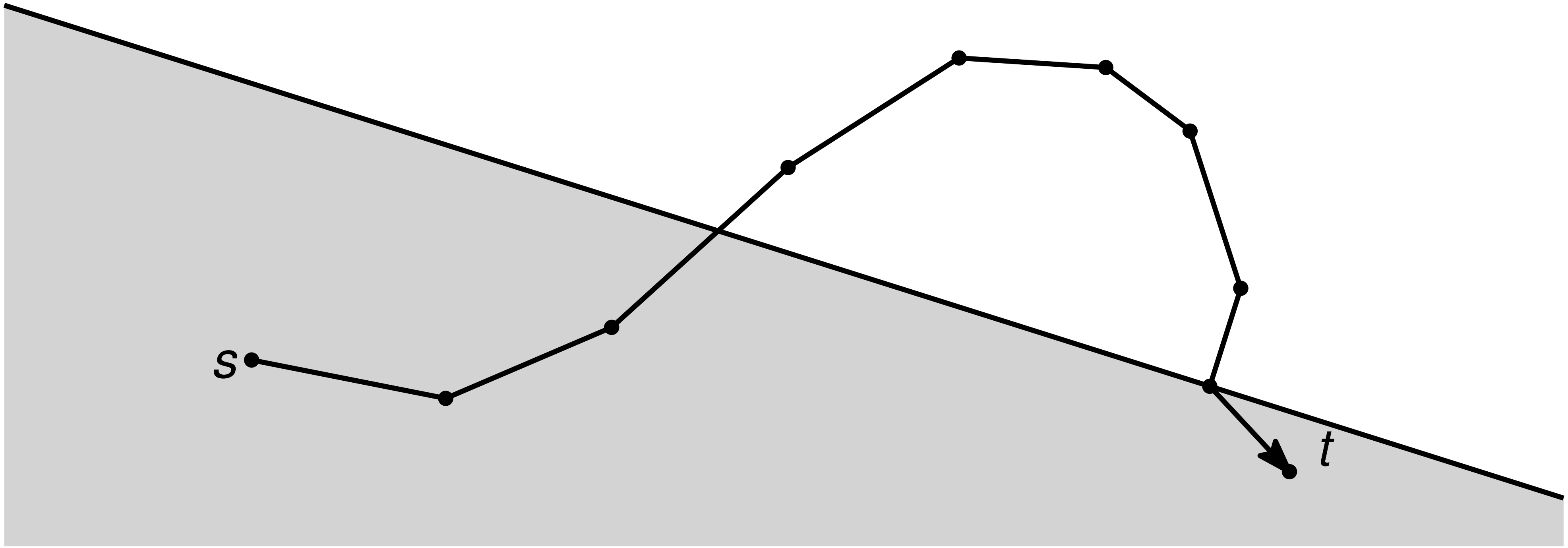

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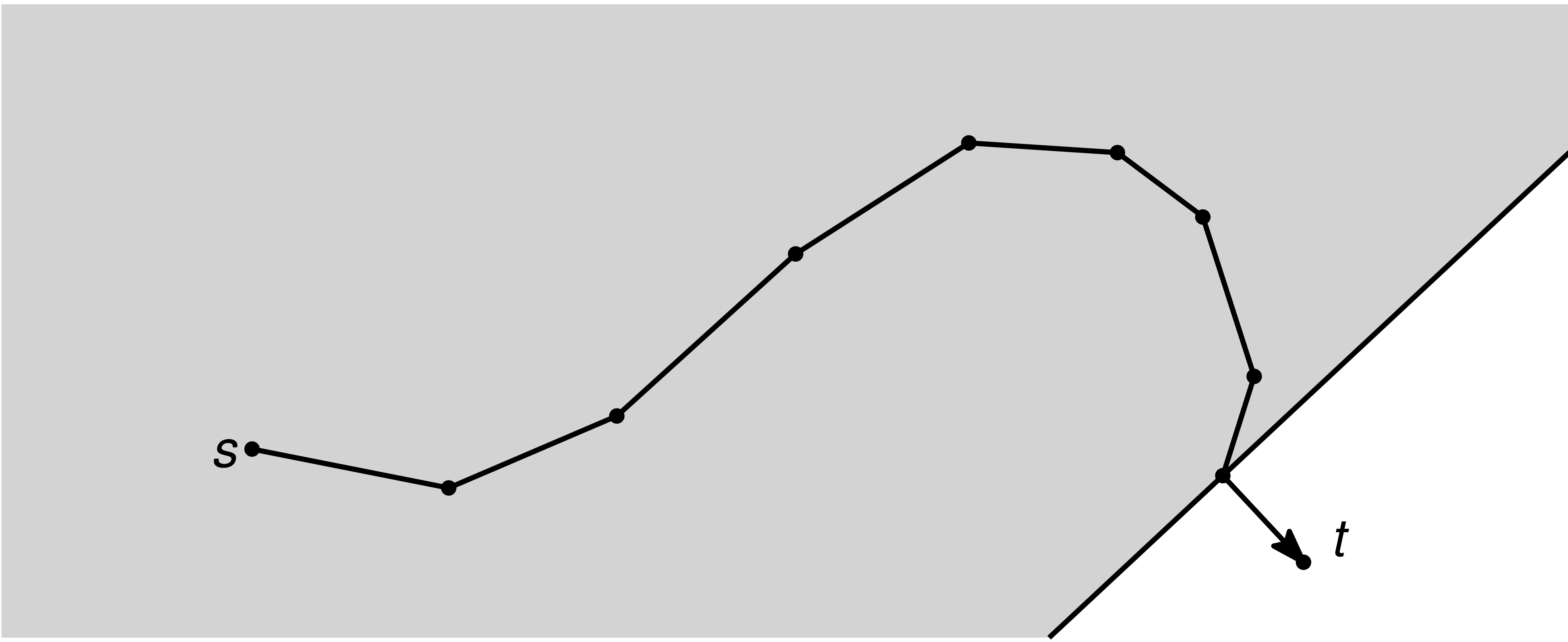

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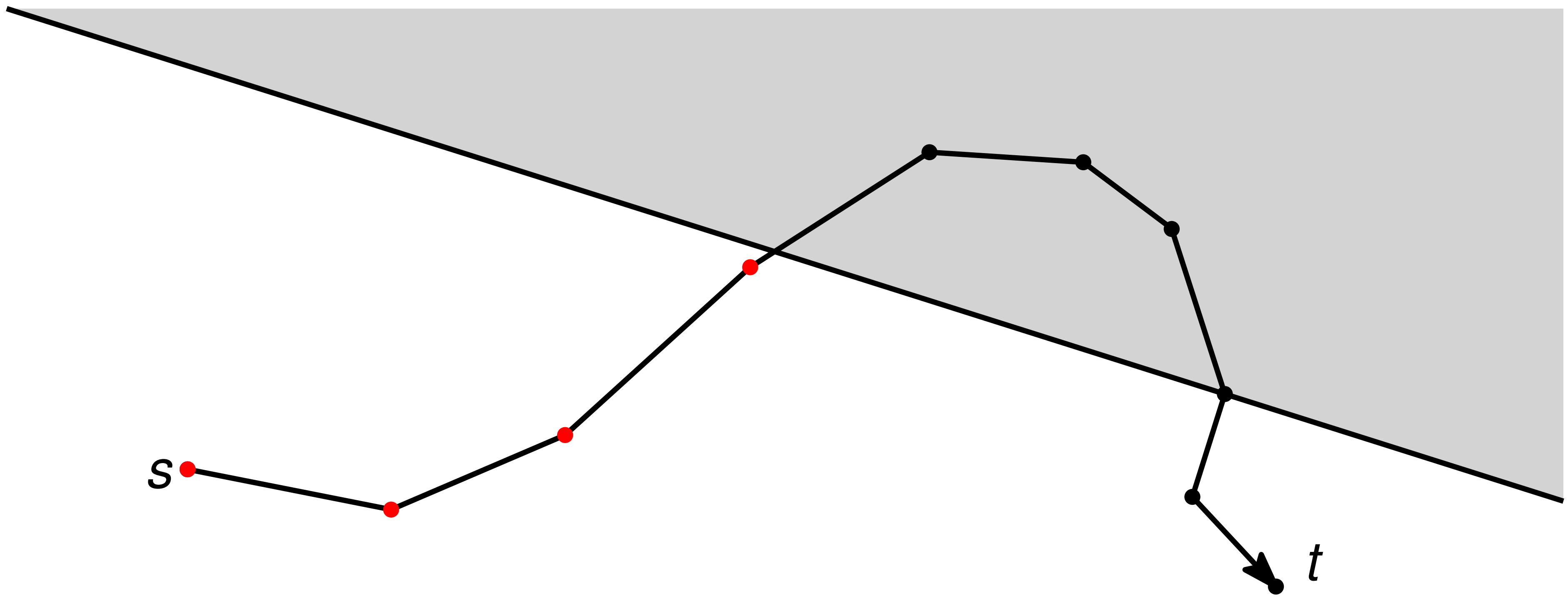

Self-Approaching Drawings

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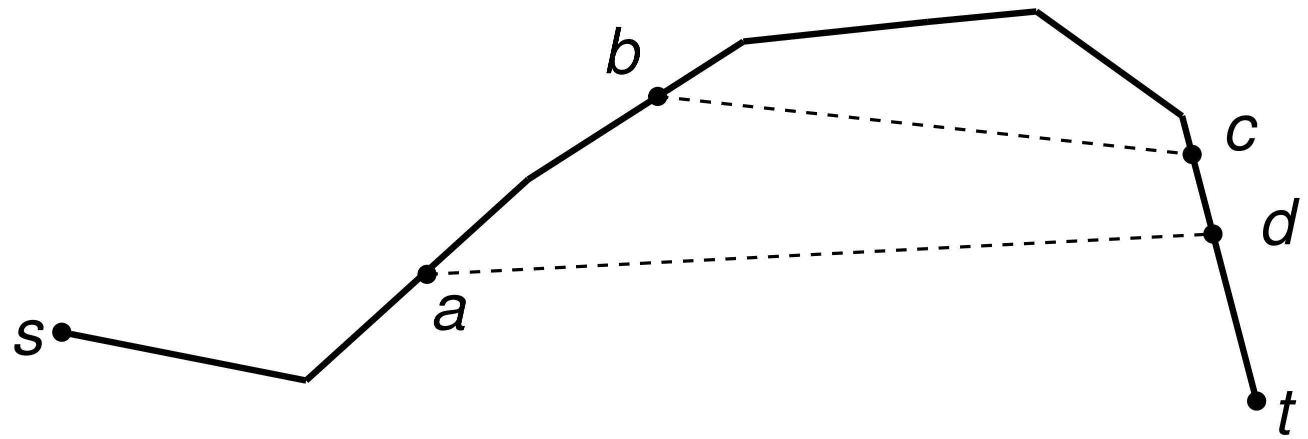
self-approaching curve: for any a, b, c along the curve, $|bc| \leq |ac|$

equivalent: no normal crosses the curve later on

increasing chords: for a, b, c, d along the curve, $|bc| \leq |ad|$

equivalent: self-approaching in both directions



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Related Work

paths have bounded detour

length $\leq 5.33|st|$ for self-approaching, [Icking et al. 1995]

$\leq 2.09|st|$ for increasing chords [Rote 1994]

characterization of trees with self-approaching drawing [Alamdari et al. 2013]

open: 3-connected planar?

planar self-approaching drawings?

Contributions

Every triangulation has an increasing-chord drawing.

- has spanning downward-triangulated binary cactus [Angelini et al. 2010]
- such cactus has increasing-chord drawing

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Some binary cactuses have no self-approaching drawing.

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- it worked for greedy drawings

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Planar 3-trees have planar increasing-chord drawings.

first construction for str. monotone/greedy drawings of pl. 3-trees

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Planar 3-trees have planar increasing-chord drawings.

first construction for str. monotone/greedy drawings of pl. 3-trees

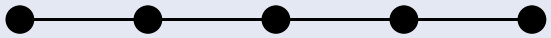
Hyperbolic plane is more powerful for increasing-chord drawings.

- characterize drawable trees
- every 3-connected planar graph is drawable

Recall: GE of 3-connected Planar Graphs

drawing spanner greedily

G has Hamiltonian path: easy



3-conn. planar are “almost” Hamiltonian: contain closed 2-walk

have spanning binary cactus

Recall: GE of 3-connected Planar Graphs

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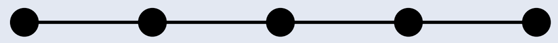
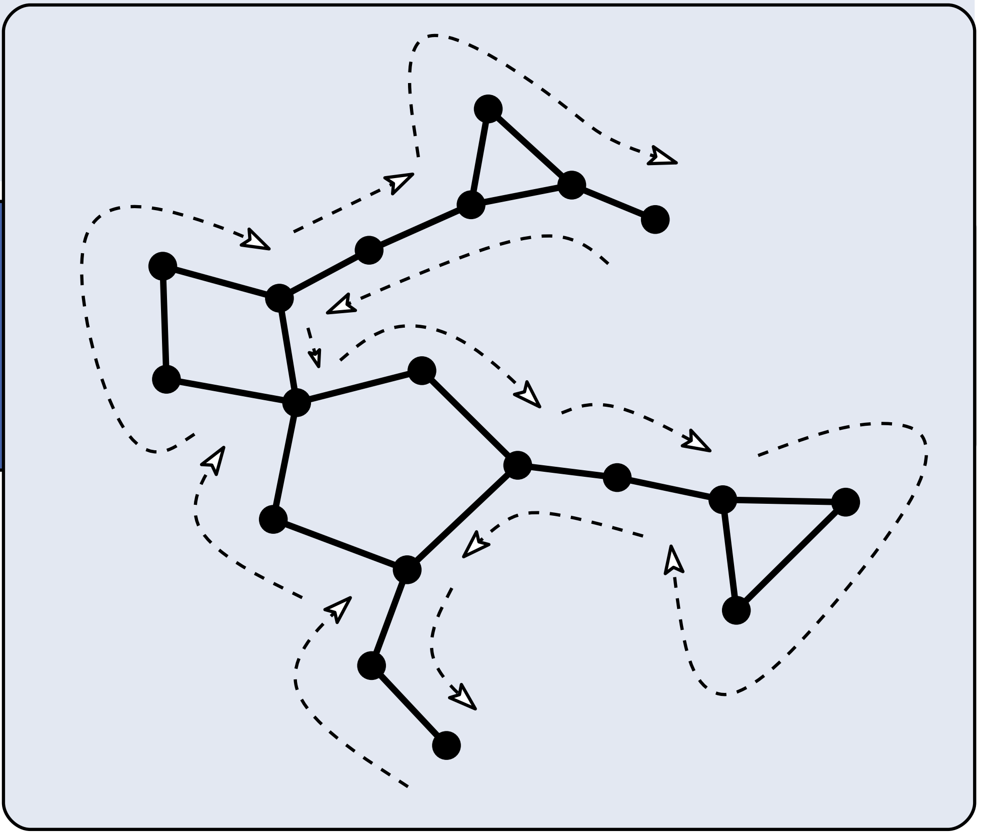


3-conn. planar are "almost" Hamiltonian: contain closed 2-walk

have spanning binary cactus

binary cactus

- each edge part of ≤ 1 cycle
- each vertex part of ≤ 2 cycles



GE of a Binary Cactus

[Leighton, Moitra 2008]
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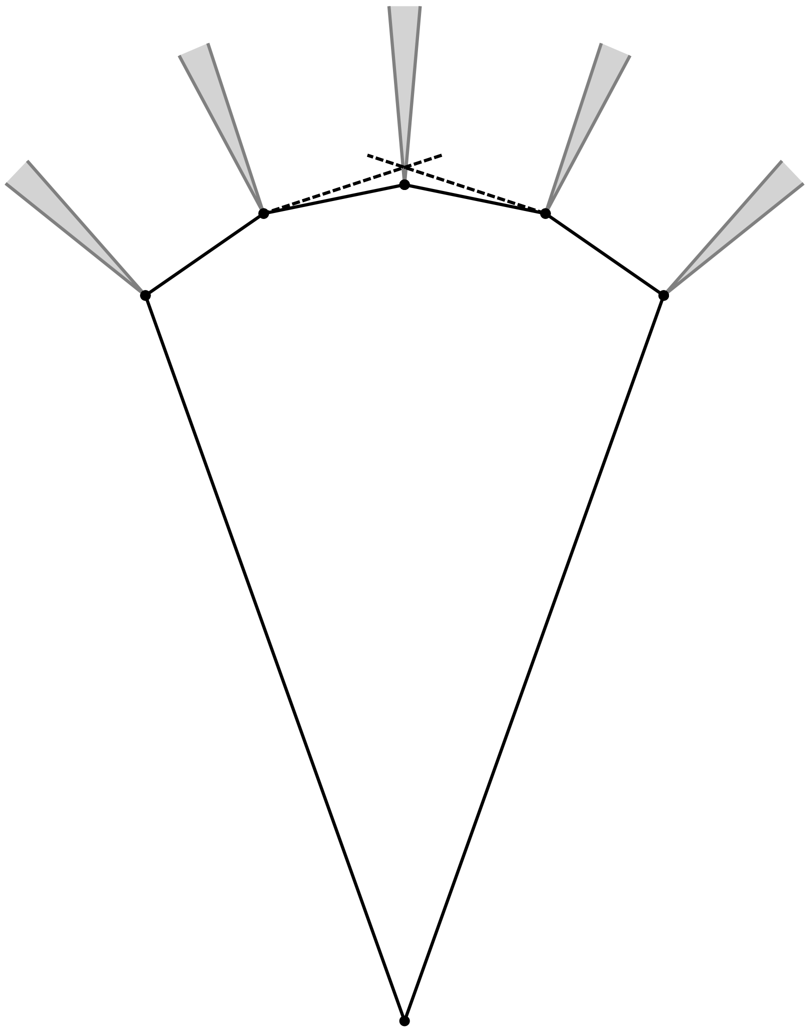

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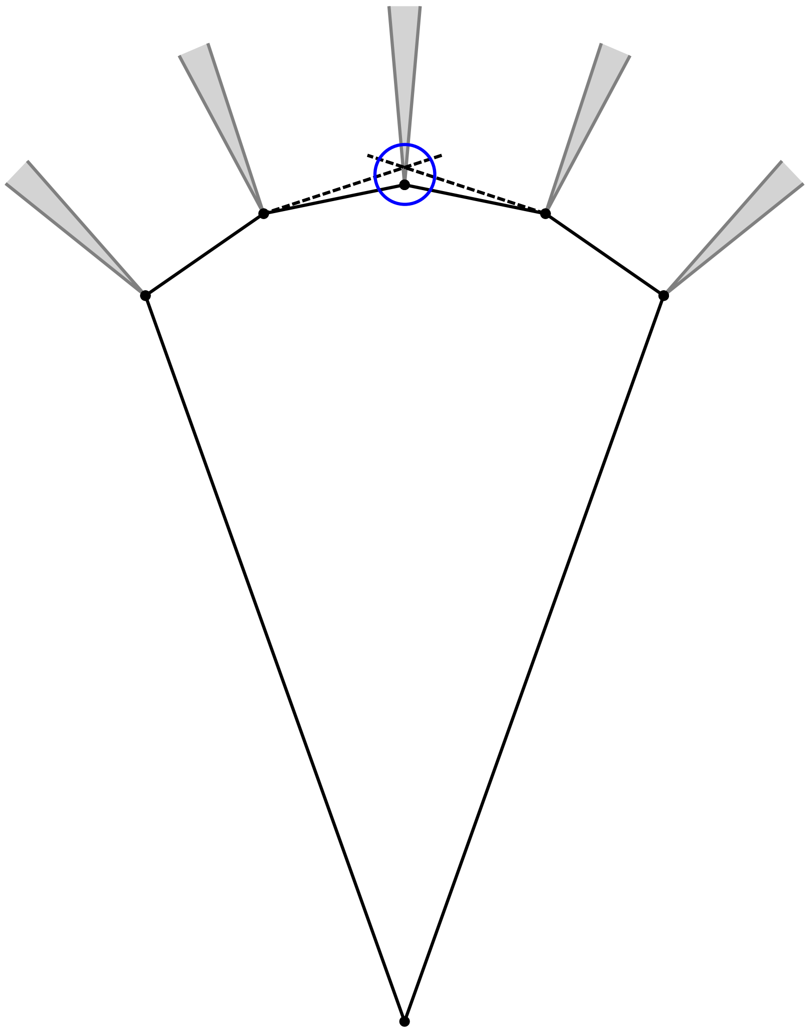

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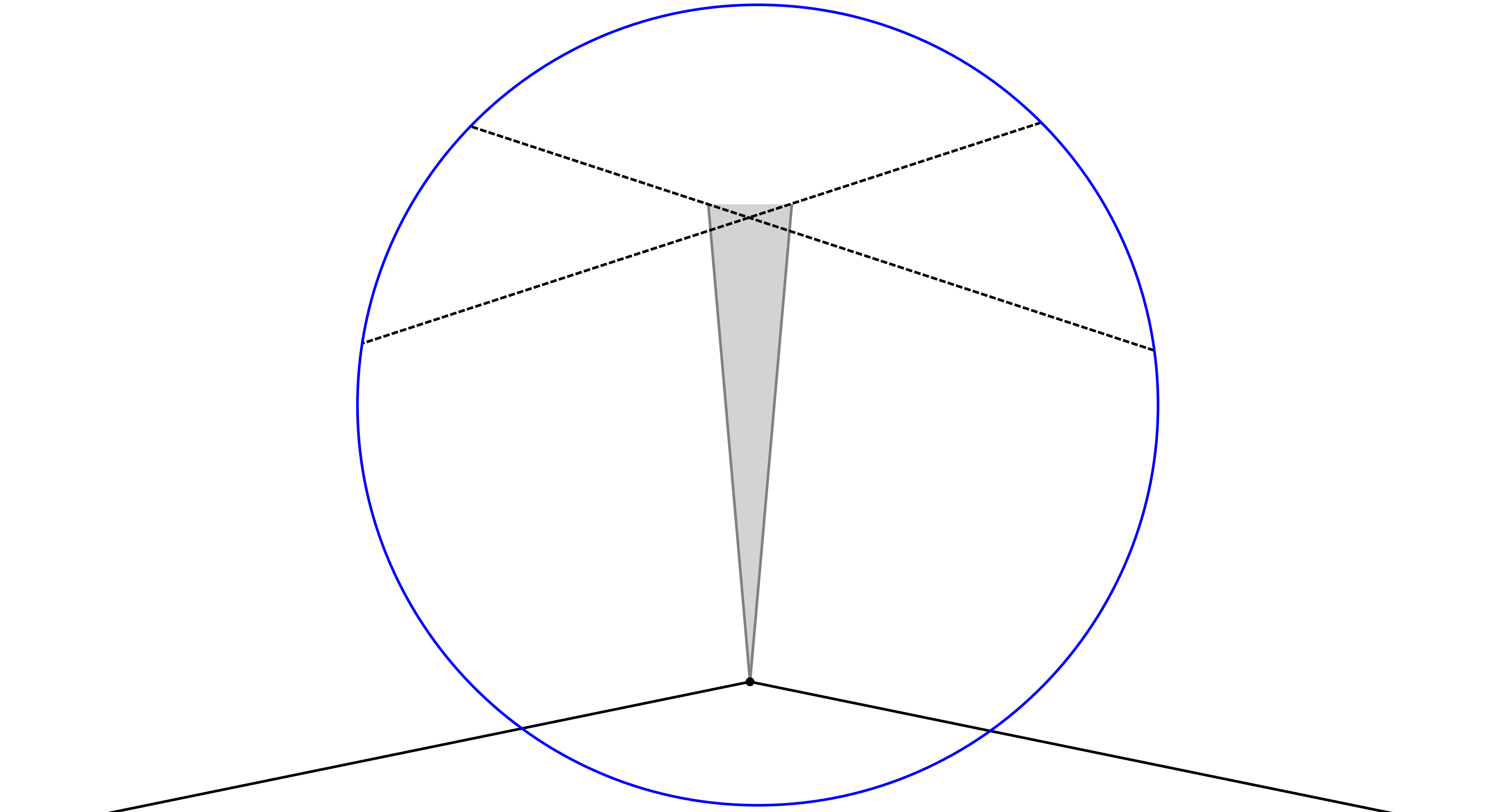

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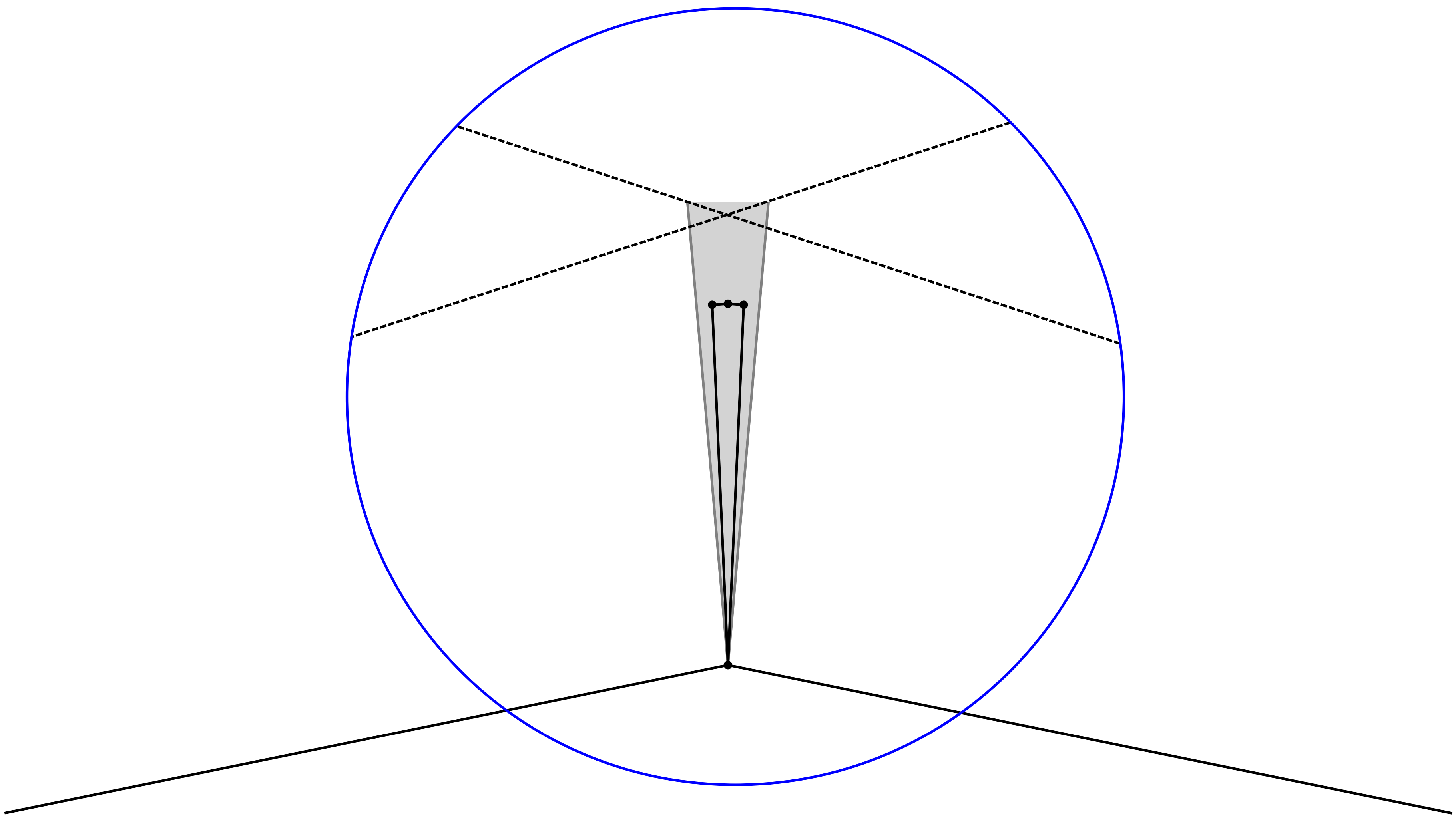
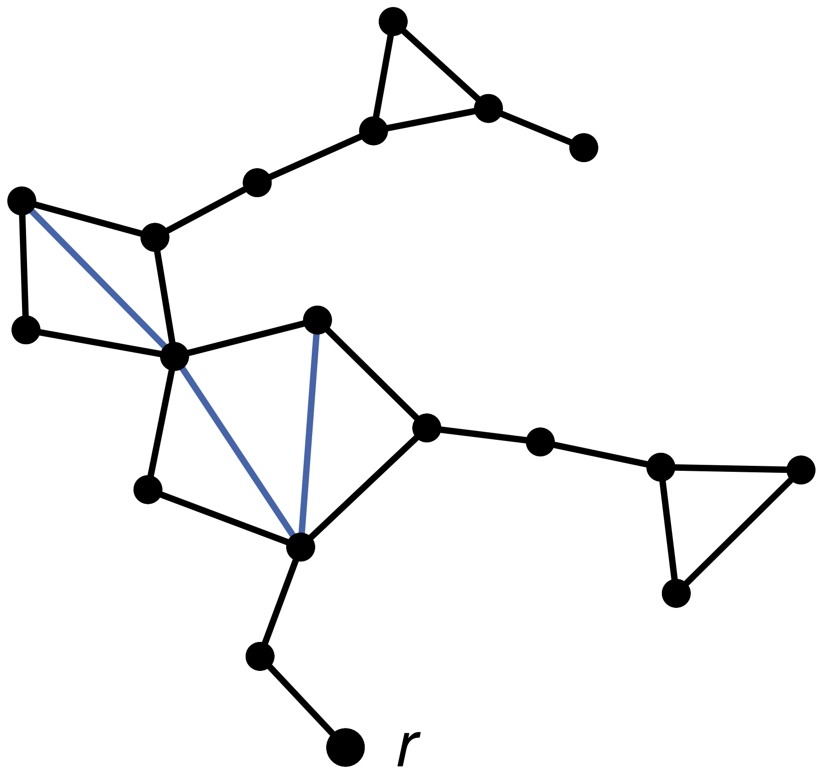


Similar Idea for Increasing Chords

Triangulations have downward-triangulated spanning binary cactus.

[Angelini et al. 2010]



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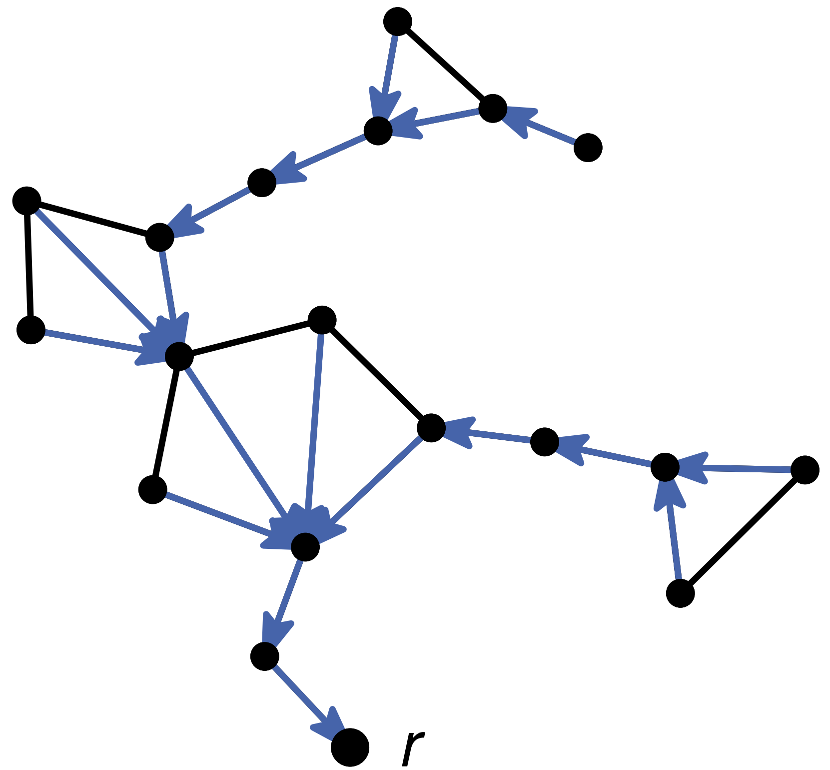


downward edges

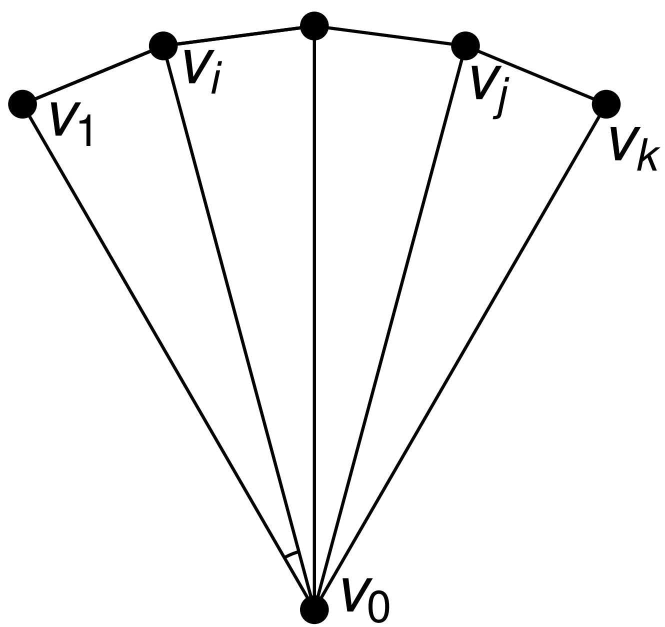
Similar Idea for Increasing Chords

Theorem

Every **triangulation** has an increasing-chords drawing.

Proof (similar to proof for GE)

By induction: every downward-triangulated binary cactus has increasing-chord drawing with almost-vertical downward edges



base case

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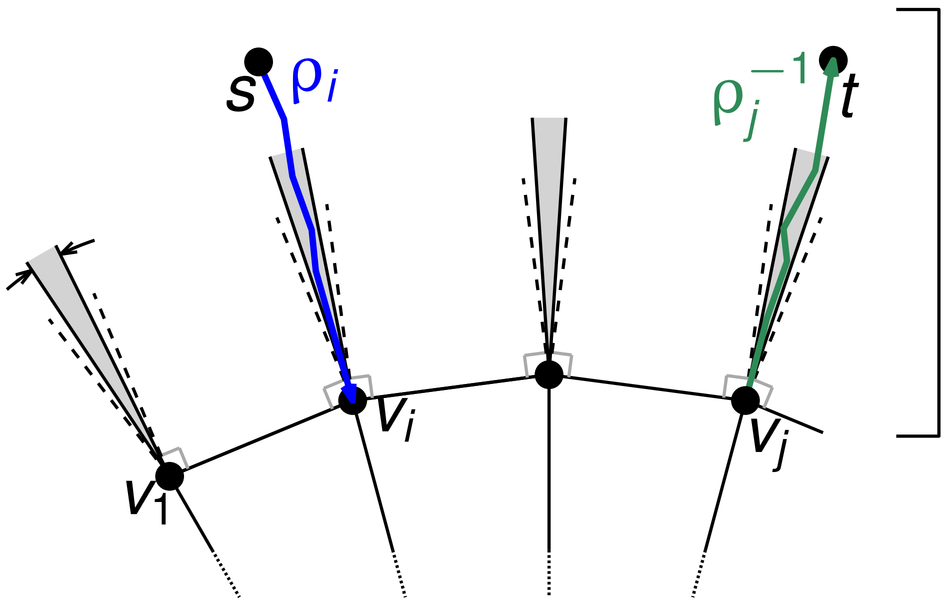

draw child cactuses inside narrow cones

induction step

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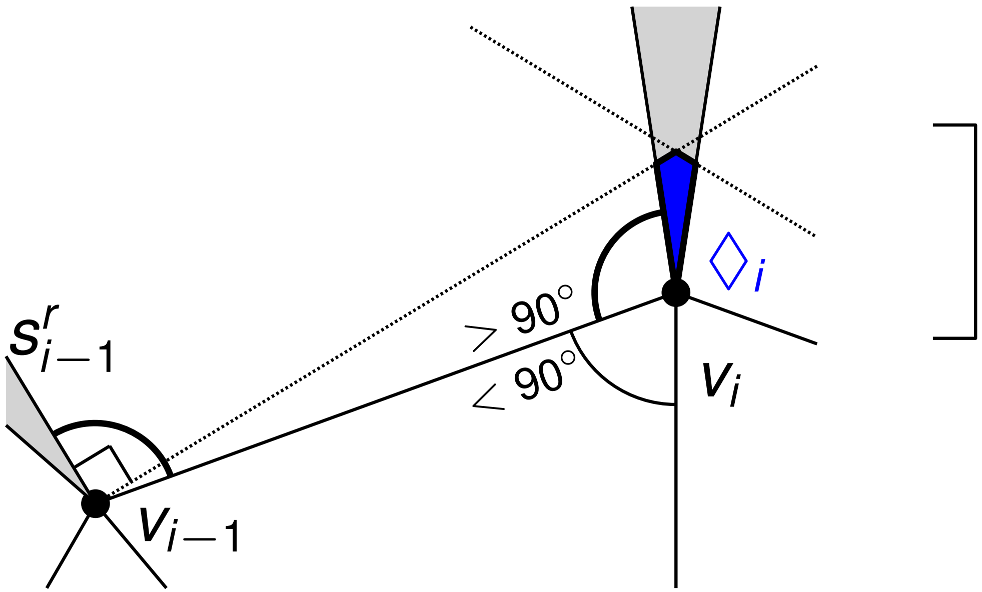

draw child cactuses sufficiently small

induction step

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Planar 3-trees have planar increasing-chord drawings.
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Hyperbolic plane is more powerful for increasing-chord drawings.
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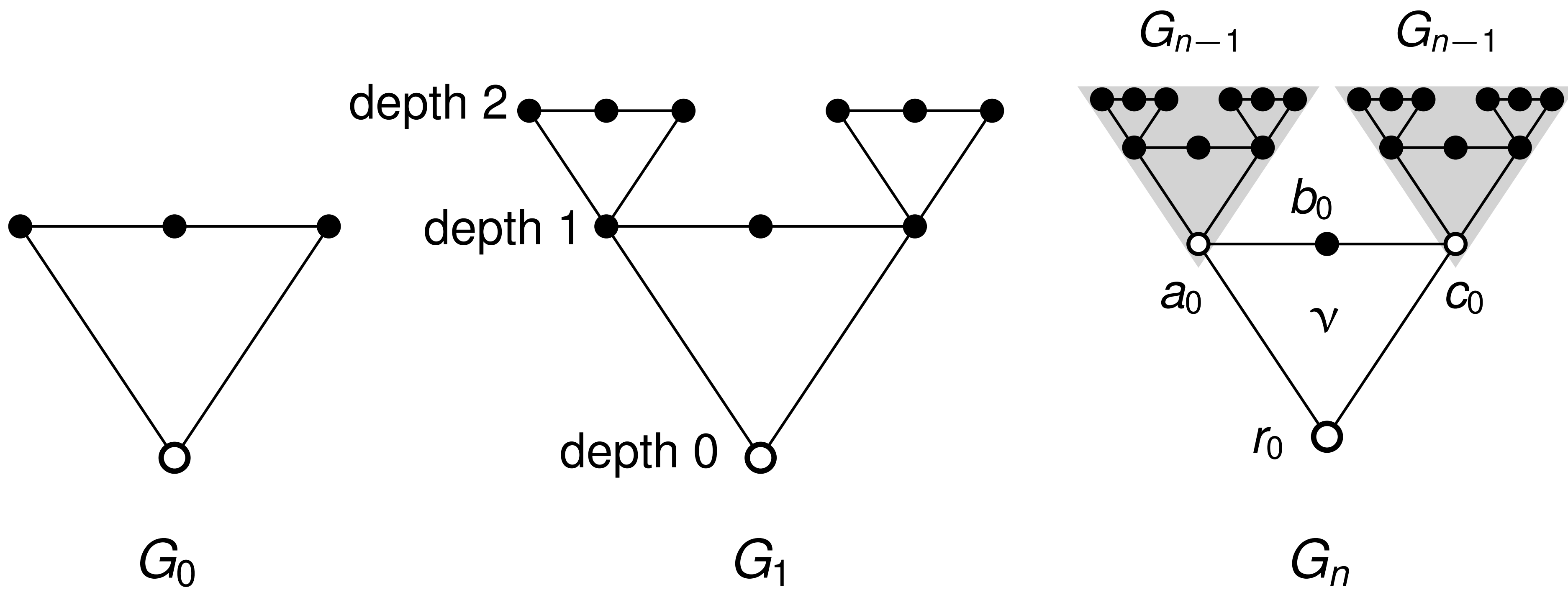
Non-Triangulated Binary Cactus

Theorem

G_9 has no self-approaching drawing.

This covers all embeddings of G_9 including non-planar.



Non-Triangulated Binary Cactus

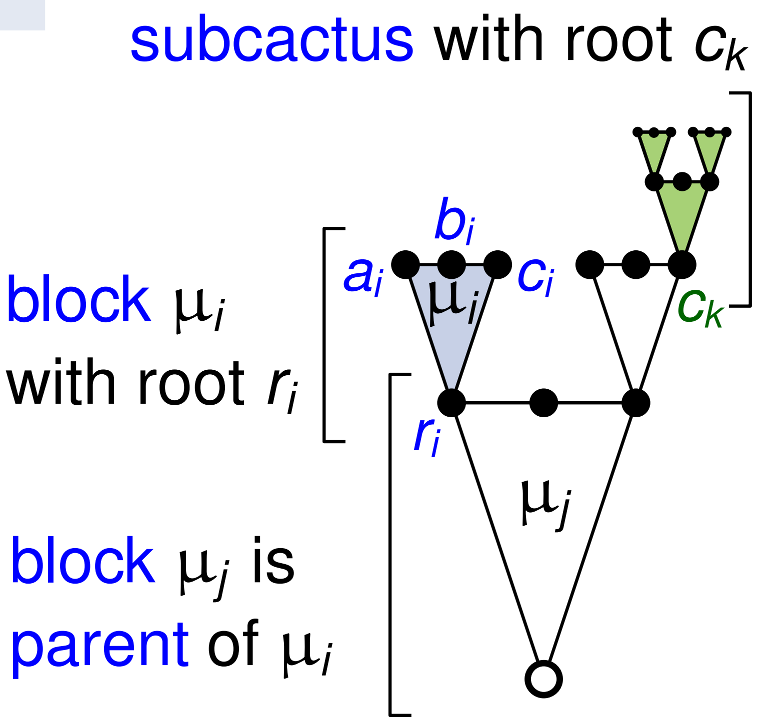
Theorem

G_9 has no self-approaching drawing.

Proof overview. Every self-approaching drawing of G_9 contains a drawing of a subcactus, in which:

Claim 1

Each block is smaller than its parent block.



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Proof overview. Every self-approaching drawing of G_9 contains a drawing of a subcactus, in which:

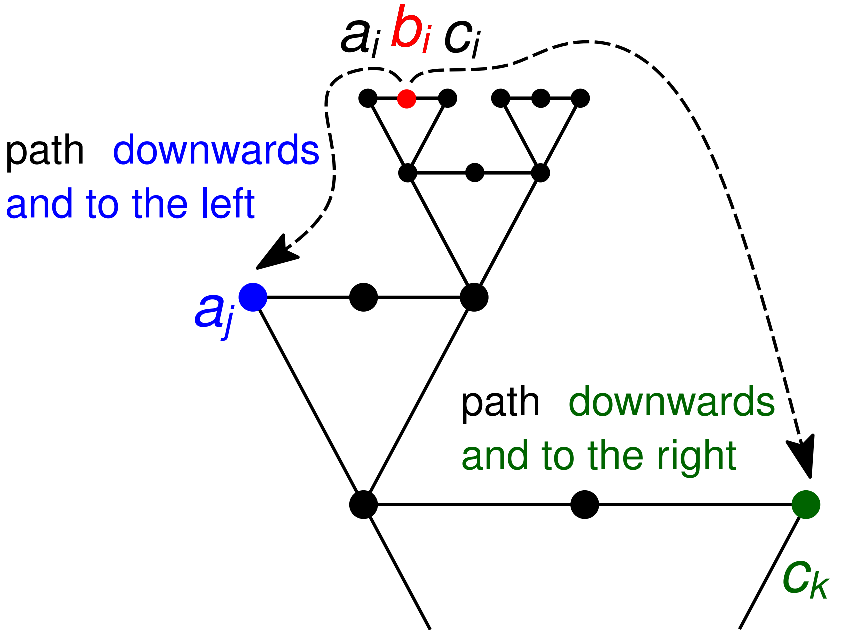
Claim 1

Each block is smaller than its parent block.

Claim 2

each self-approaching path from b_i downwards and to the left uses a_i;

each self-approaching path from b_i downwards and to the right uses c_i;



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Claim 2

each self-approaching path from b_i downwards and to the left uses a_i;

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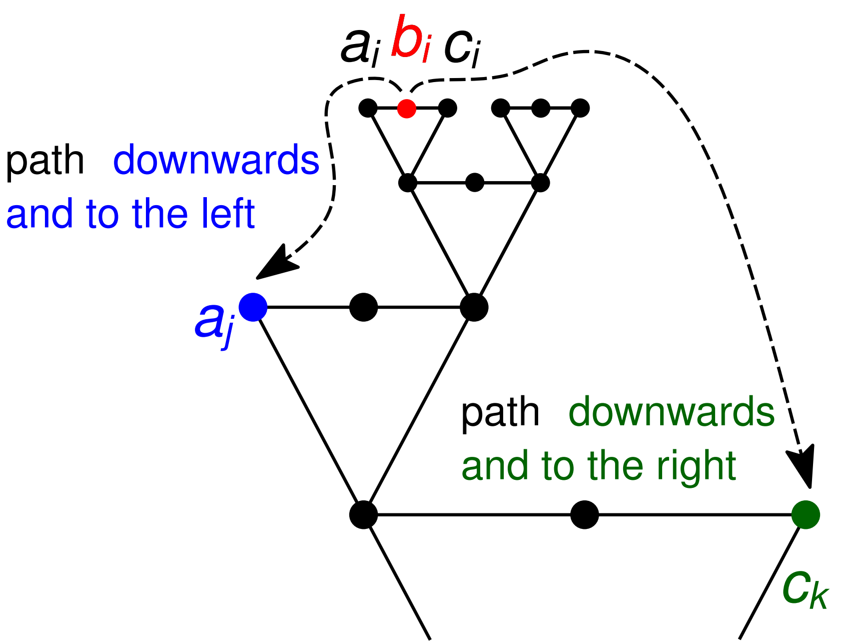
Claim 3

Claim 2 $\Rightarrow$ some block is bigger than its parent block; ↯ to **Claim 1**.



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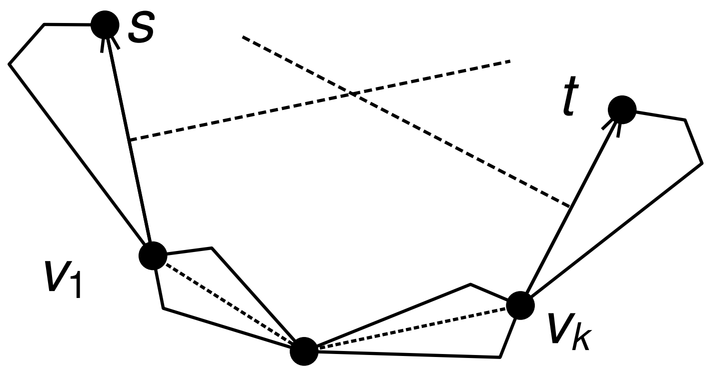
Divergence of Blocks, Small Angles

Lemma

Consider greedy drawing of a cactus, vertices s, t and cutvertices $v_1, \ldots, v_k$ on each st path. It holds:

$(s, v_1, \ldots, v_k, t)$ is drawn greedily, i.e., each of its subpaths is greedy;

rays from v_1 through s and from v_k through t diverge.



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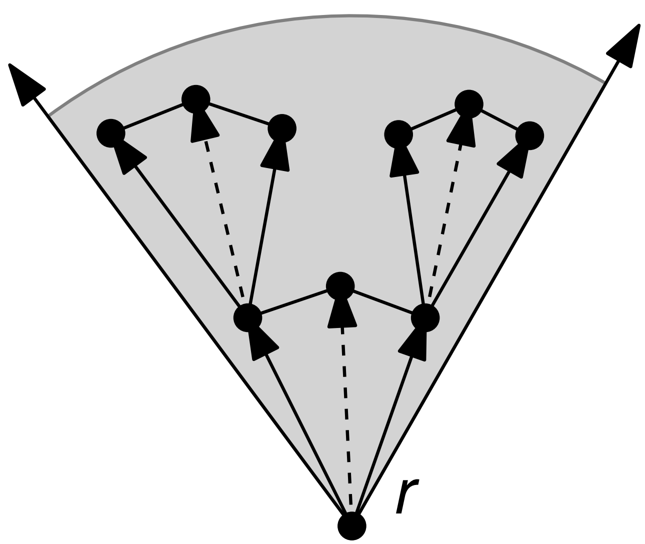
Def. Cone U_r of upward directions of subcactus rooted at r



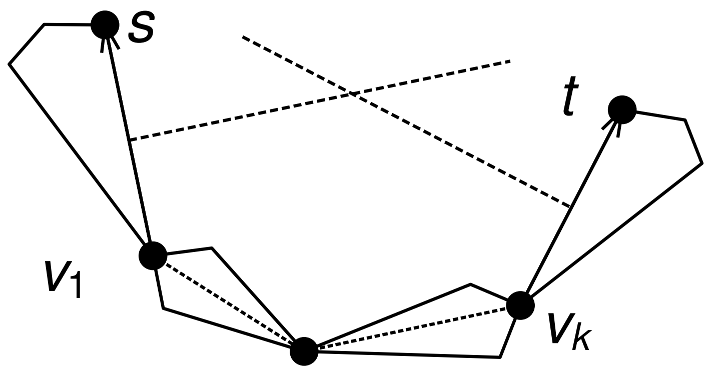
Lemma

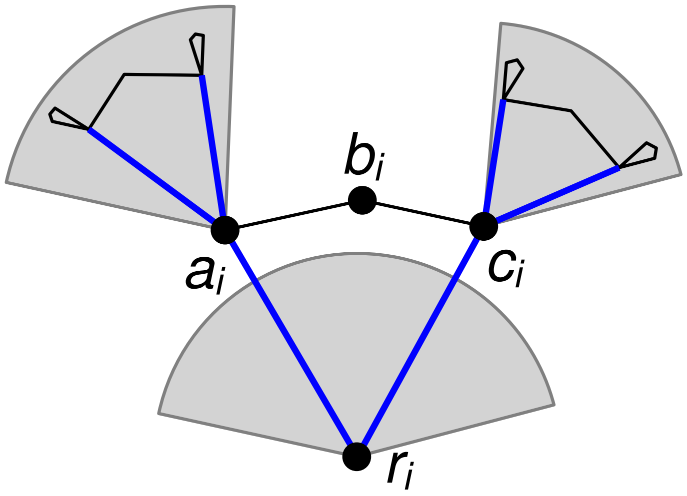
Consider self-appr. drawing of G_9.
If $|U_{r_i}| < 180°$, then $U_{a_i} \cap U_{c_i} = \emptyset$.
There exists a cutvertex r at depth 4 and $|U_r| < 22.5°$ (sufficiently small for our proof).

From now on, consider G_r.

Divergence of Blocks, Small Angles

Wlog, in subcactus G_r rooted at r, all $r_i a_i$, $r_i c_i$ are almost vertical.





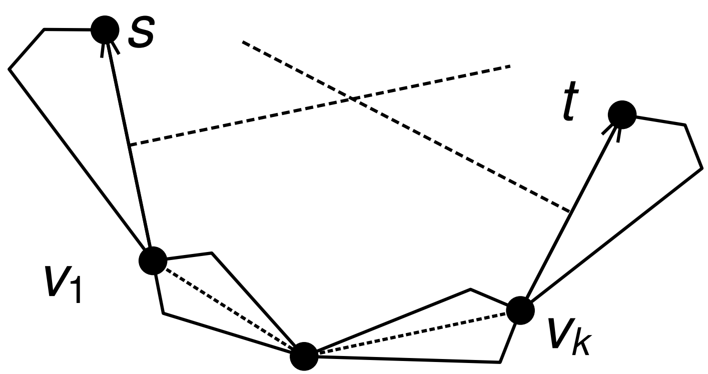
Divergence of Blocks, Small Angles

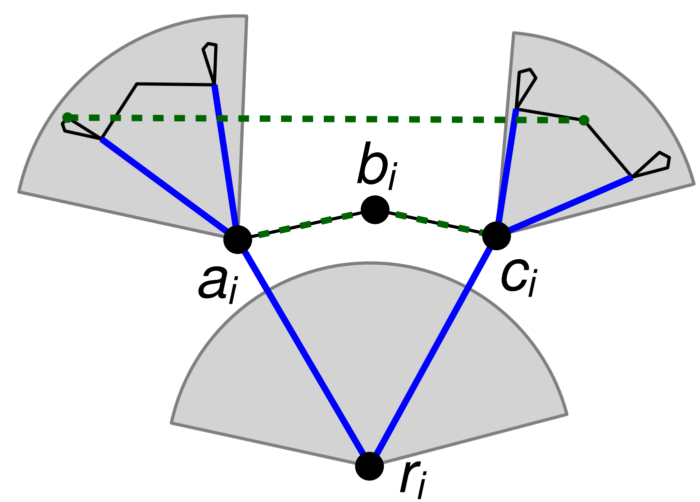
Wlog, in subcactus G_r rooted at r, all $r_i a_i$, $r_i c_i$ are almost vertical.

Lemma

All $\overrightarrow{a_i b_i}$, $\overrightarrow{b_i c_i}$ are almost horizontal and point rightwards.

A line between points of sibling subcactuses is almost horizontal.





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Blocks Become Smaller

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Claim 1

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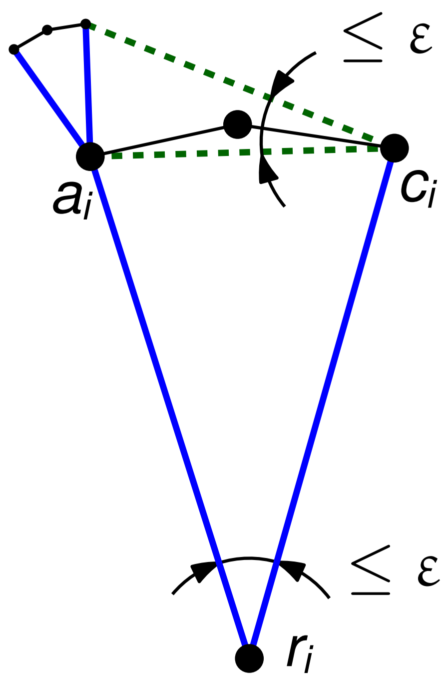

Self-approaching Downward Left/Right Paths

Claim 2

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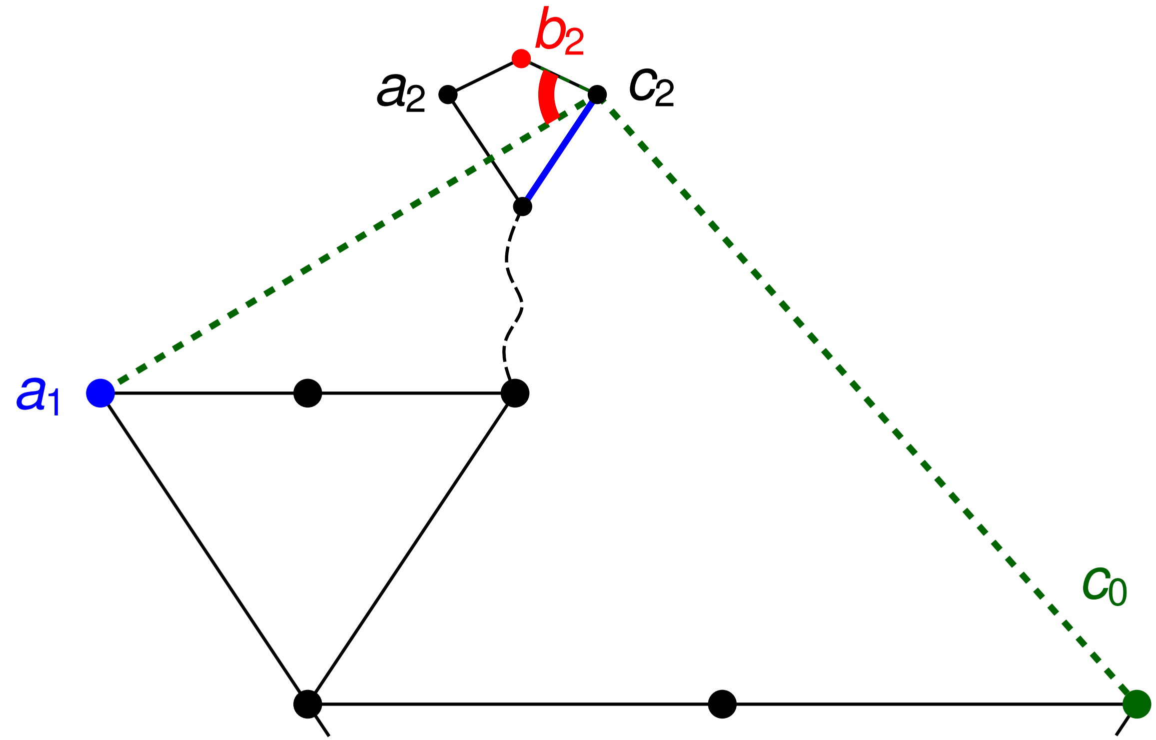

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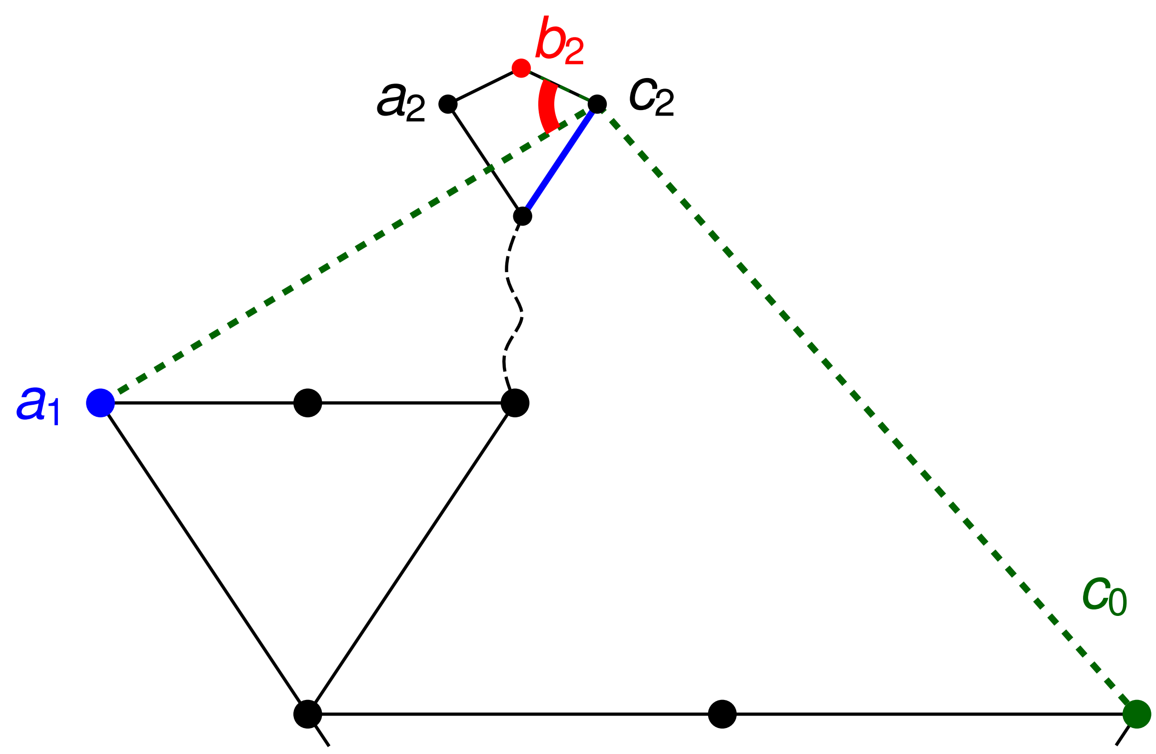
Proof

$\angle a_1 c_1 b_2 < 90^\circ$

$\Rightarrow$ $b_2 c_2$ can not lie on a self-appr. b_2-a_1 path.



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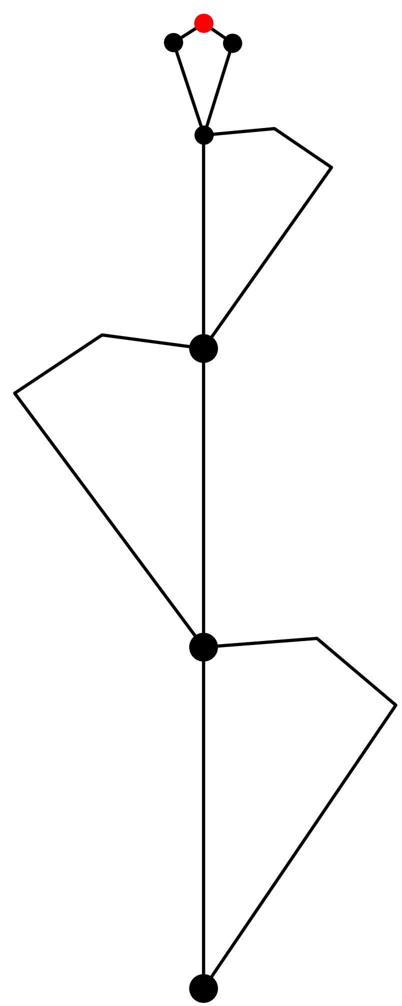
Deriving Contradiction

Claim 3

Claim 2 $\Rightarrow$ some block is bigger than its parent block.



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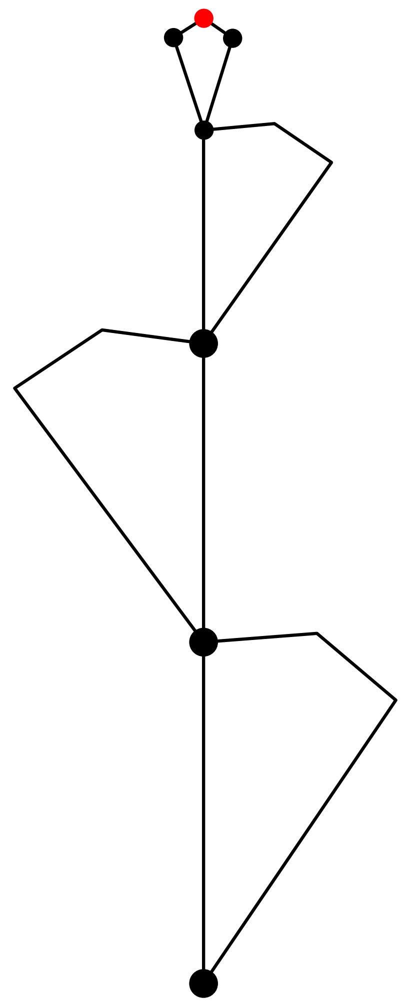

Key Idea

consider common cutvertices of self-approaching downward paths

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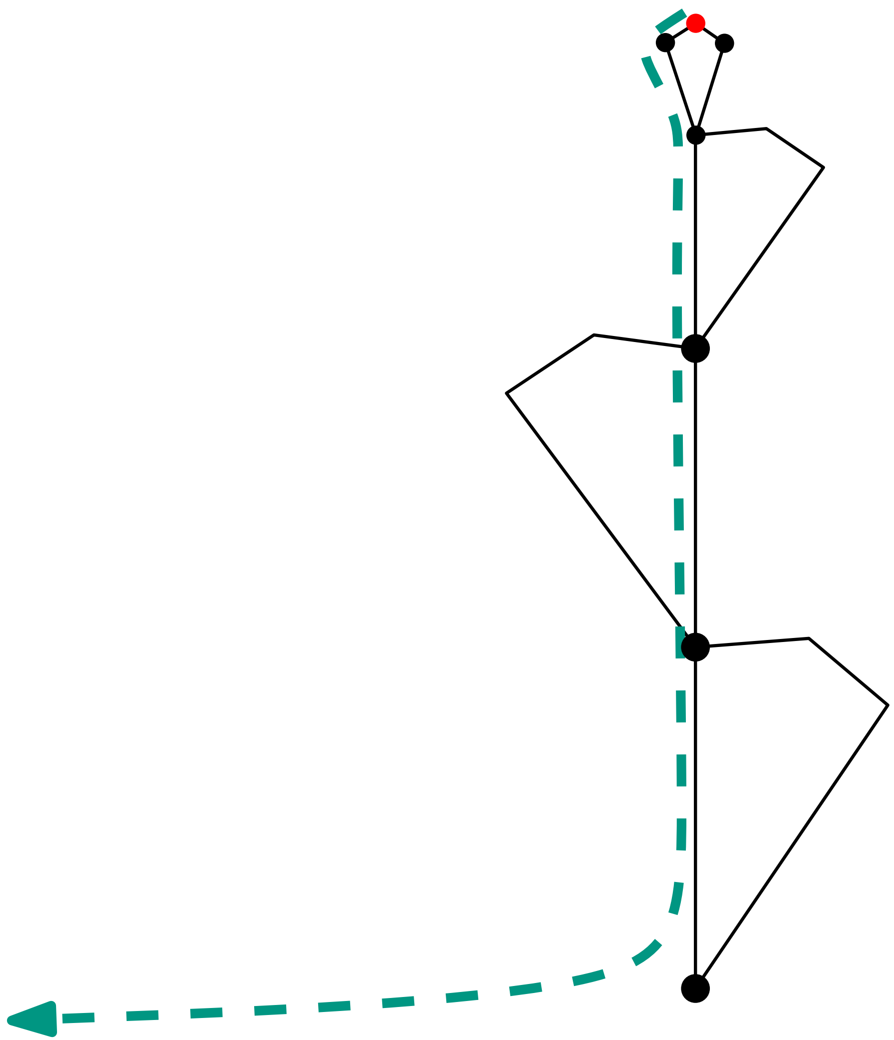

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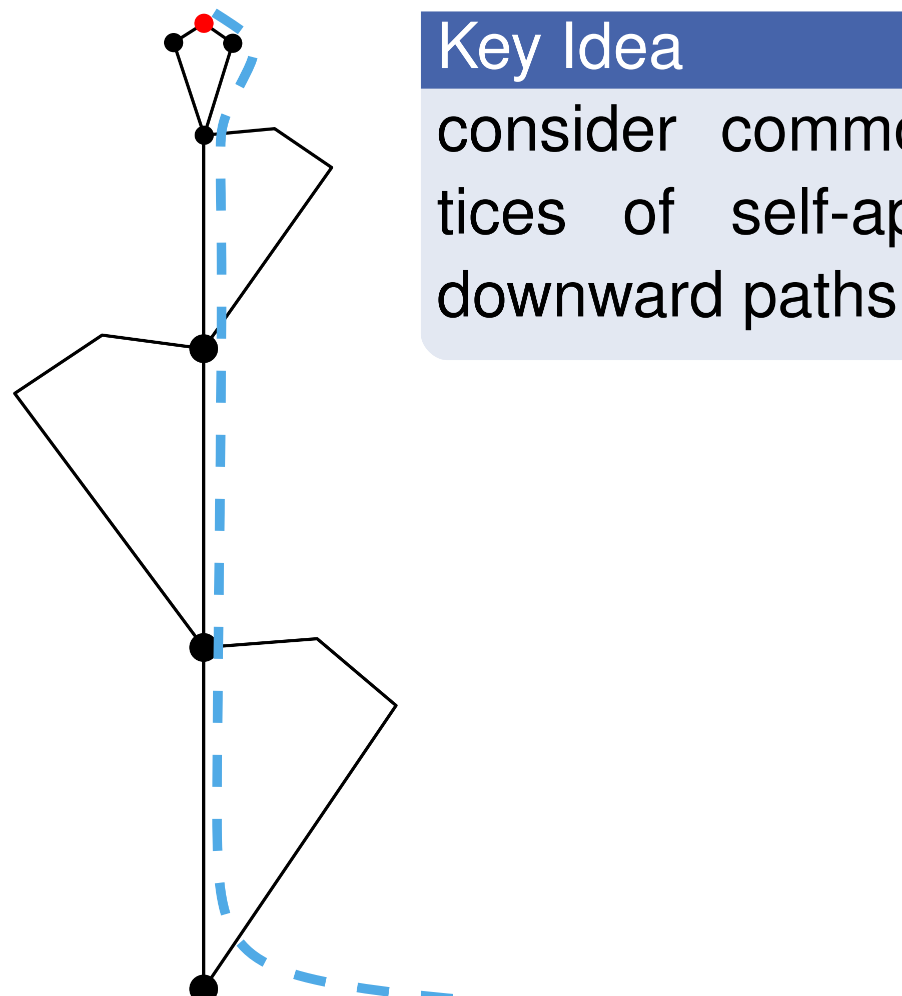

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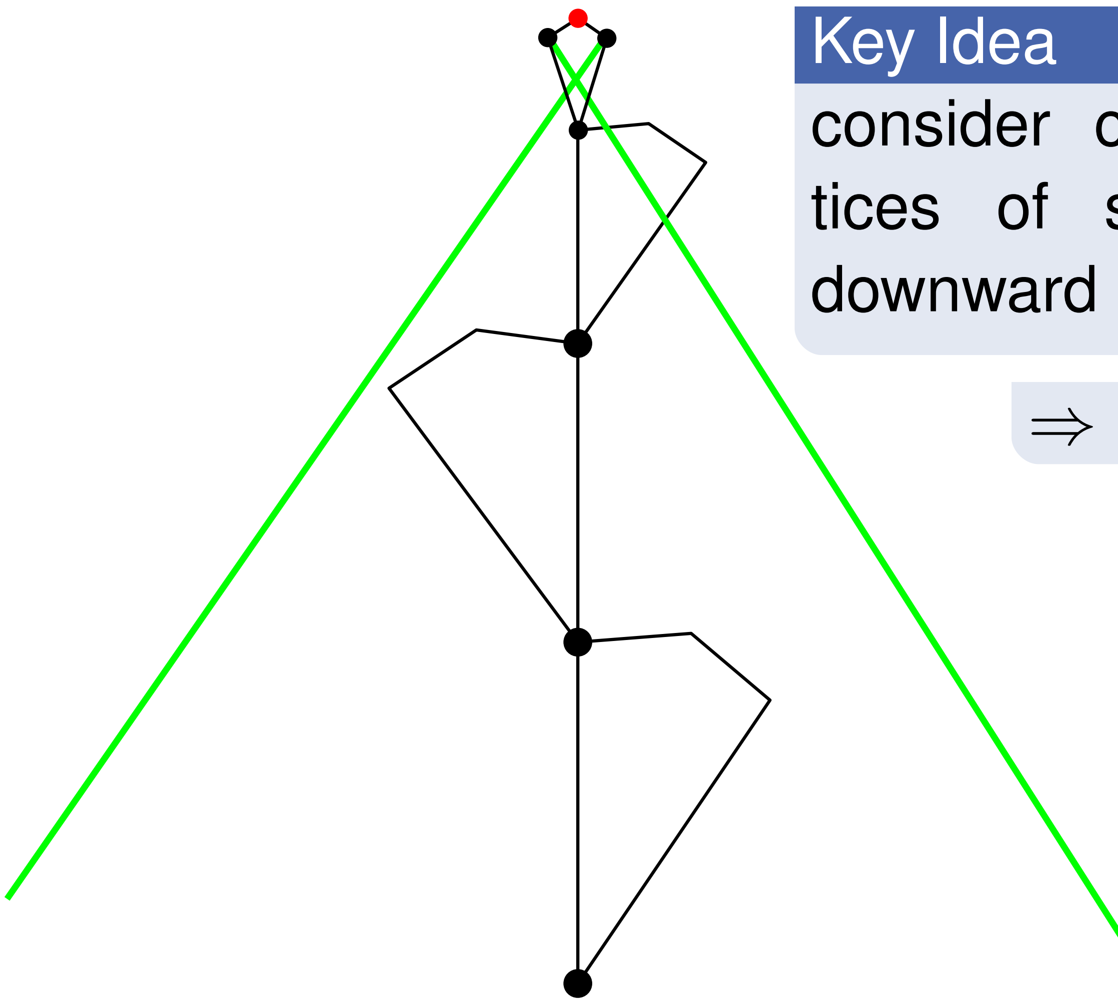

Key Idea

consider common cutvertices of self-approaching downward paths

⇒ lie inside cone

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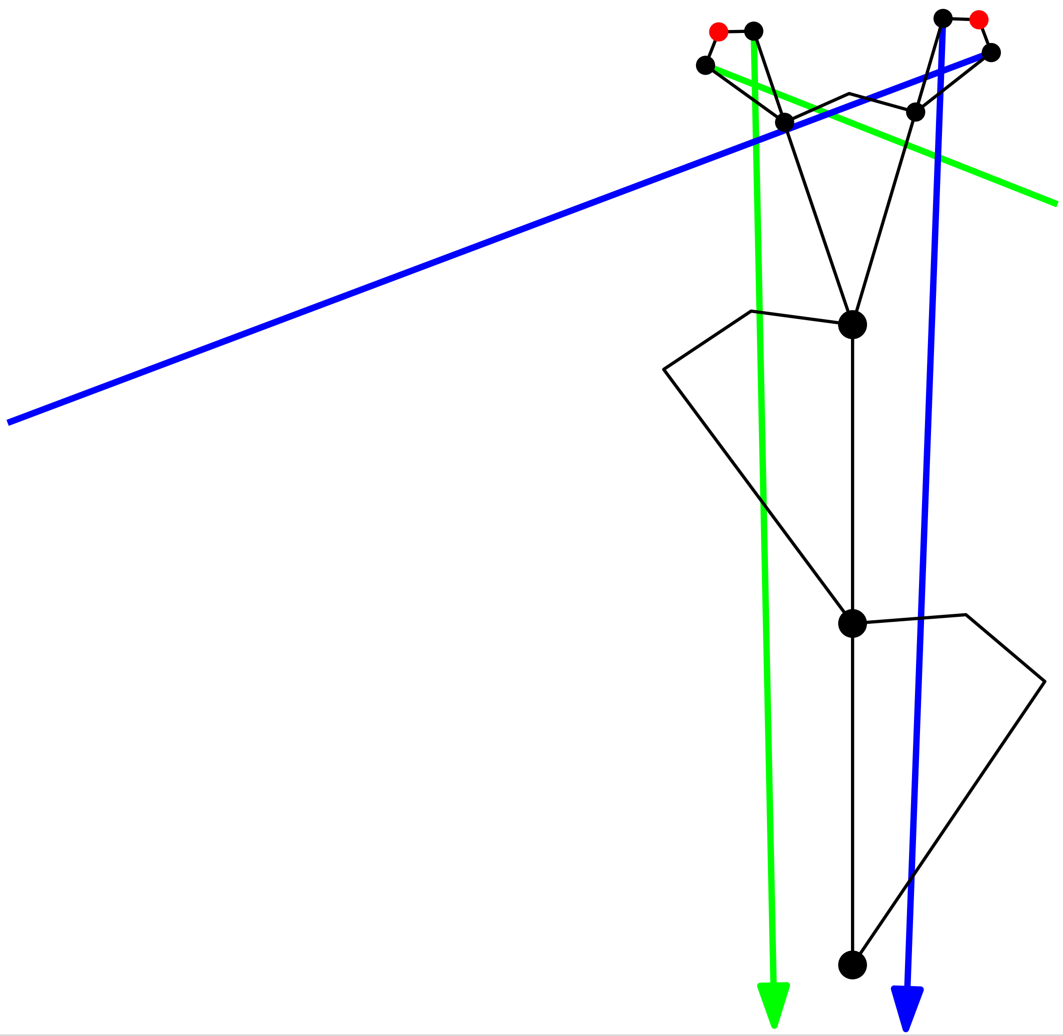

Key Idea

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$\Rightarrow$ lie inside cone

$\Rightarrow$ lie inside 2 cones

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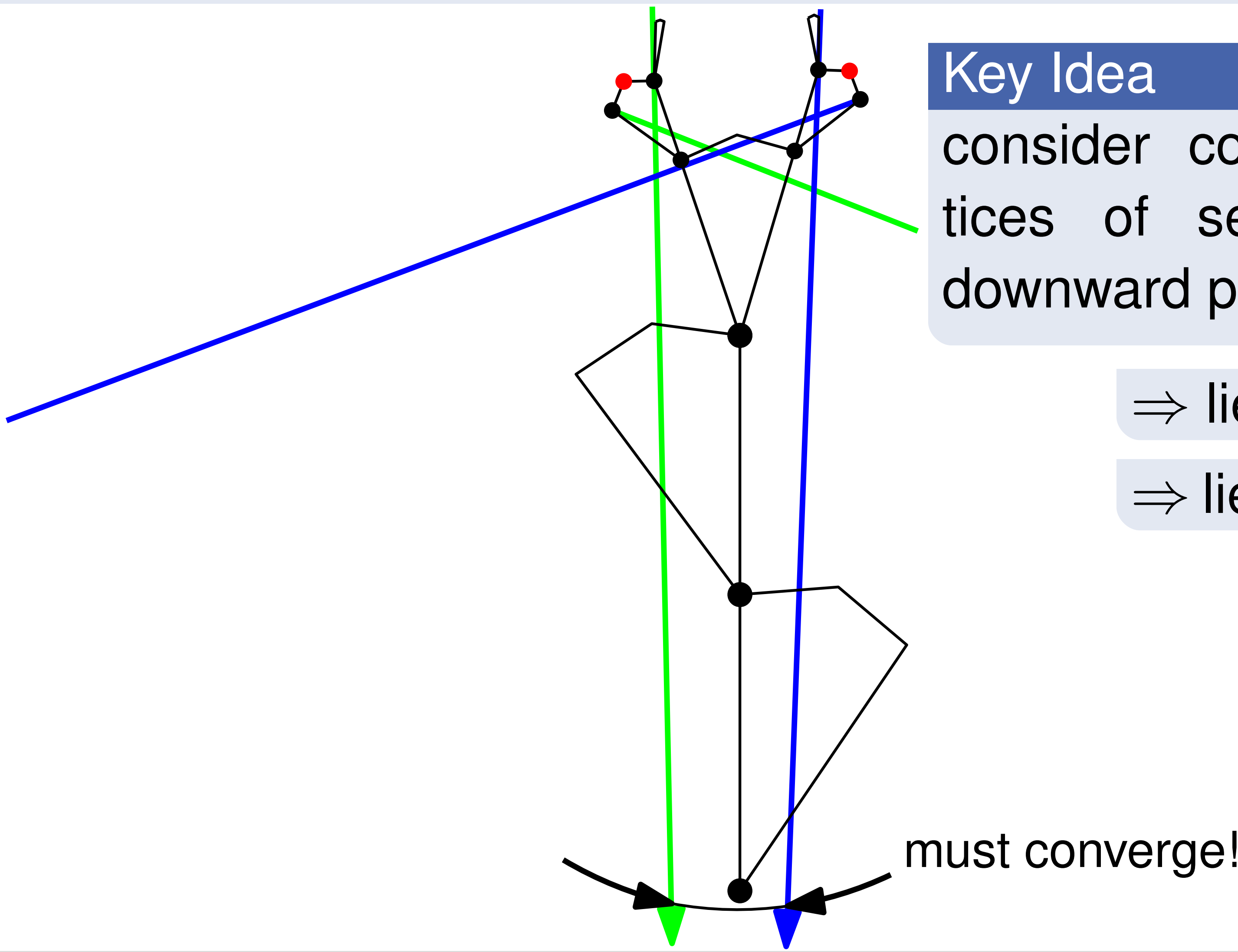

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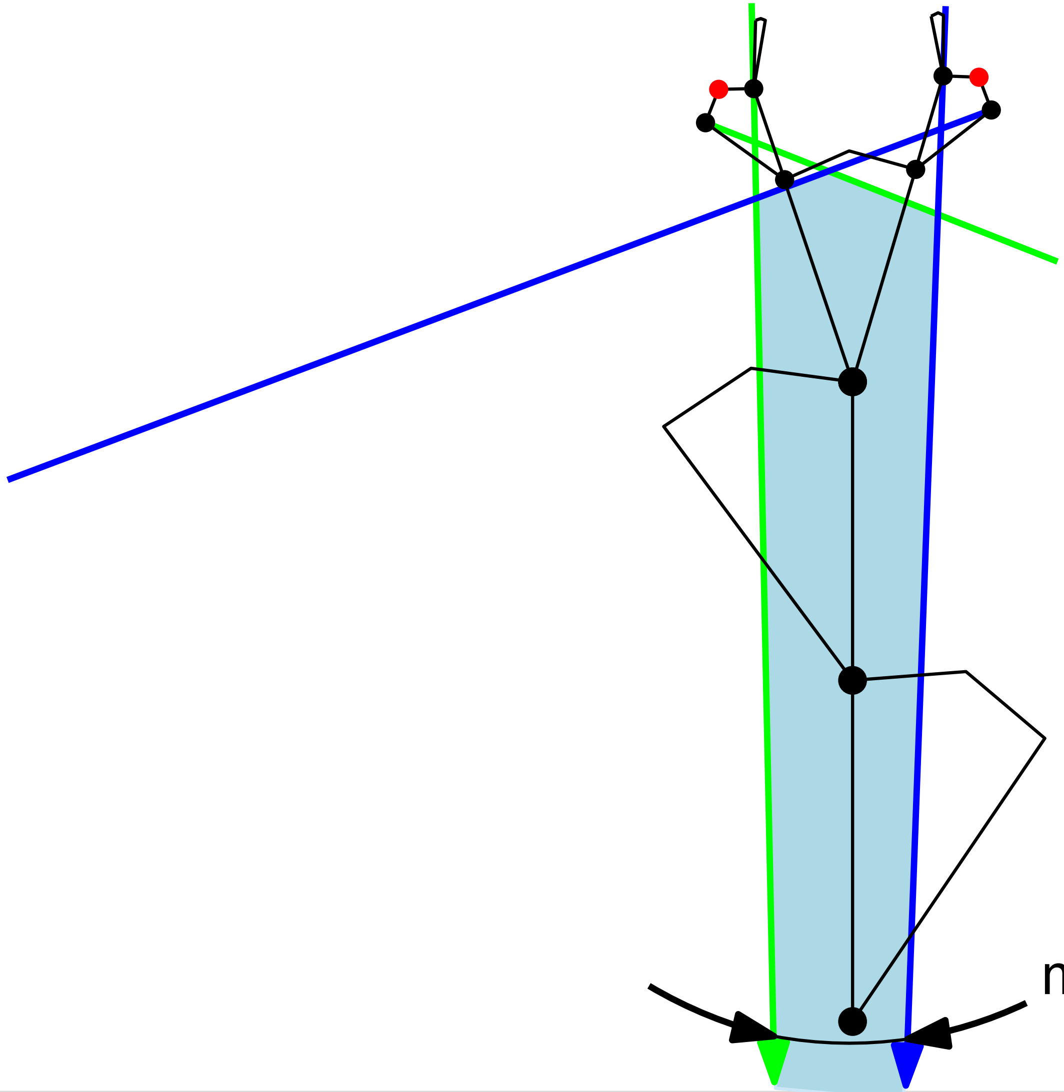

Key Idea

consider common cutvertices of self-approaching downward paths

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$\Rightarrow$ lie inside 2 cones

$\Rightarrow$ lie inside a strip

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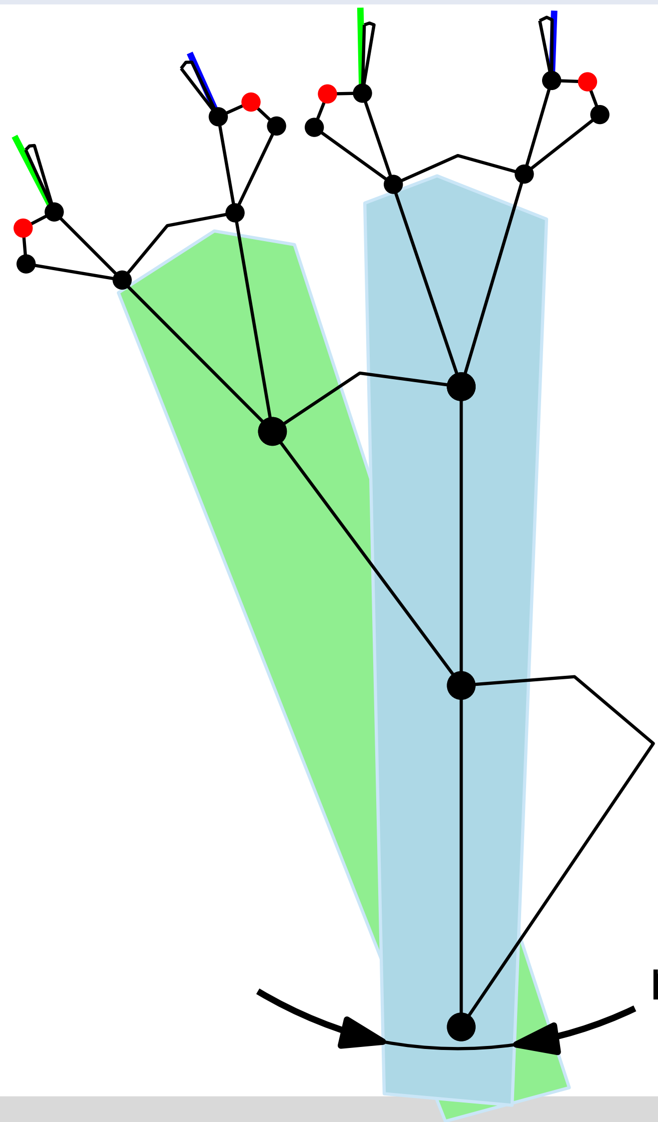

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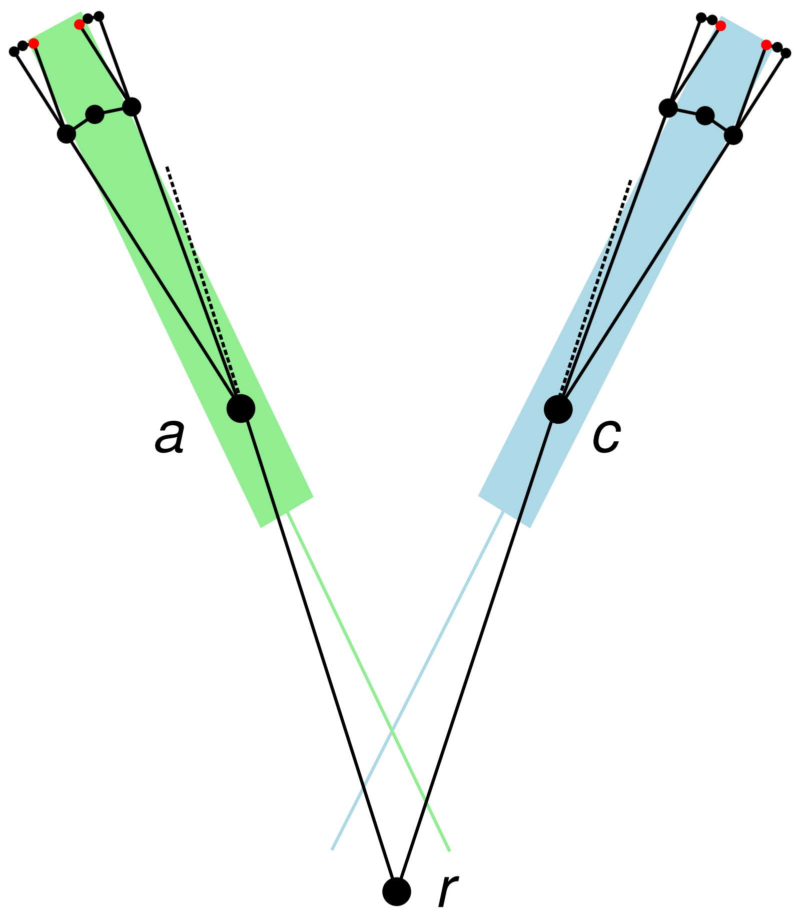

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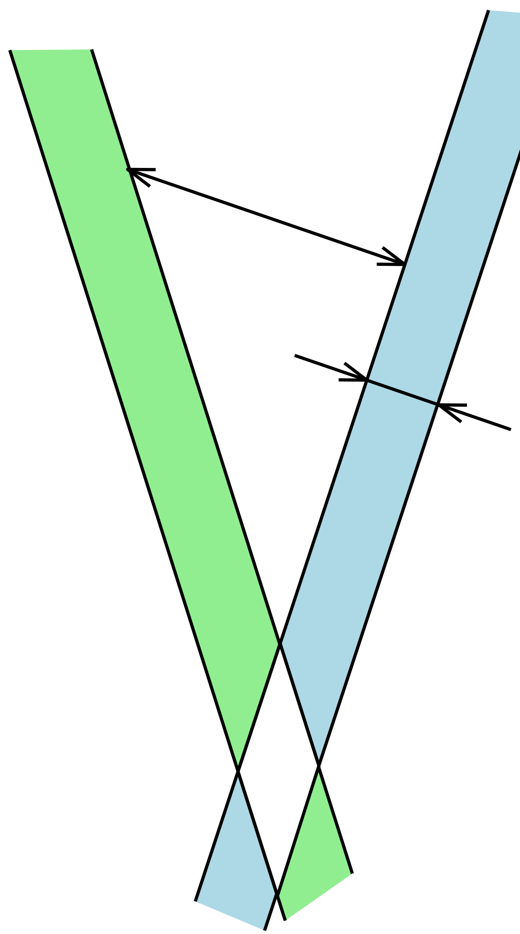

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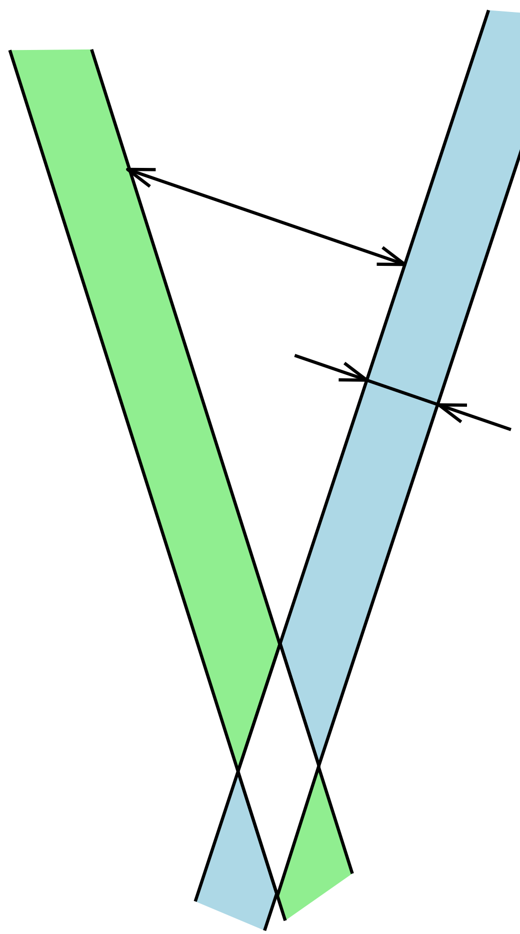

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$\Rightarrow$ lie inside a strip

$\Rightarrow$ lie inside 2 strips

$\Rightarrow$ parent block is small

Contributions

Every triangulation has an increasing-chord drawing.

- has spanning downward-triangulated binary cactus [Angelini et al. 2010]
- such cactus has increasing-chord drawing

Some binary cactuses have no self-approaching drawing.

- above proof strategy does not work :(
- it worked for greedy drawings

Planar 3-trees have planar increasing-chord drawings.

- first construction for str. monotone/greedy drawings of pl. 3-trees

Hyperbolic plane is more powerful for increasing-chord drawings.

- characterize drawable trees
- every 3-connected planar graph is drawable

Planar Increasing-Chord Drawings of 3-Trees

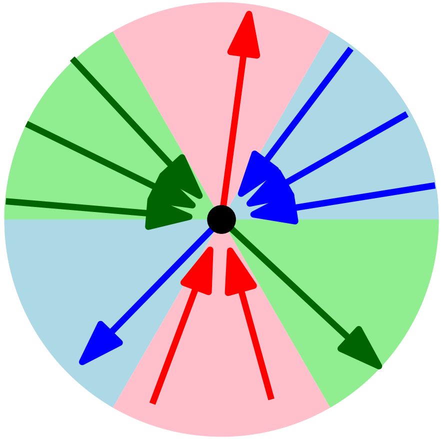
Schnyder labeling of a triangulation

coloring and orientation of edges

external vertices *r*, *g*, *b*: all edges incoming

internal: one outgoing in each color, cyclic order

counting triangles in red, green, blue regions gives coordinates of plane drawing



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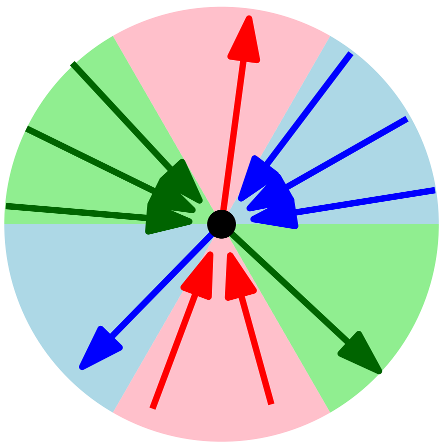
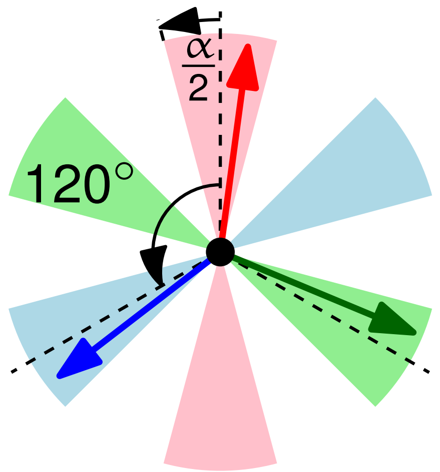


α-Schnyder drawings for $\alpha \in [0, 60^\circ]$

outgoing edges are inside cones of size α



Planar Increasing-Chord Drawings of 3-Trees

Lemma

30°-Schnyder drawings are increasing-chords.

Planar Increasing-Chord Drawings of 3-Trees

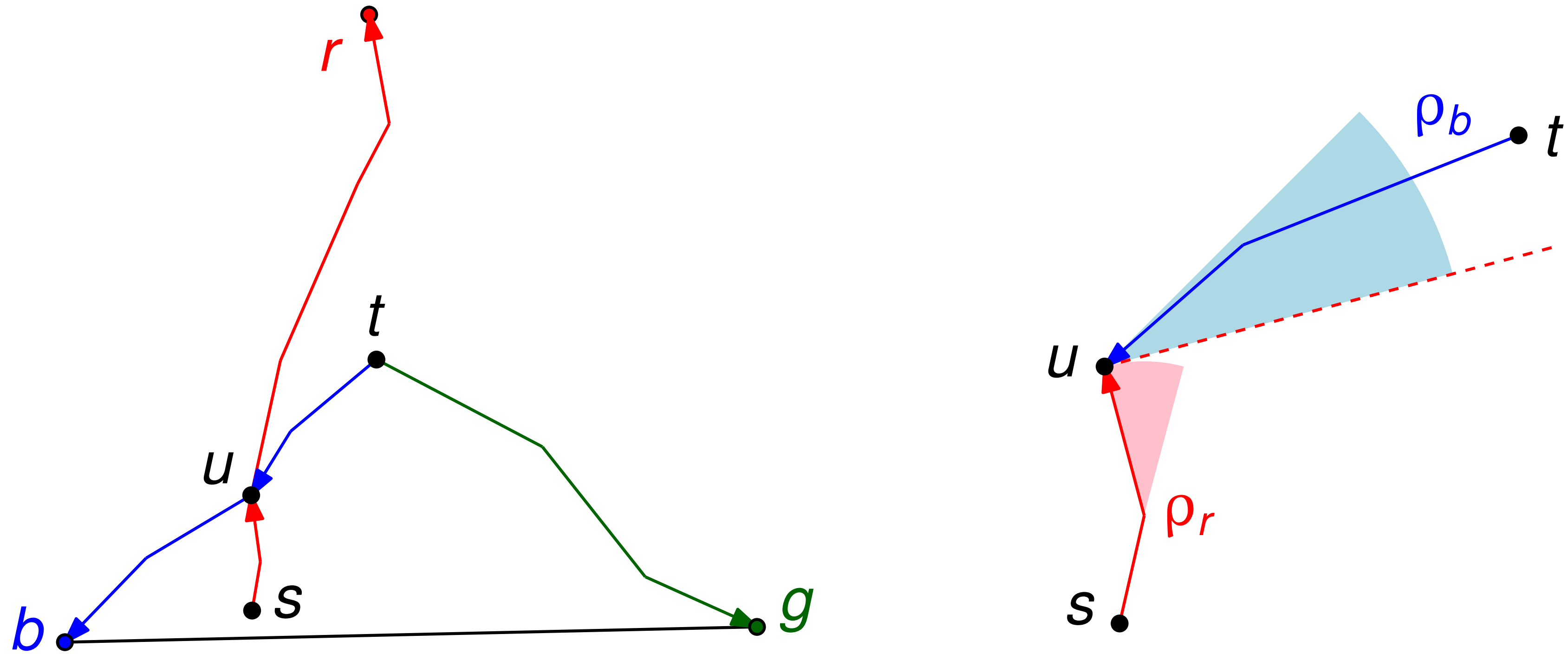
Lemma

30°-Schnyder drawings are increasing-chords.

Proof

consider paths from *s*, *t* to external vertices *r*, *g*, *b*

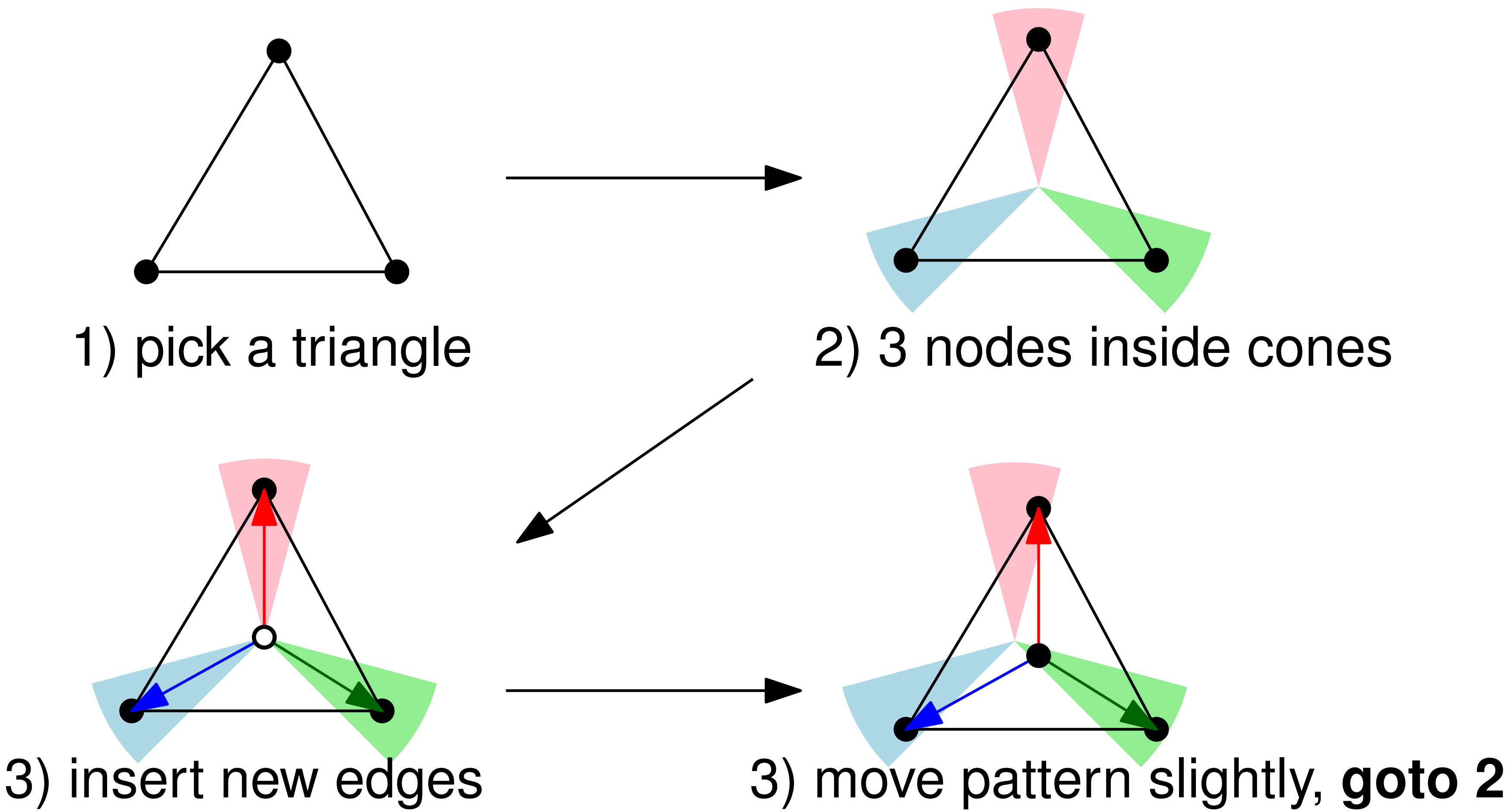
combine ρ_r, ρ_b: no normal crosses another edge



Planar Increasing-Chord Drawings of 3-Trees

Theorem

Planar 3-trees have ε-Schnyder drawings $\forall \varepsilon > 0$ and, thus, have increasing-chords drawings.



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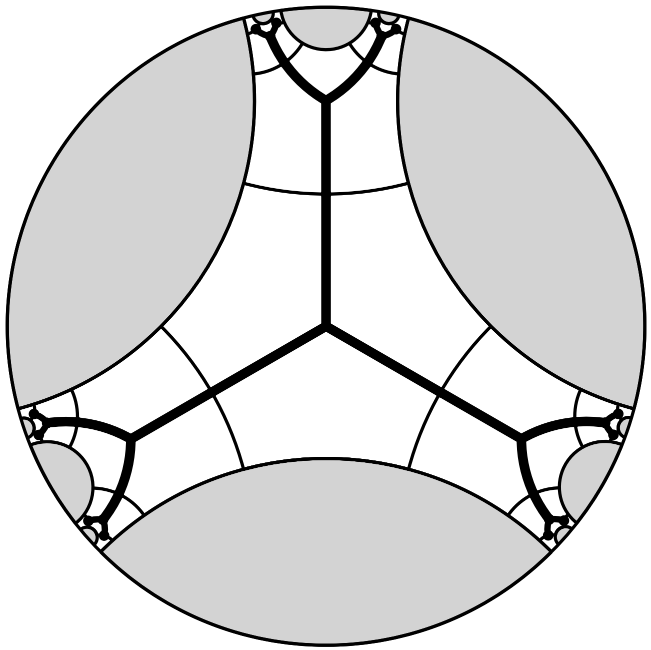
Increasing-Chord in the Hyperbolic Plane



increasing-chord drawing of complete binary tree in $\mathbb{H}^2$

Theorem

A tree has a self-approaching/increasing-chord drawing in $\mathbb{H}^2$ iff it has max. degree 3 or is a subdivision of $K_{1,4}$

$\Rightarrow$ 3-conn. planar graphs have increasing-chord drawings in $\mathbb{H}^2$.

Binary cactuses have *planar* increasing-chord drawings in $\mathbb{H}^2$.

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Conclusion

Every triangulation has an increasing-chord drawing.

Some binary cactuses have no self-approaching drawing.

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Open questions

graphs with self-appr. but without incr.-chord drawing?

self-approaching/increasing-chord drawings for 3-conn. planar?

- if yes, not just by drawing cactus spanner

planar self-approaching/incr.-chords drawings of triangulations?