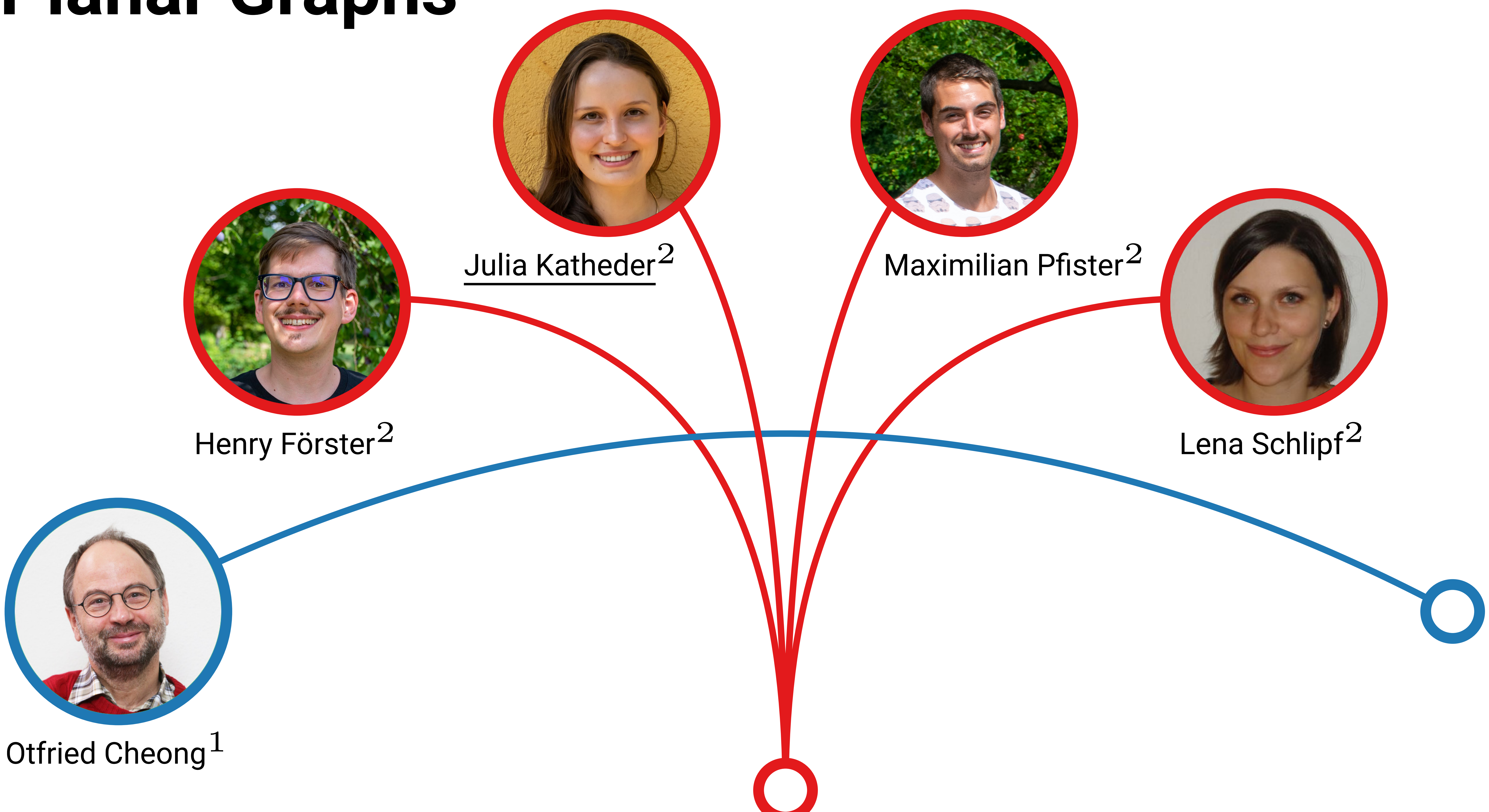
GD 2023

Weakly and Strongly Fan-Planar Graphs



Julia Katheder[2]
Maximilian Pfister[2]
Henry Förster[2]
Lena Schlipf[2]
Otfried Cheong[1]
[1] SCALGO
[2] University of Tübingen, Germany

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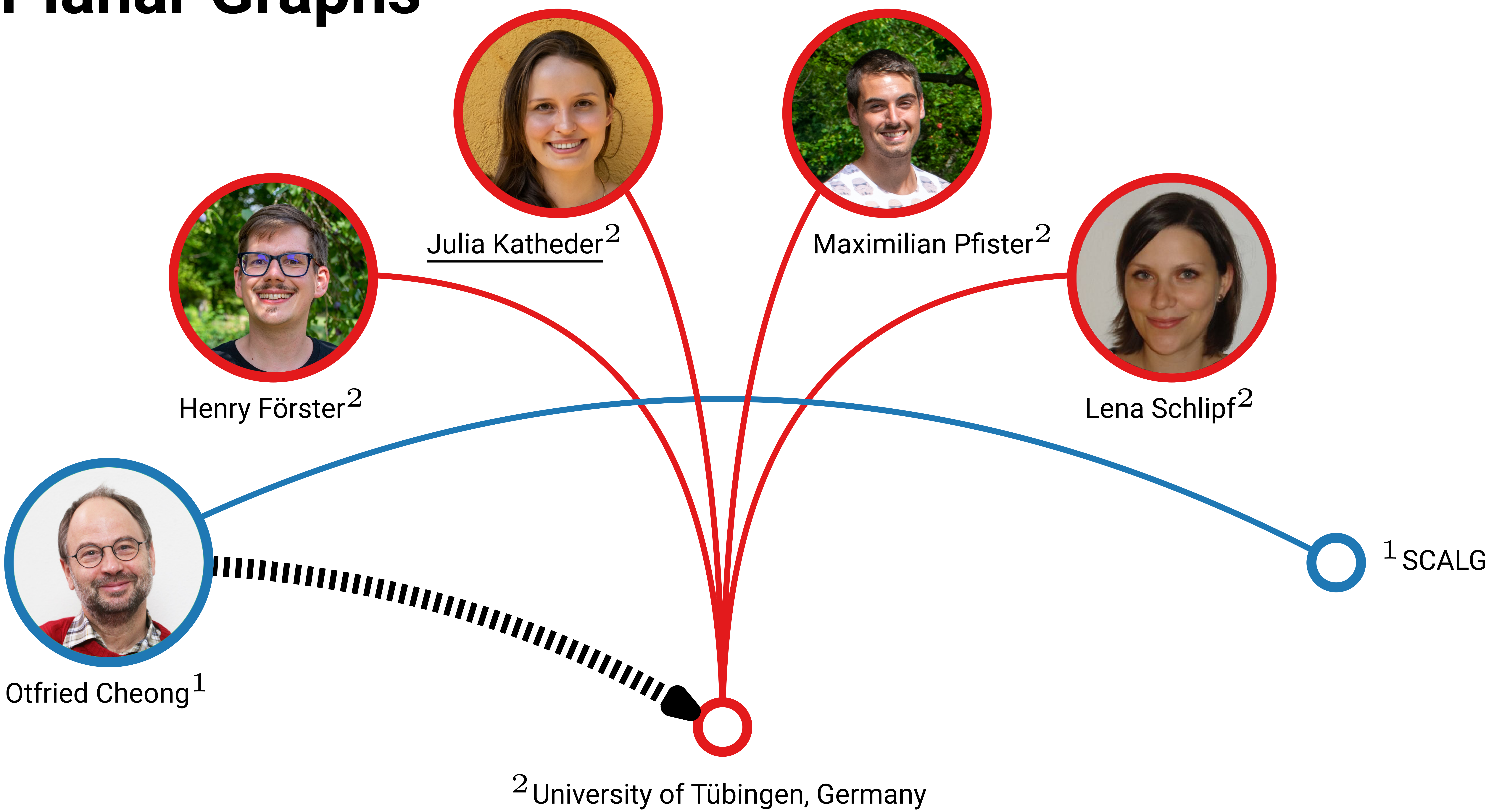
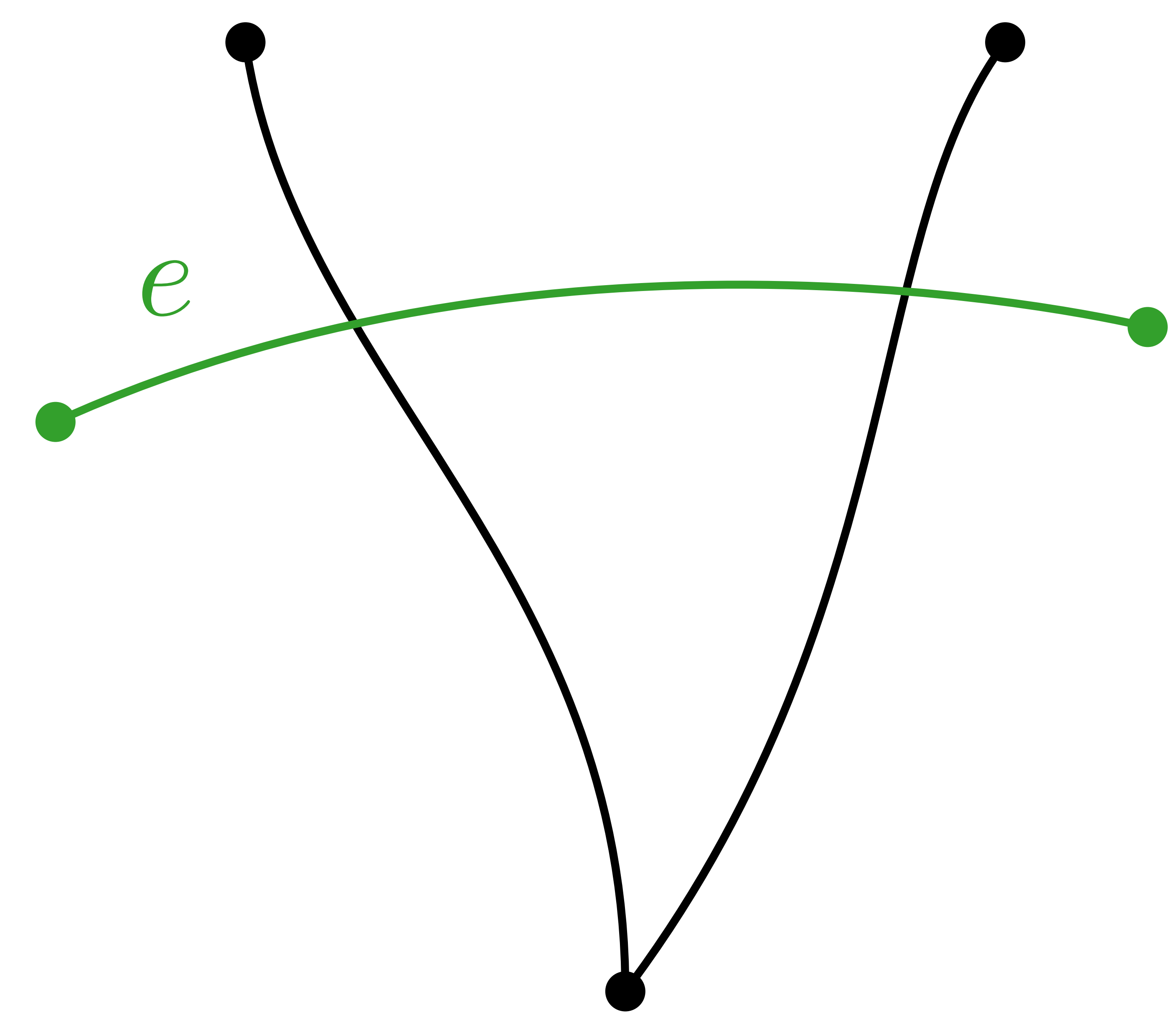


Fan-Planarity

Each edge e can only be crossed by a *fan* of edges, i.e., a bundle of edges sharing a common endpoint that all cross e from the same side.



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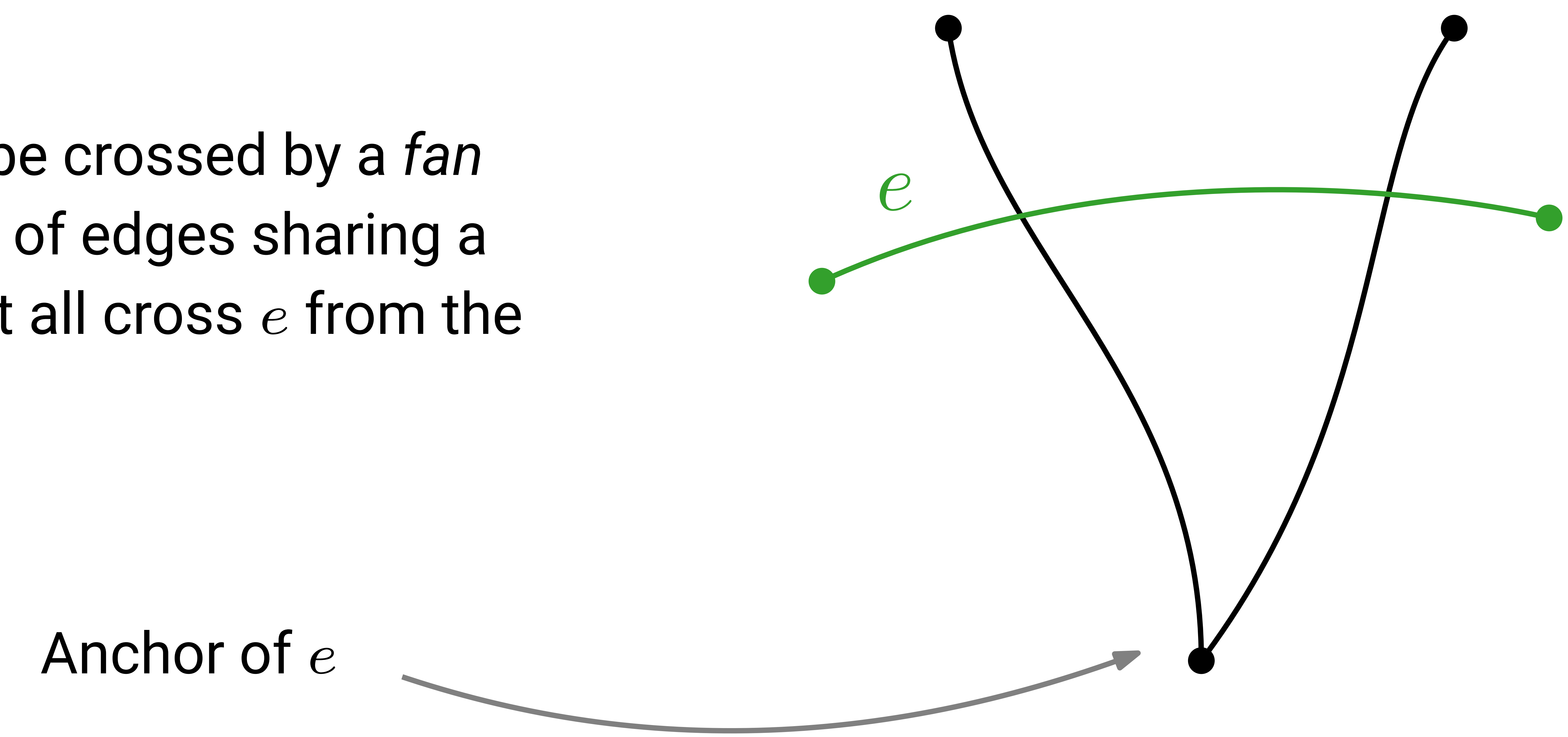
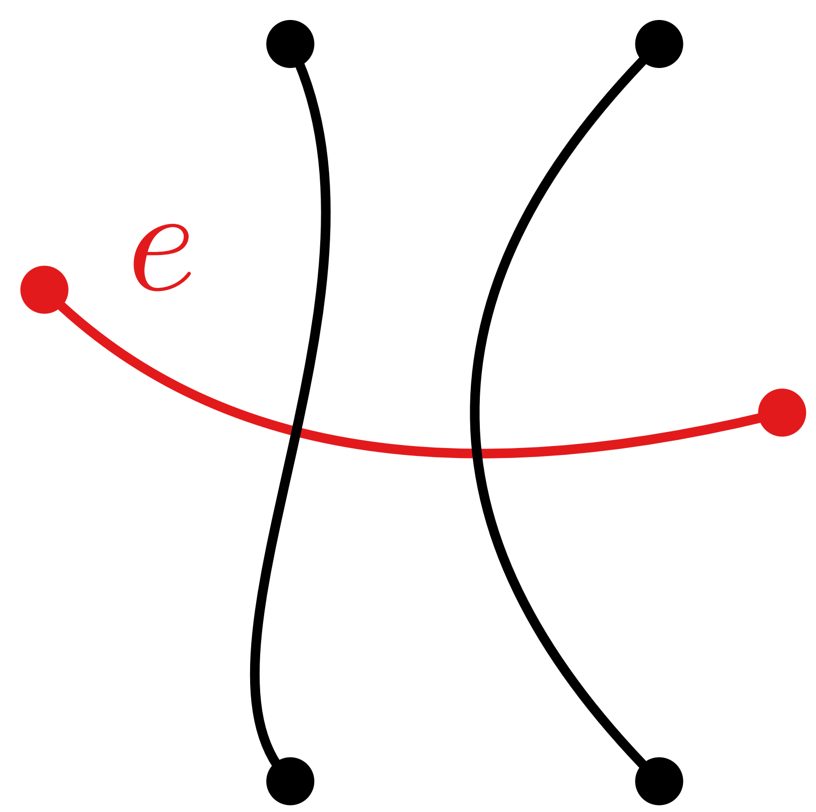
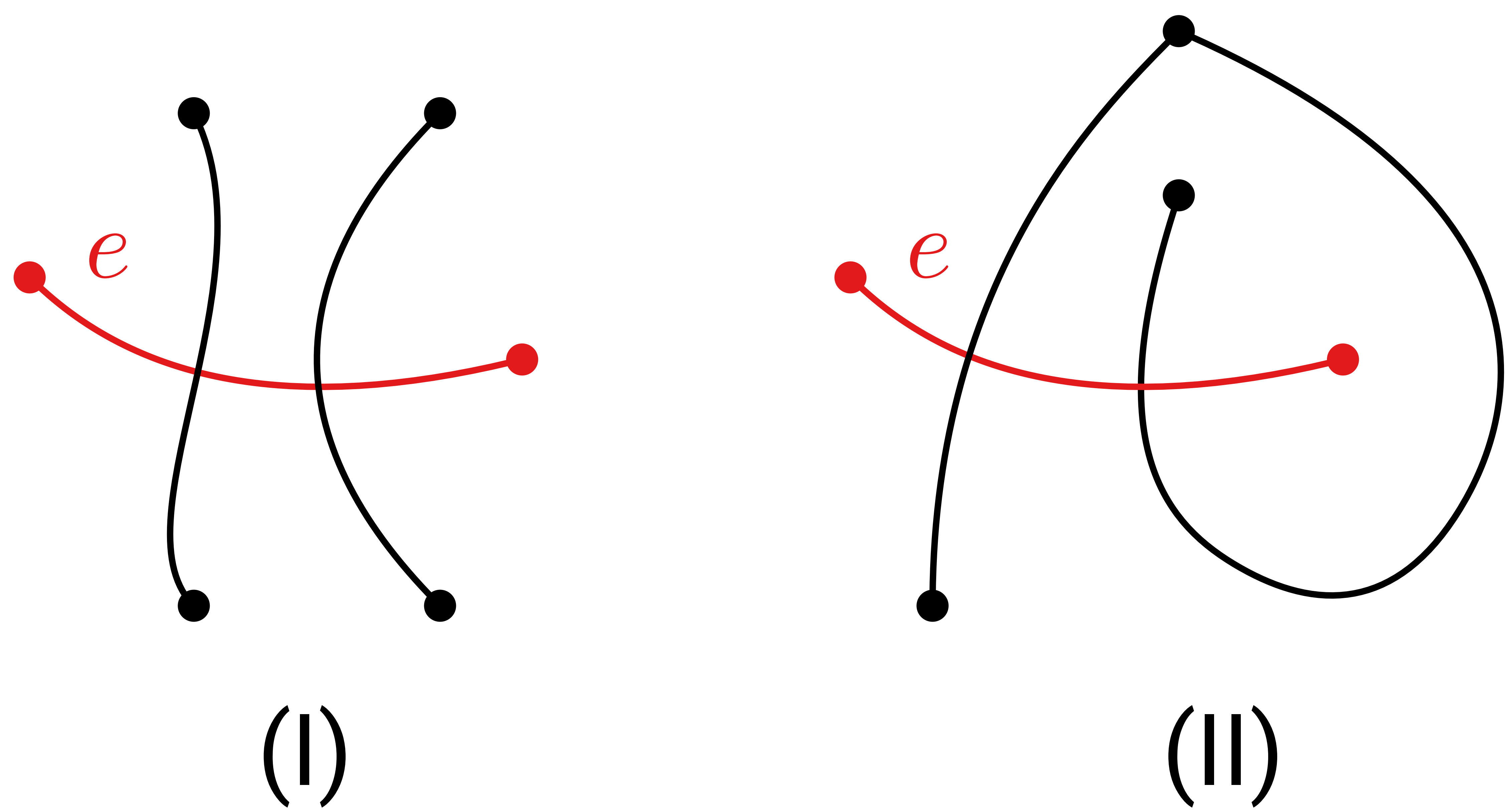


Fan-Planarity: Forbidden Crossing Patterns



(I)

Fan-Planarity: Forbidden Crossing Patterns



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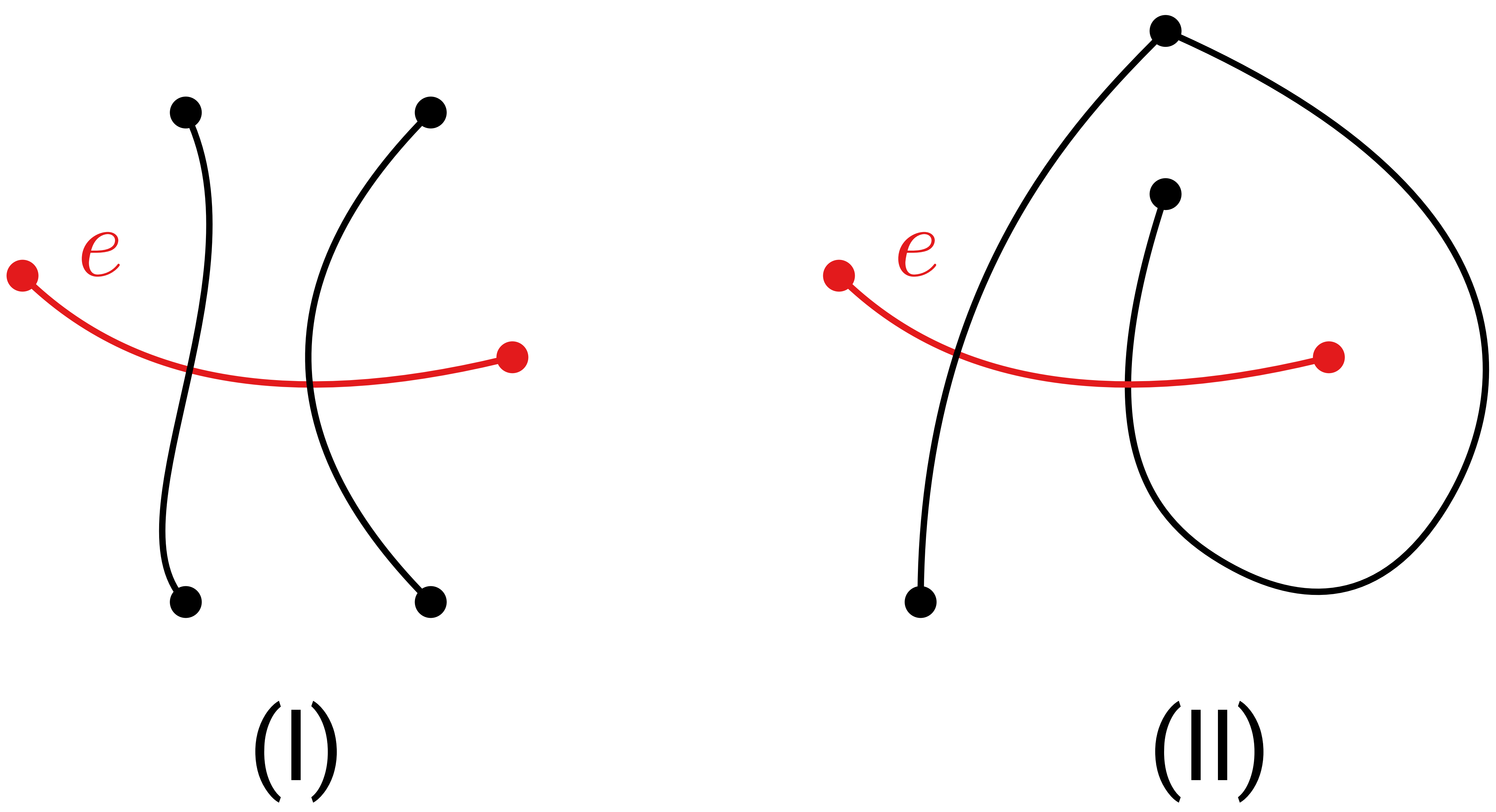

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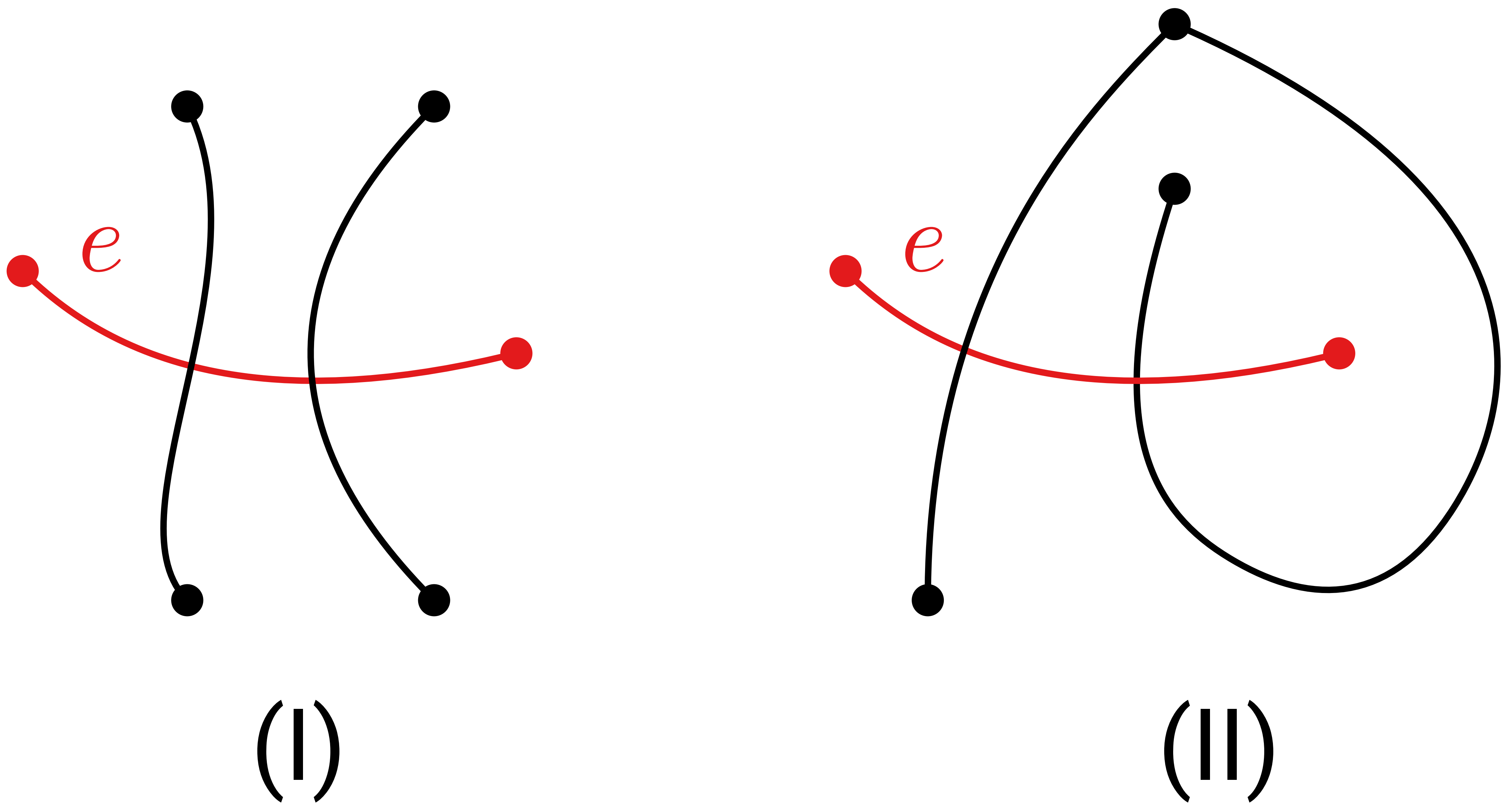

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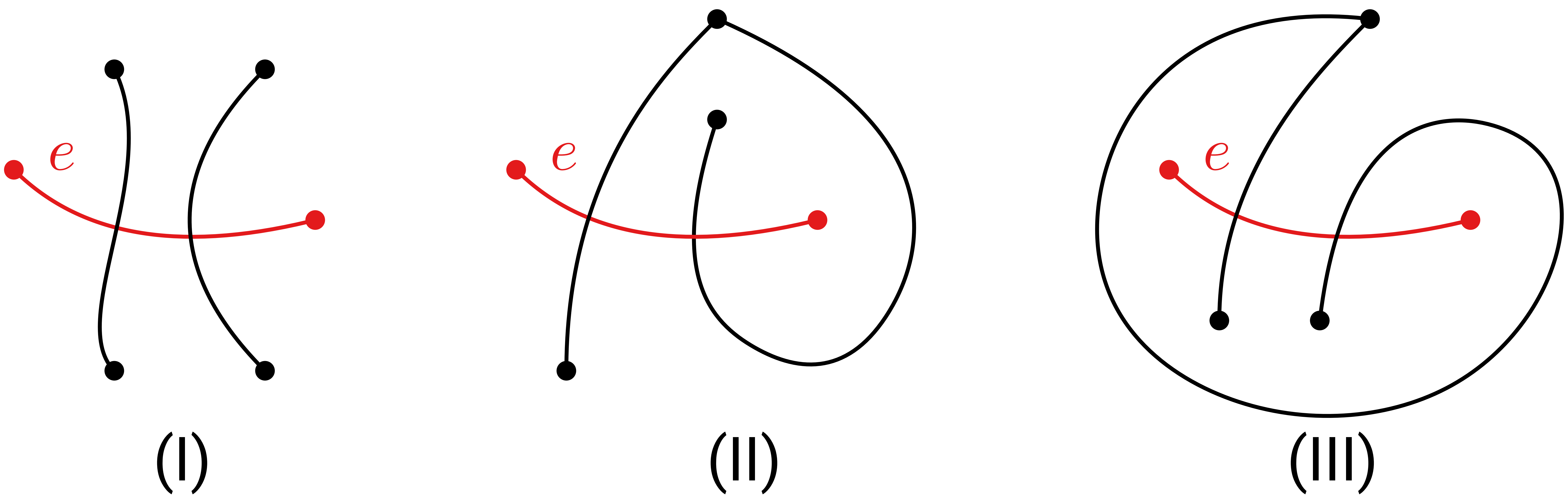

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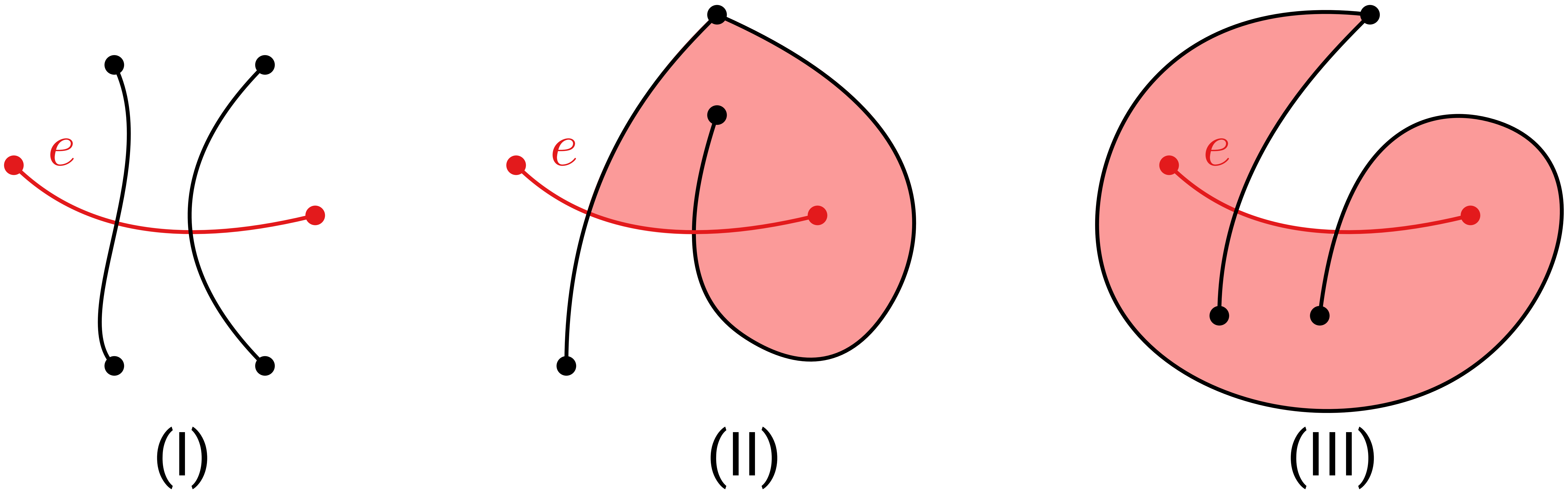

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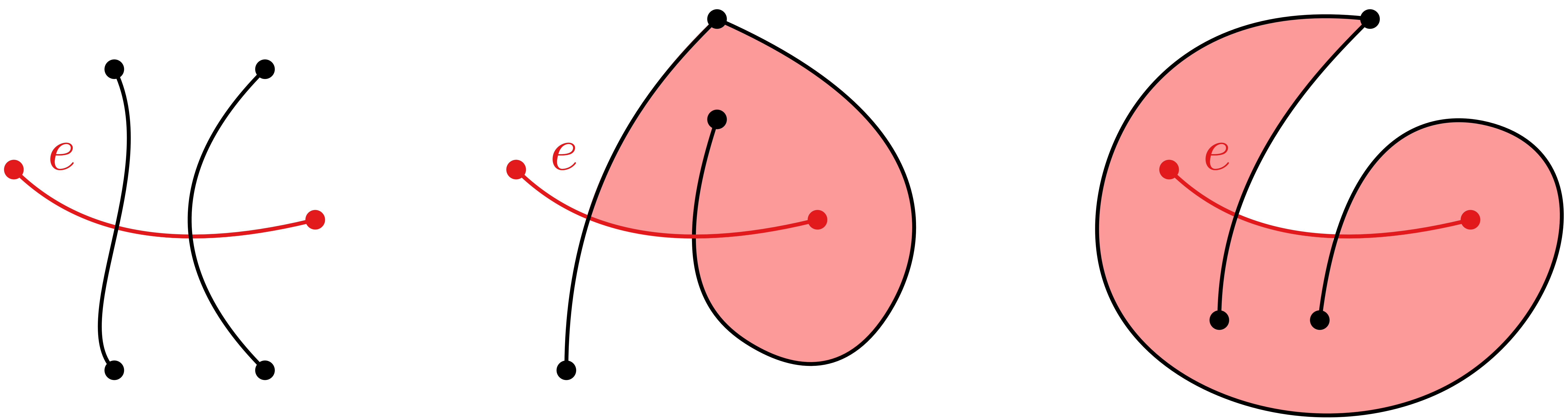

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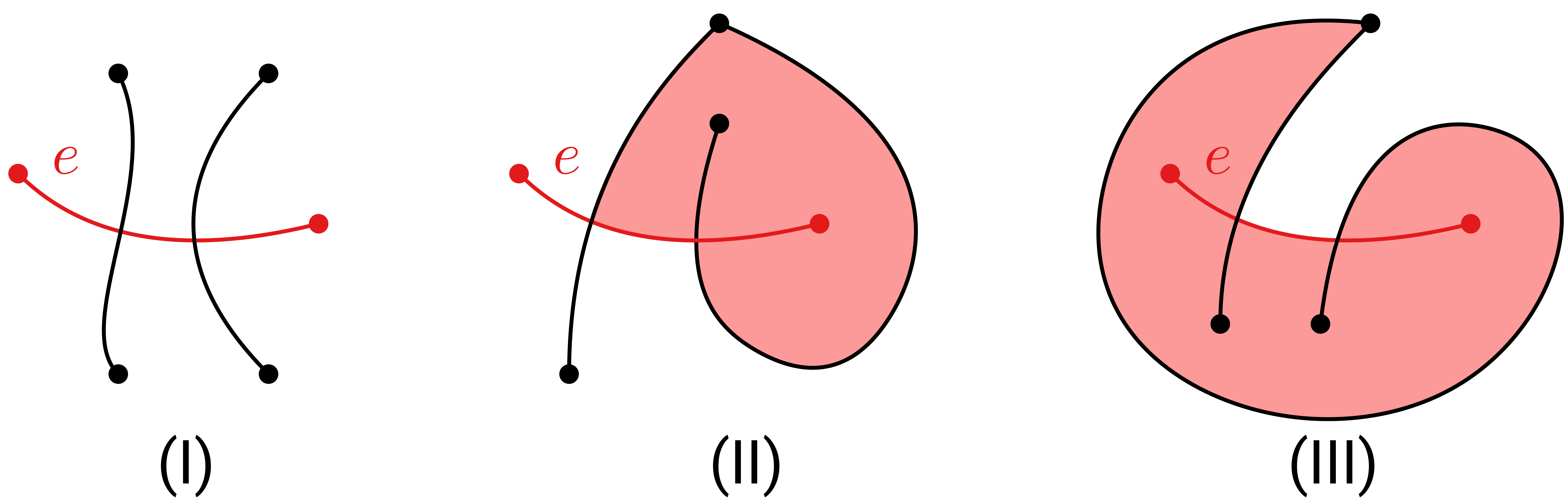
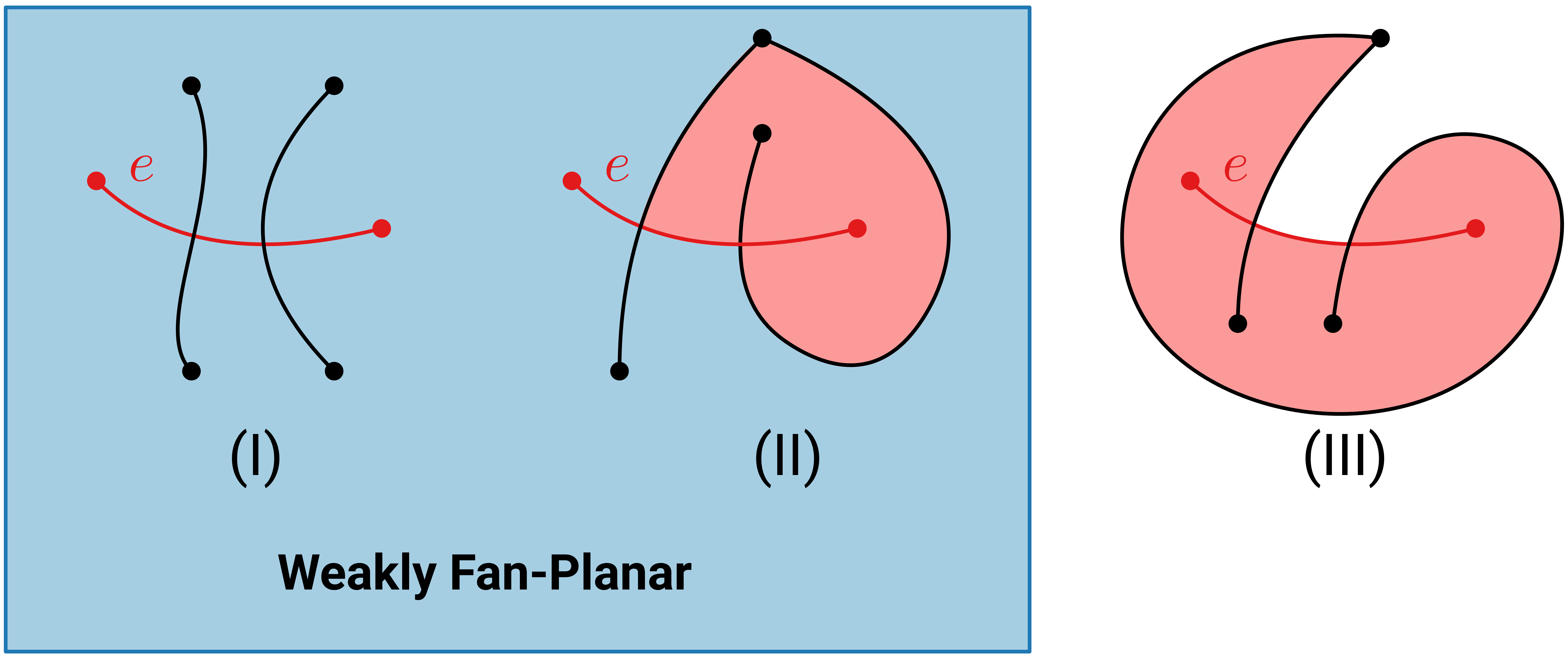


Fan-Planarity: Forbidden Crossing Patterns

[Cheong et al., 2022]



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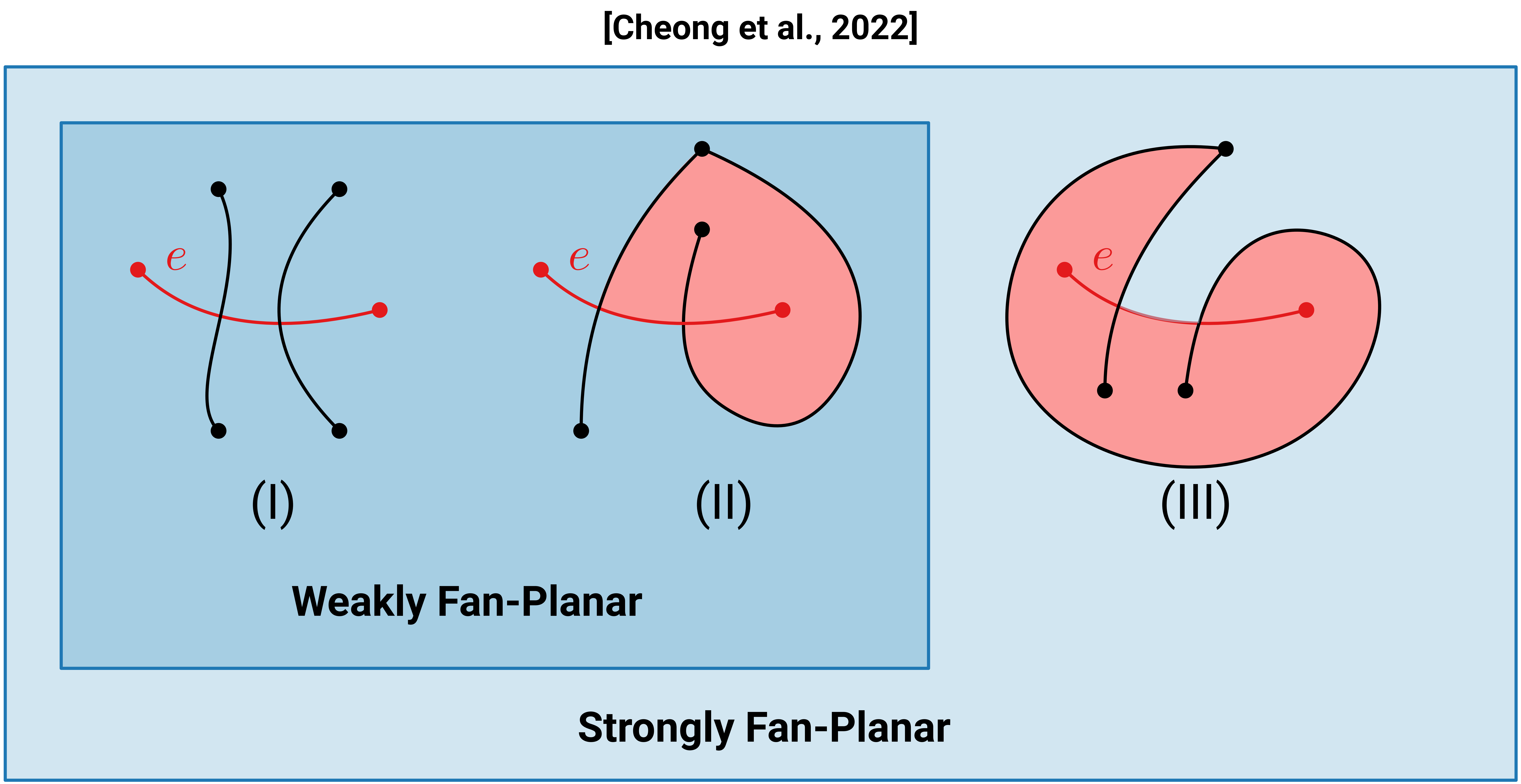

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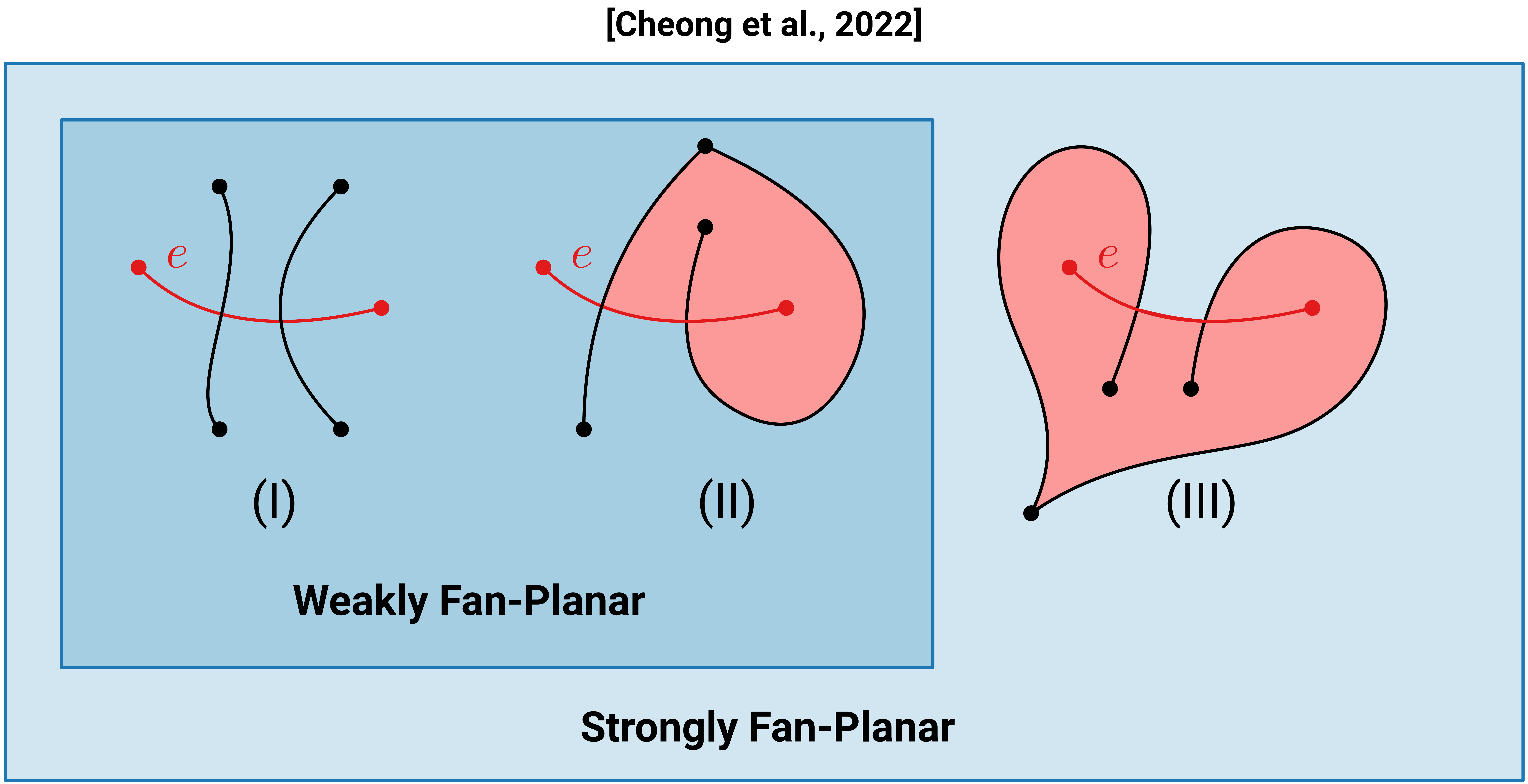
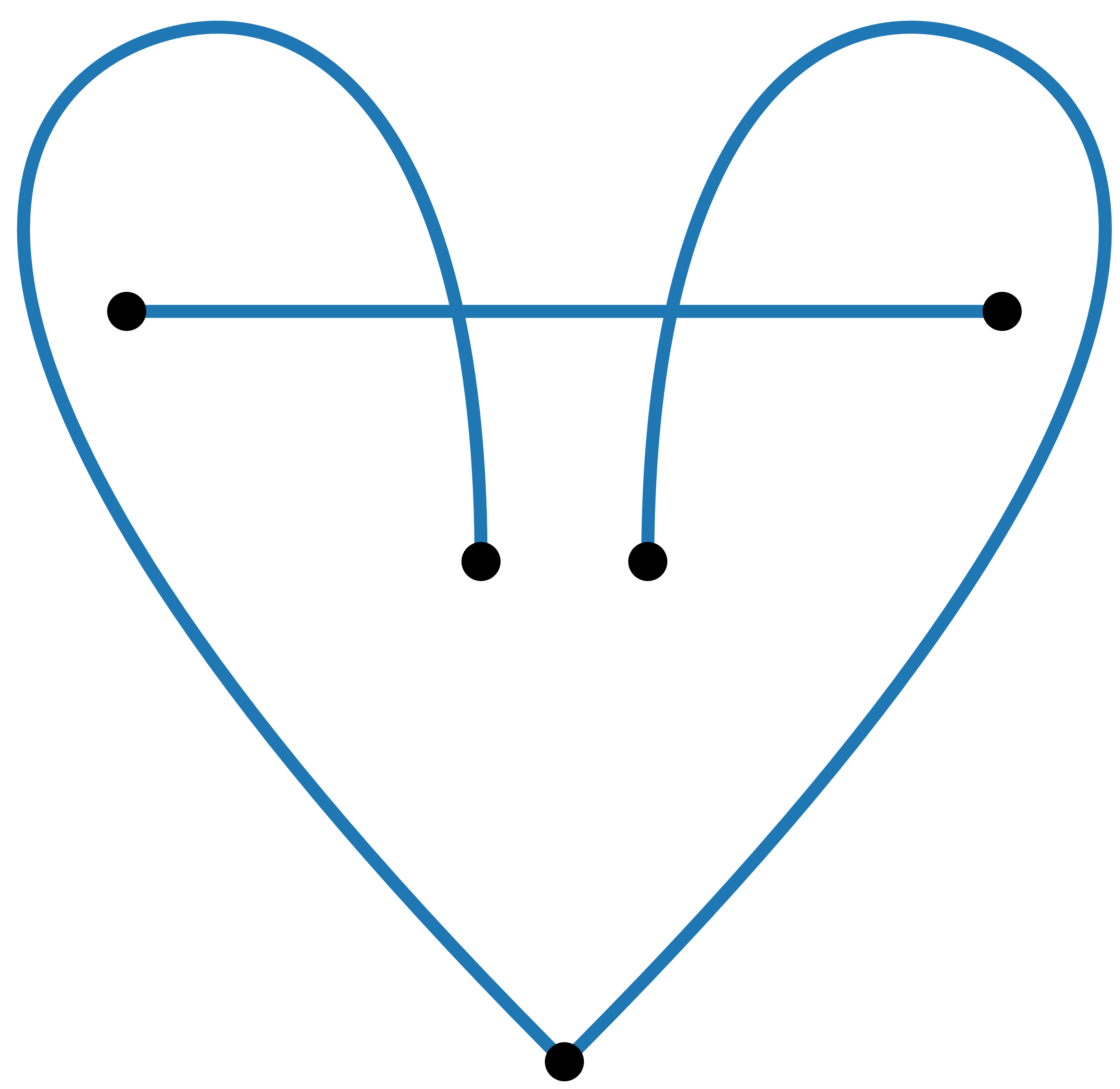


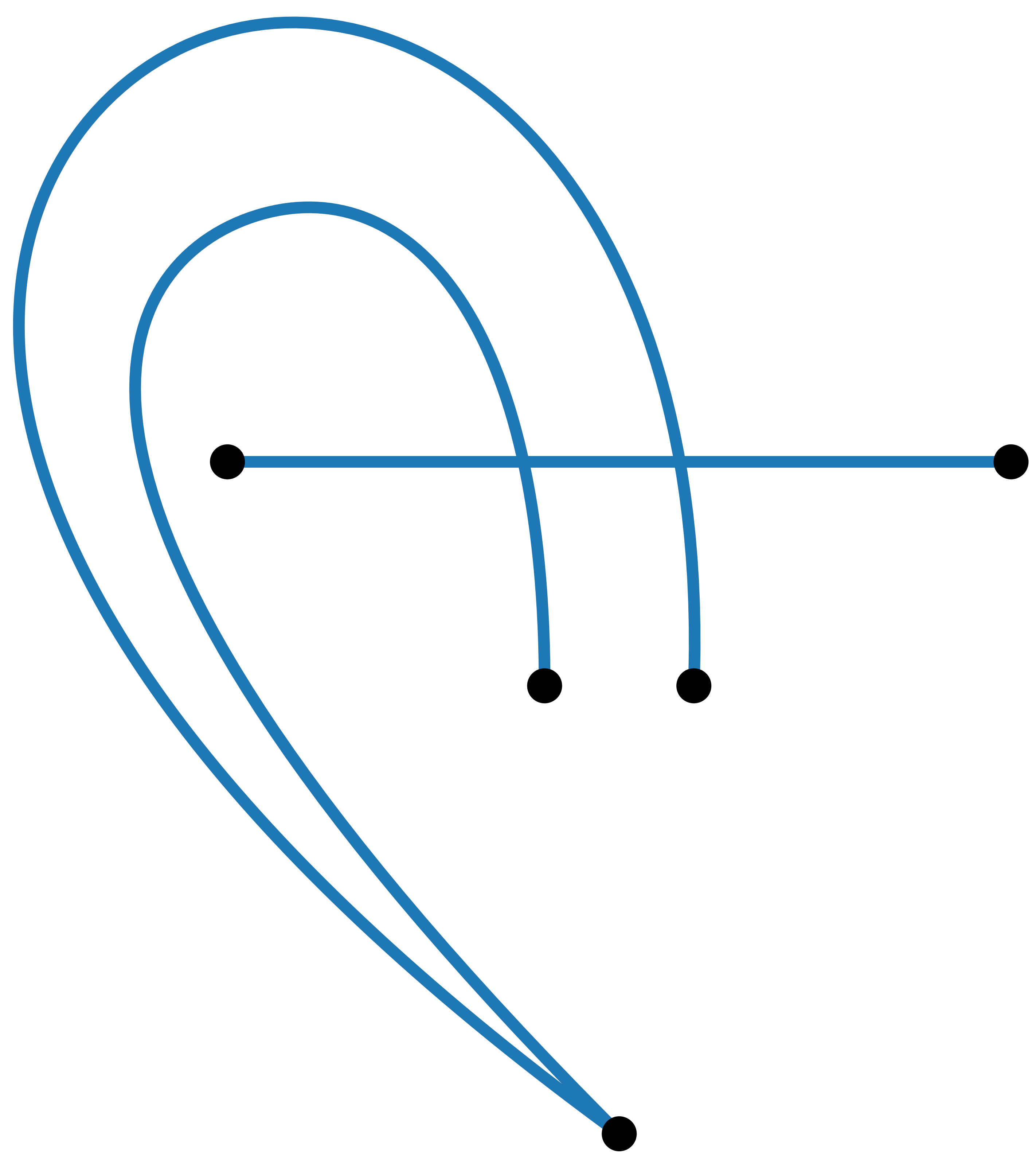
Do the graph classes of
weak & strong fan-planarity coincide?

Weak Fan-Planarity: Forcing Hearts



We start with a pattern (III) ...

Weak Fan-Planarity: Forcing Hearts



We start with a pattern (III)
and want to prevent an **edge flip**

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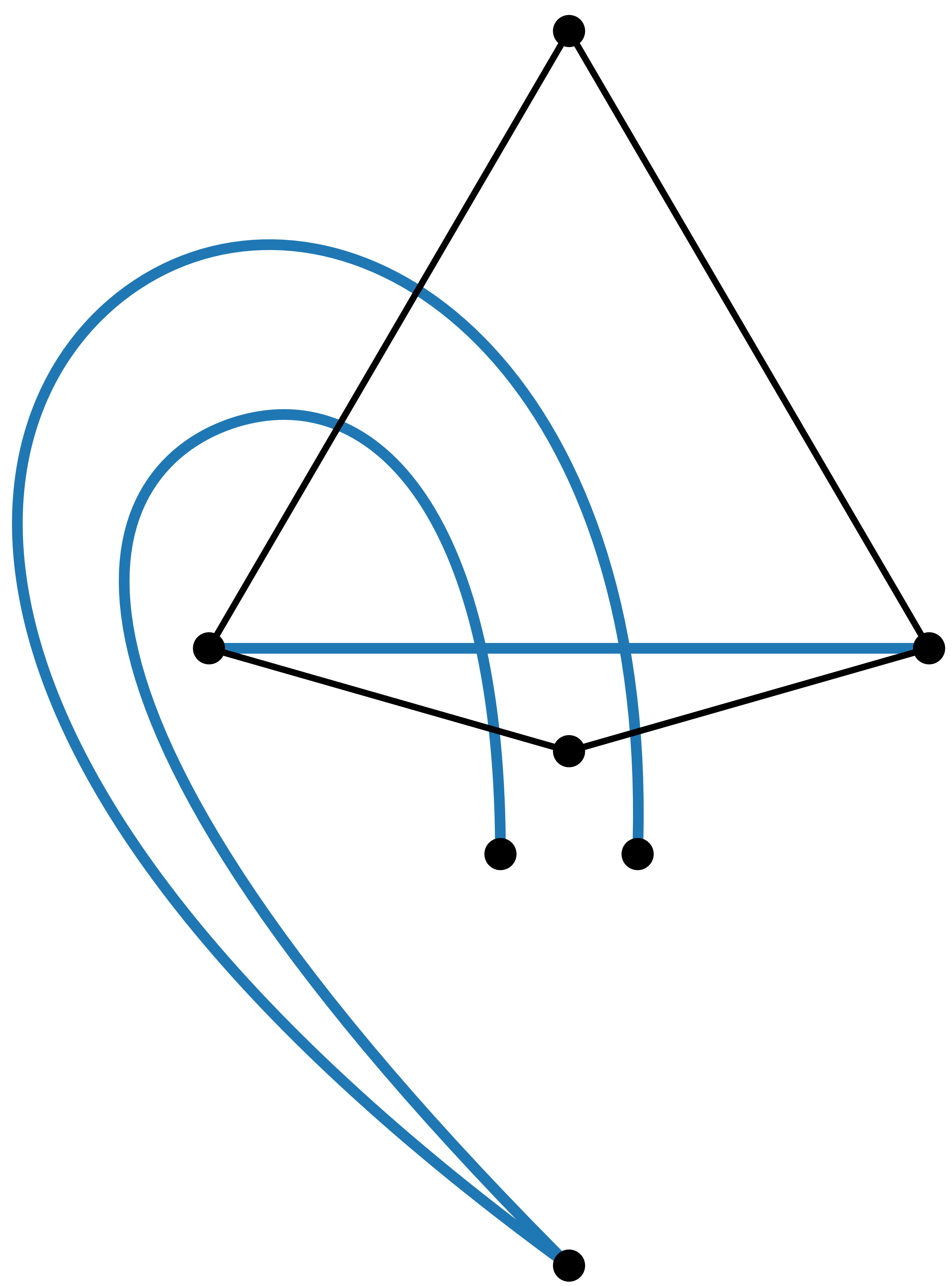

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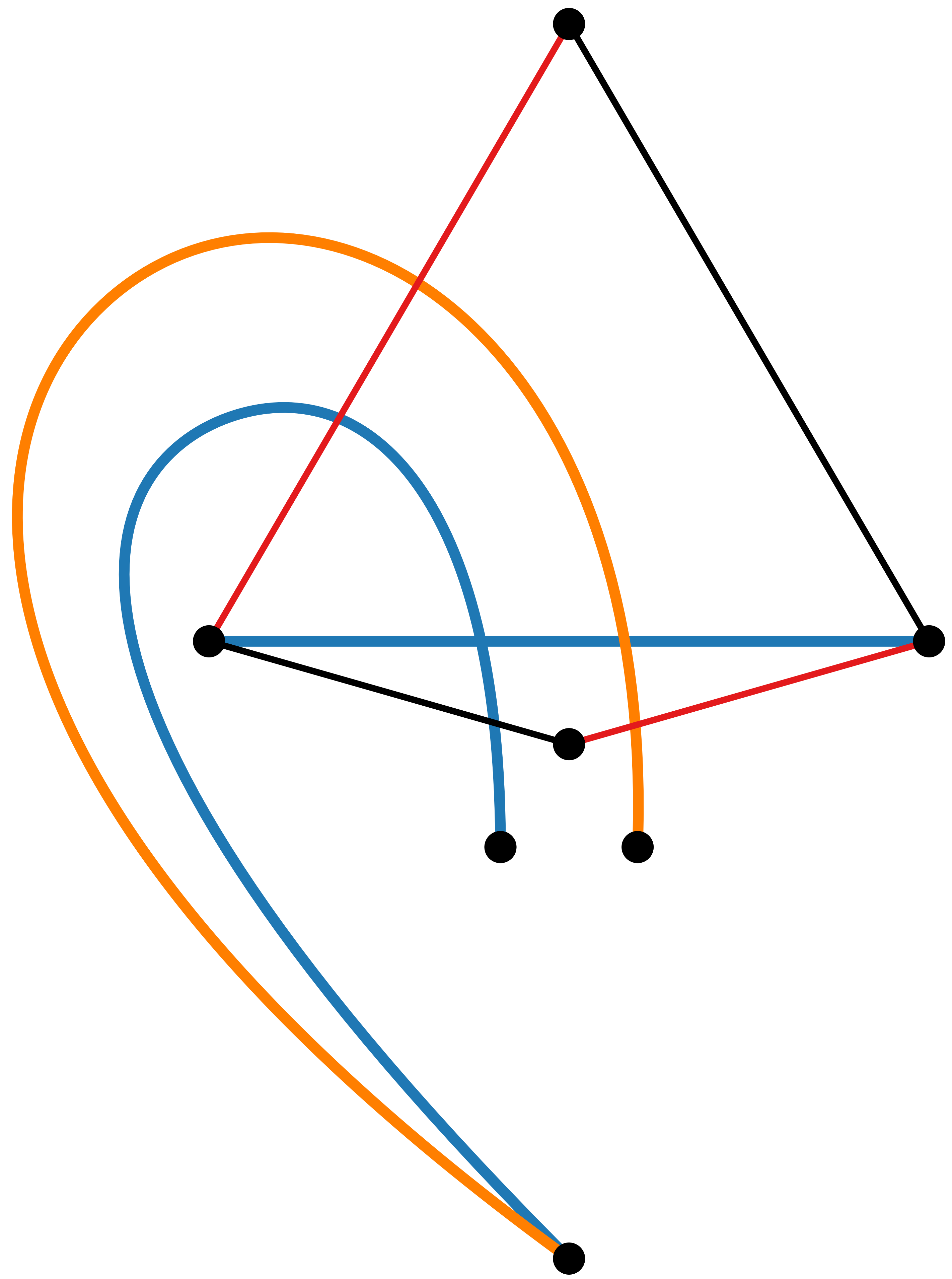

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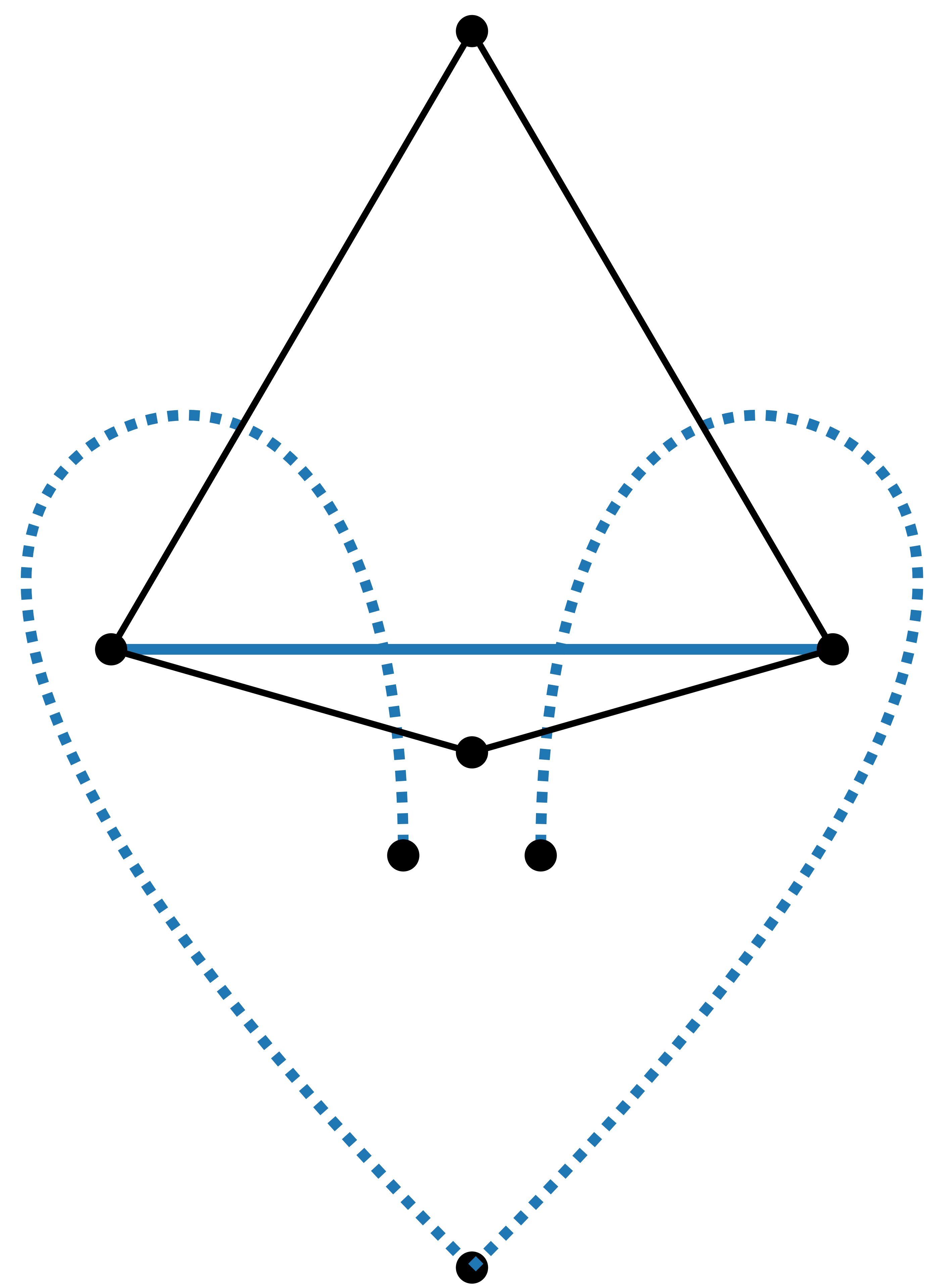

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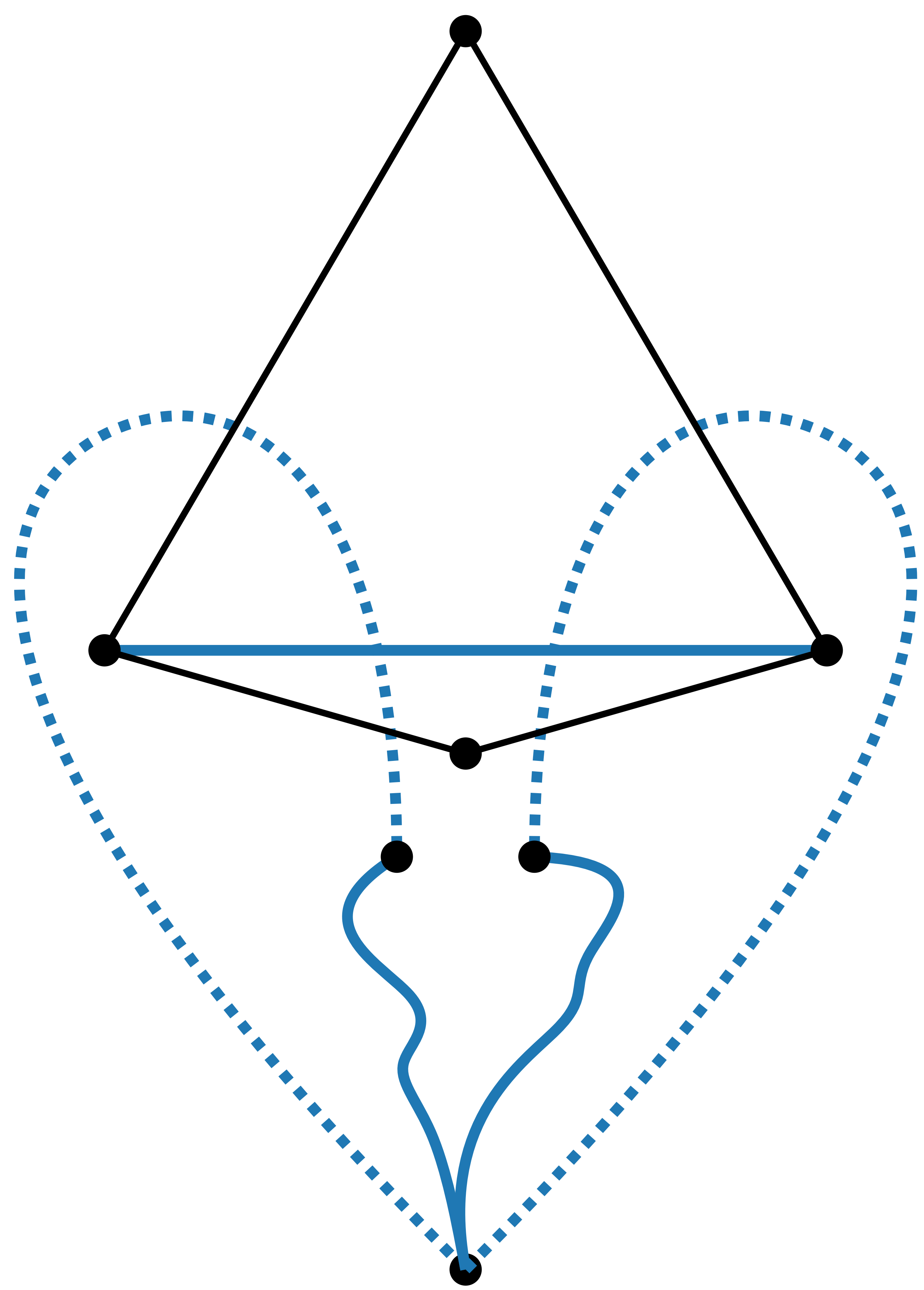

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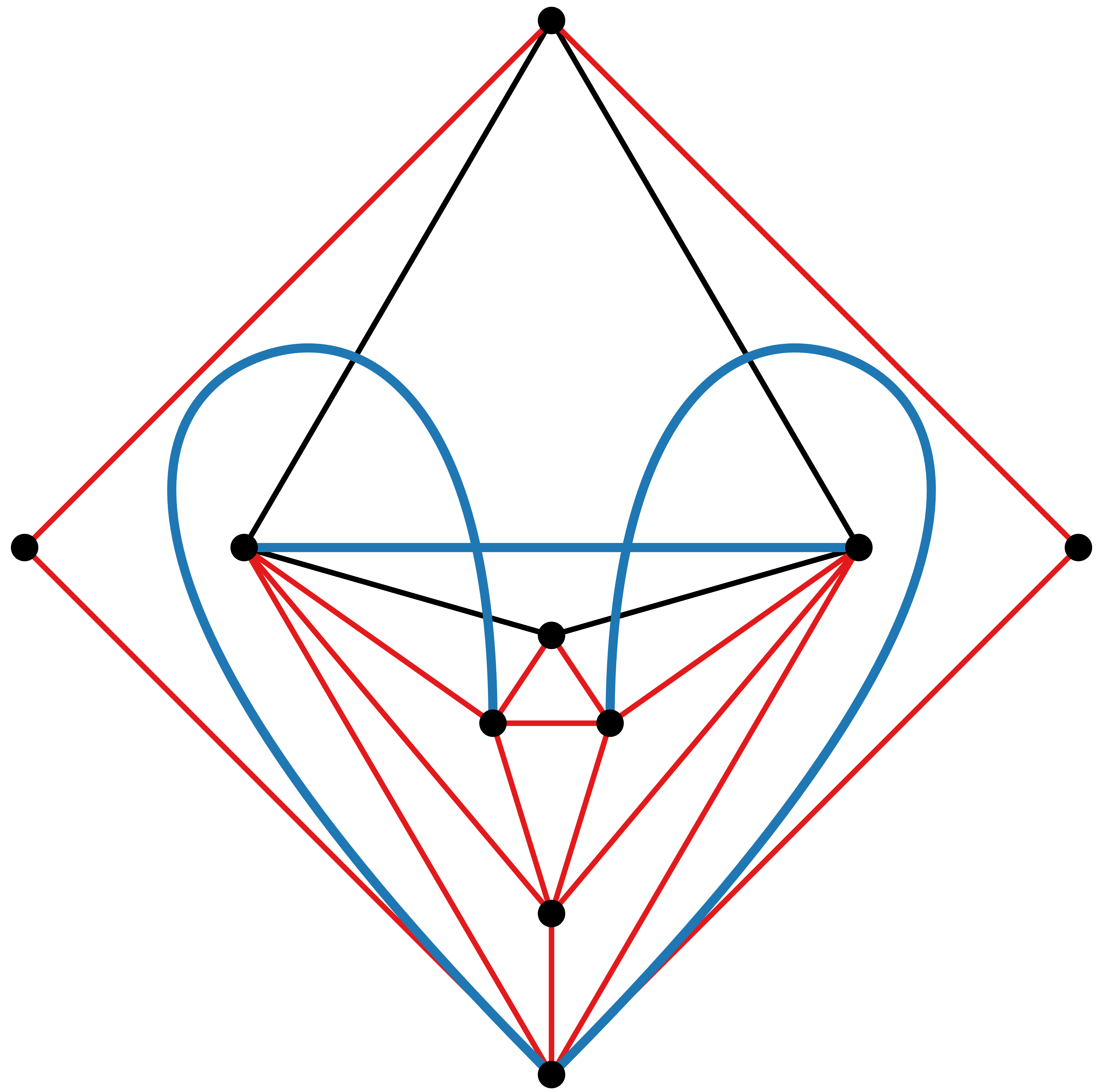

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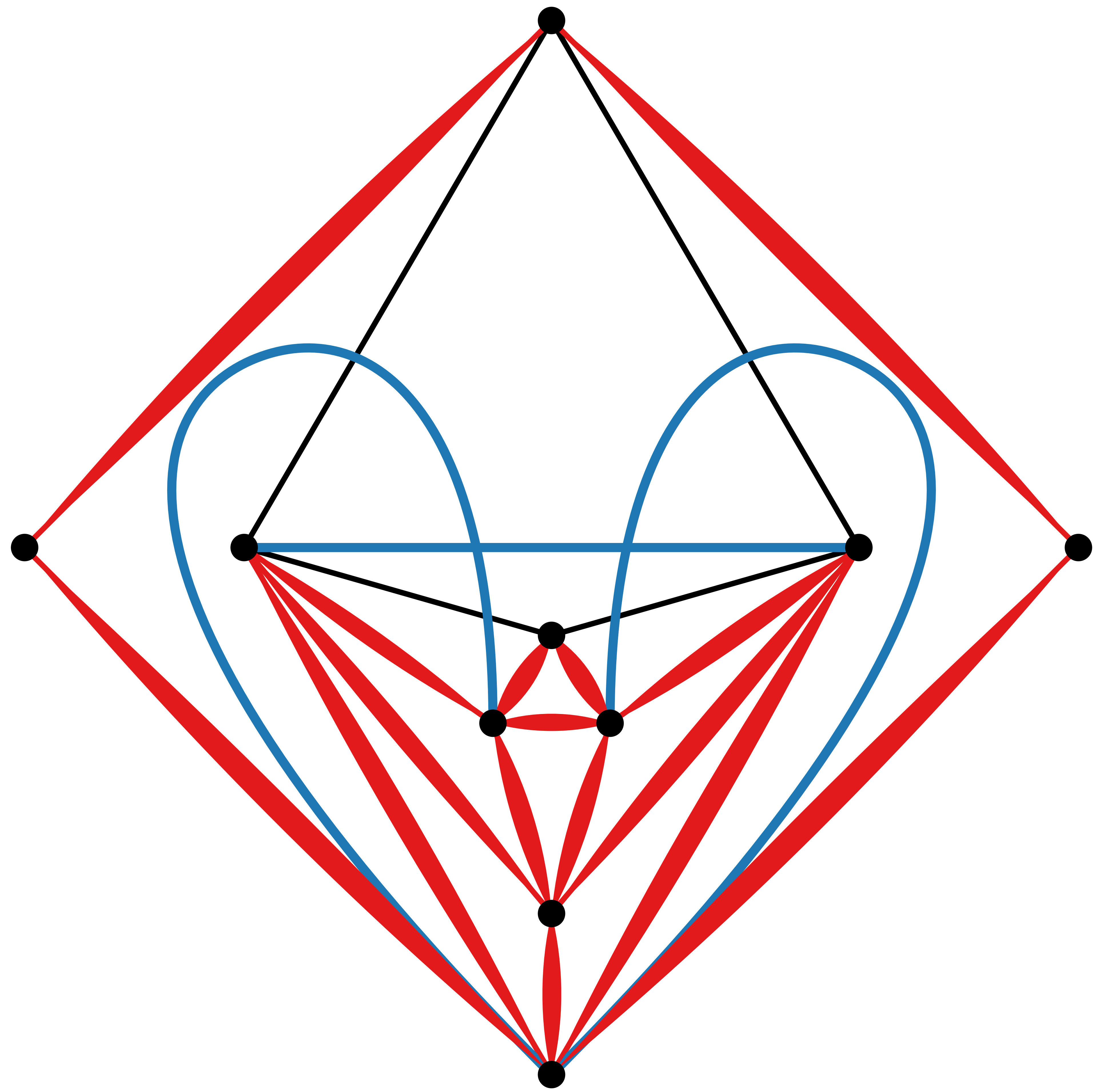

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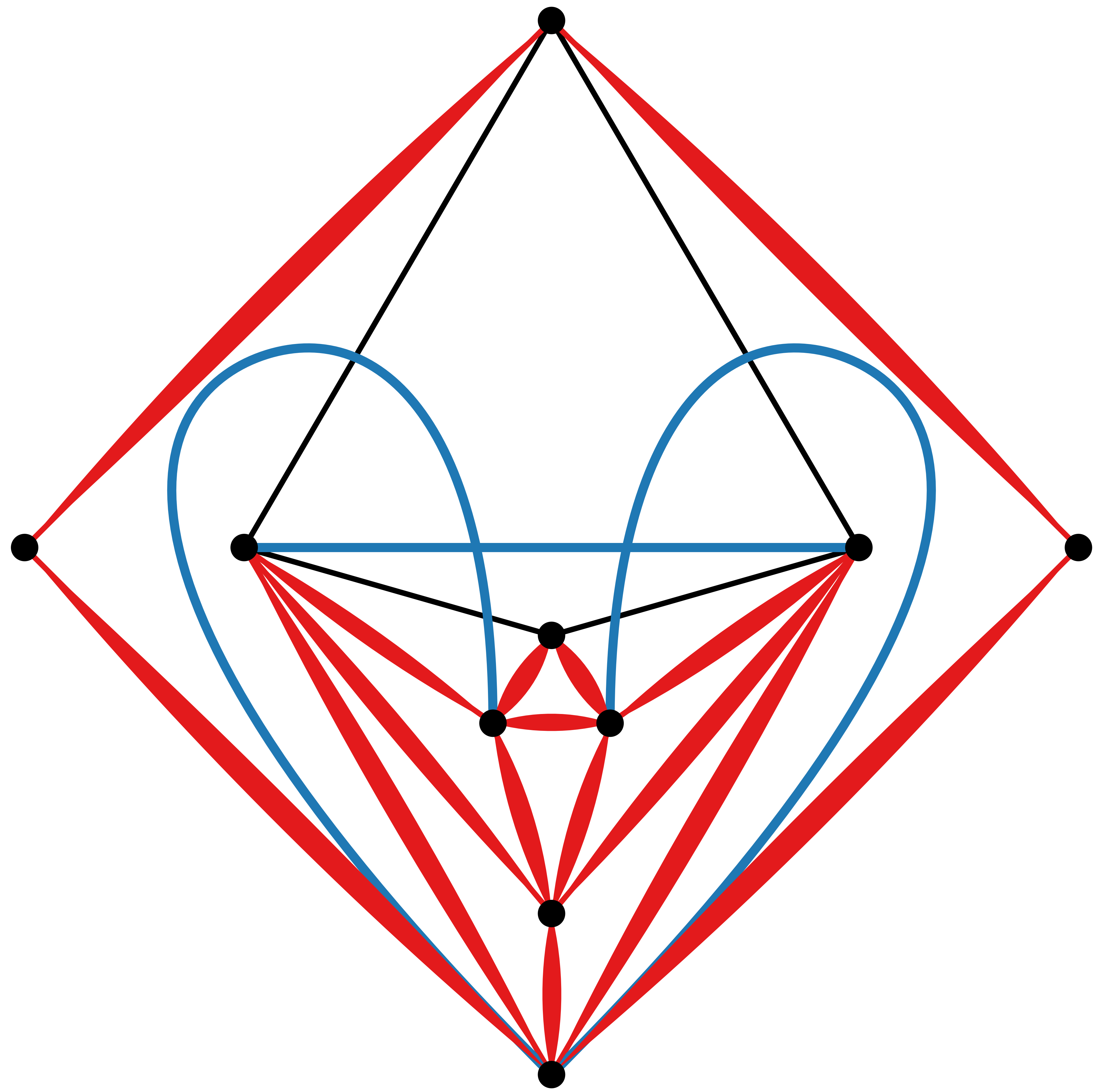
Red edges are replaced by K_7

Weak Fan-Planarity: Forcing Hearts



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Lemma [Binucci et al. 2015].
Let $\mathcal{P}$ be the planarization of any weakly fan-planar drawing of K_7. Then, between any pair of vertices of K_7, there exists a path in $\mathcal{P}$ that contains no real edge of K_7.

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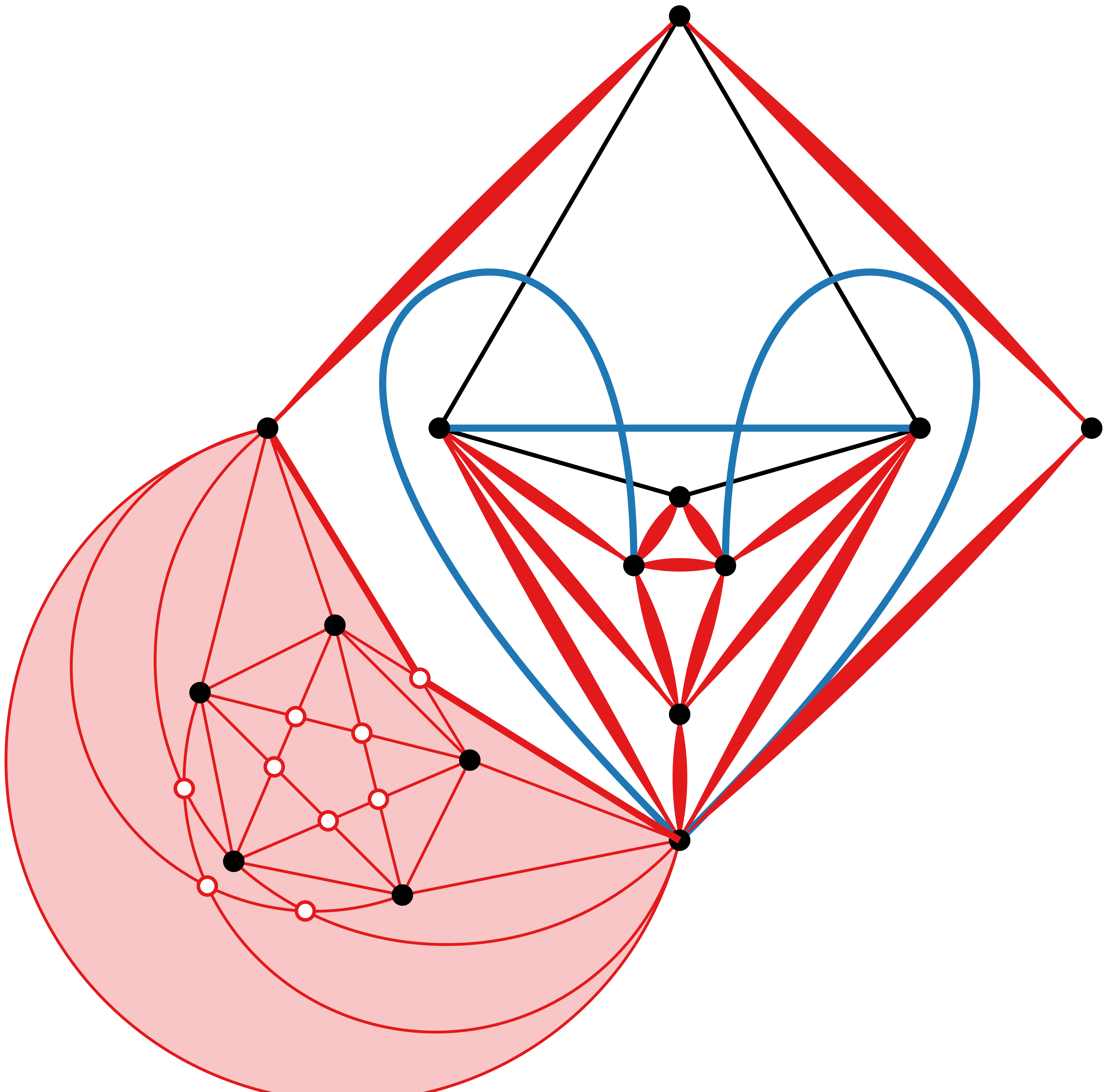

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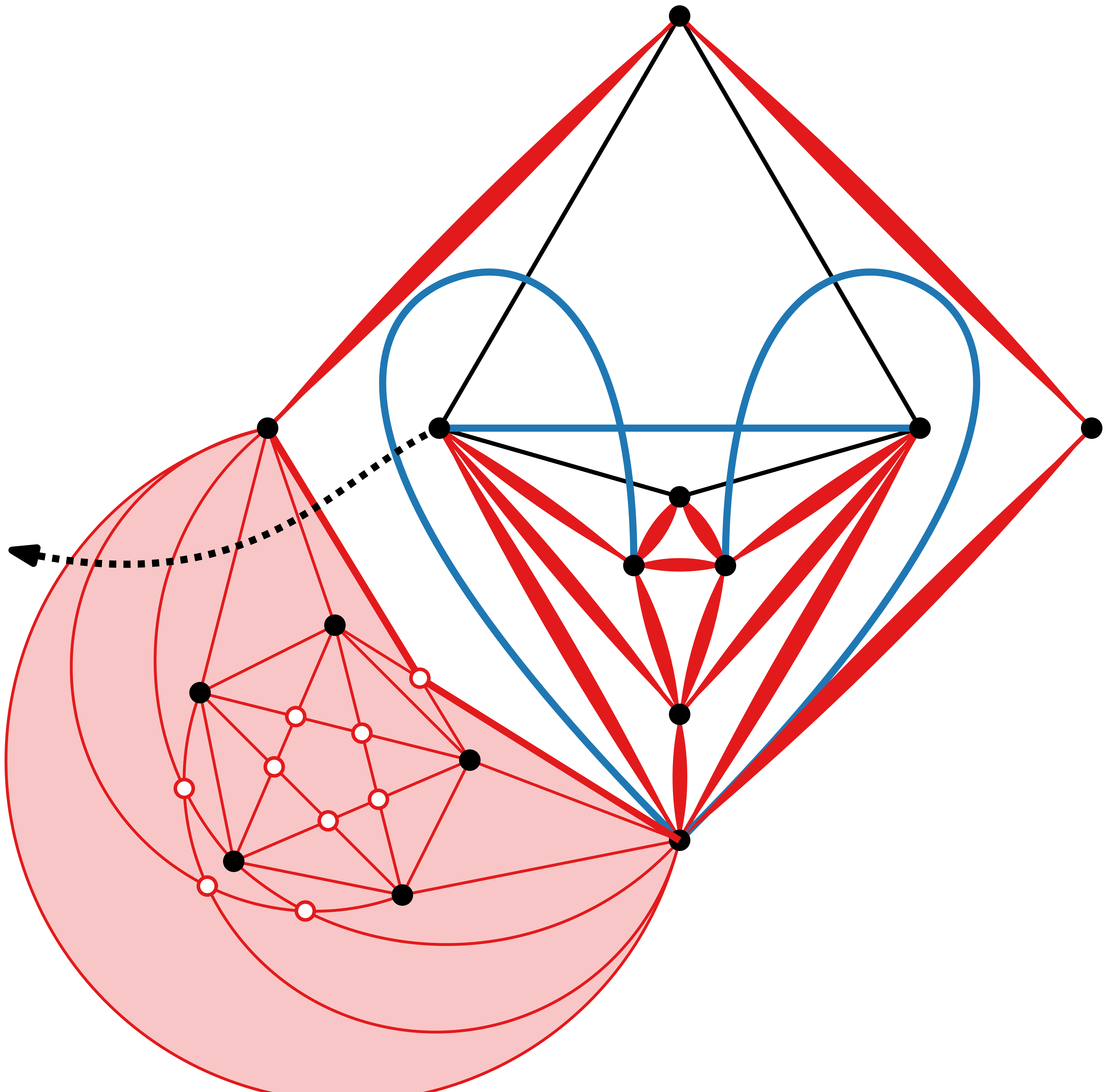

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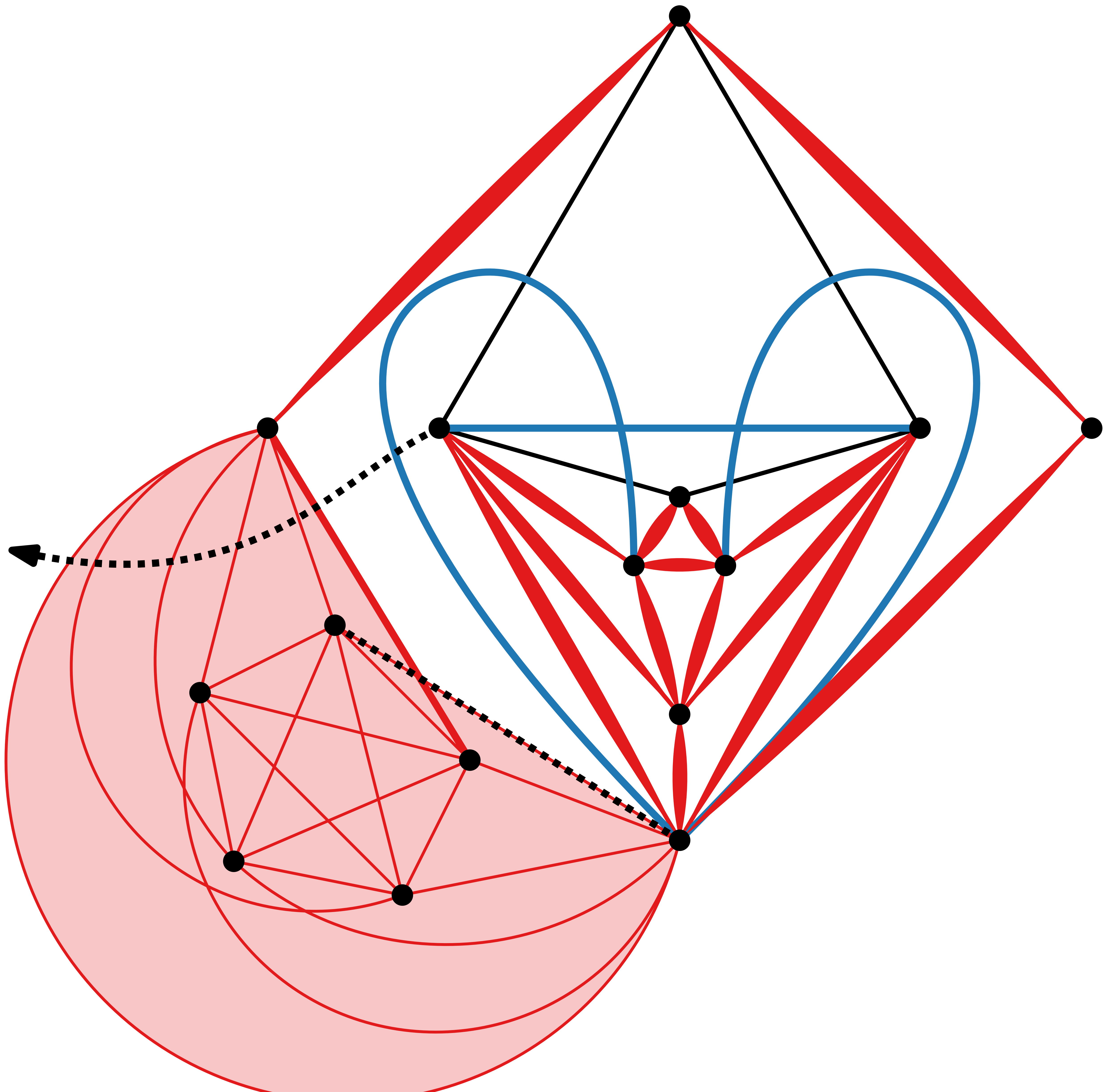

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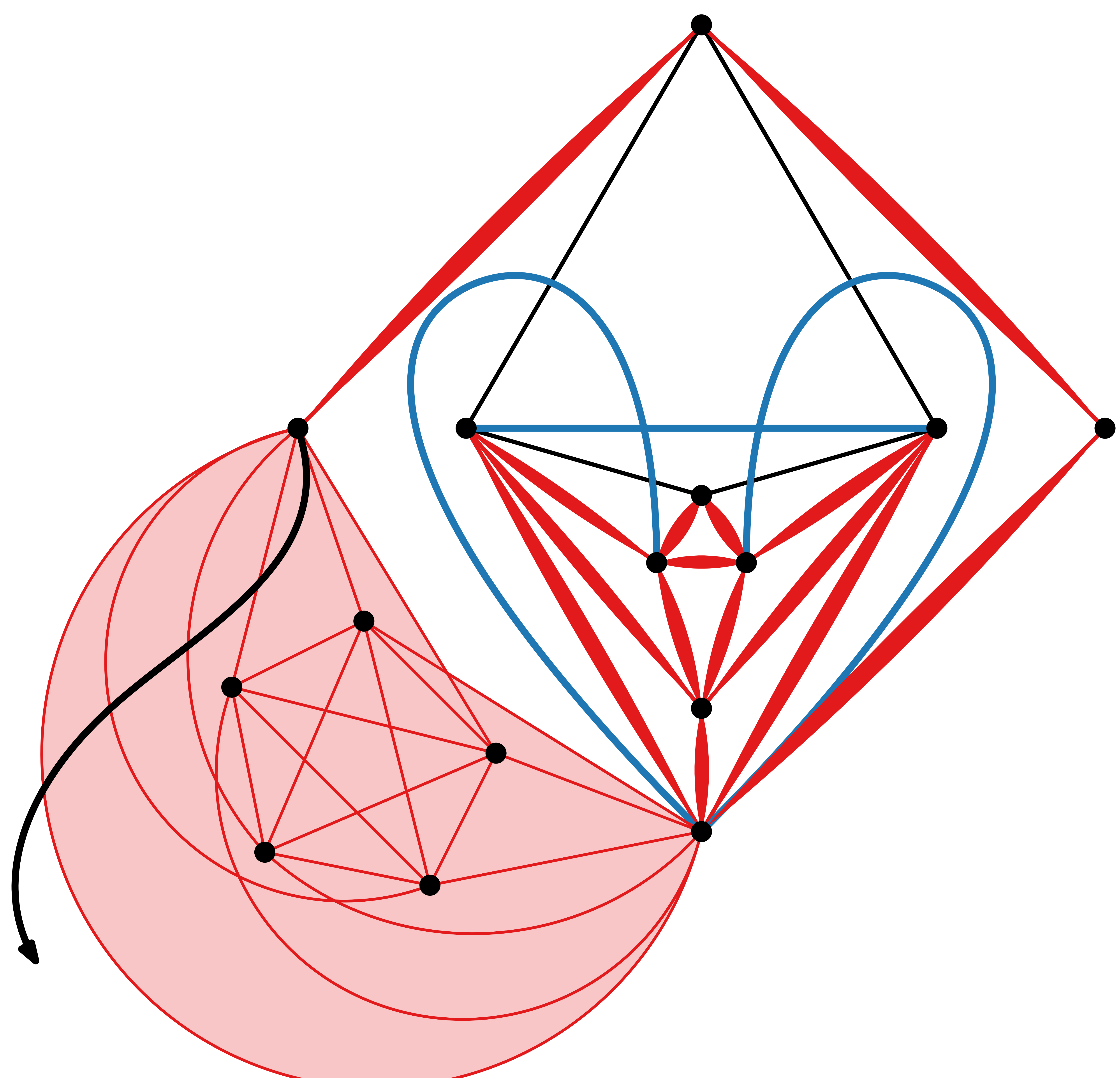

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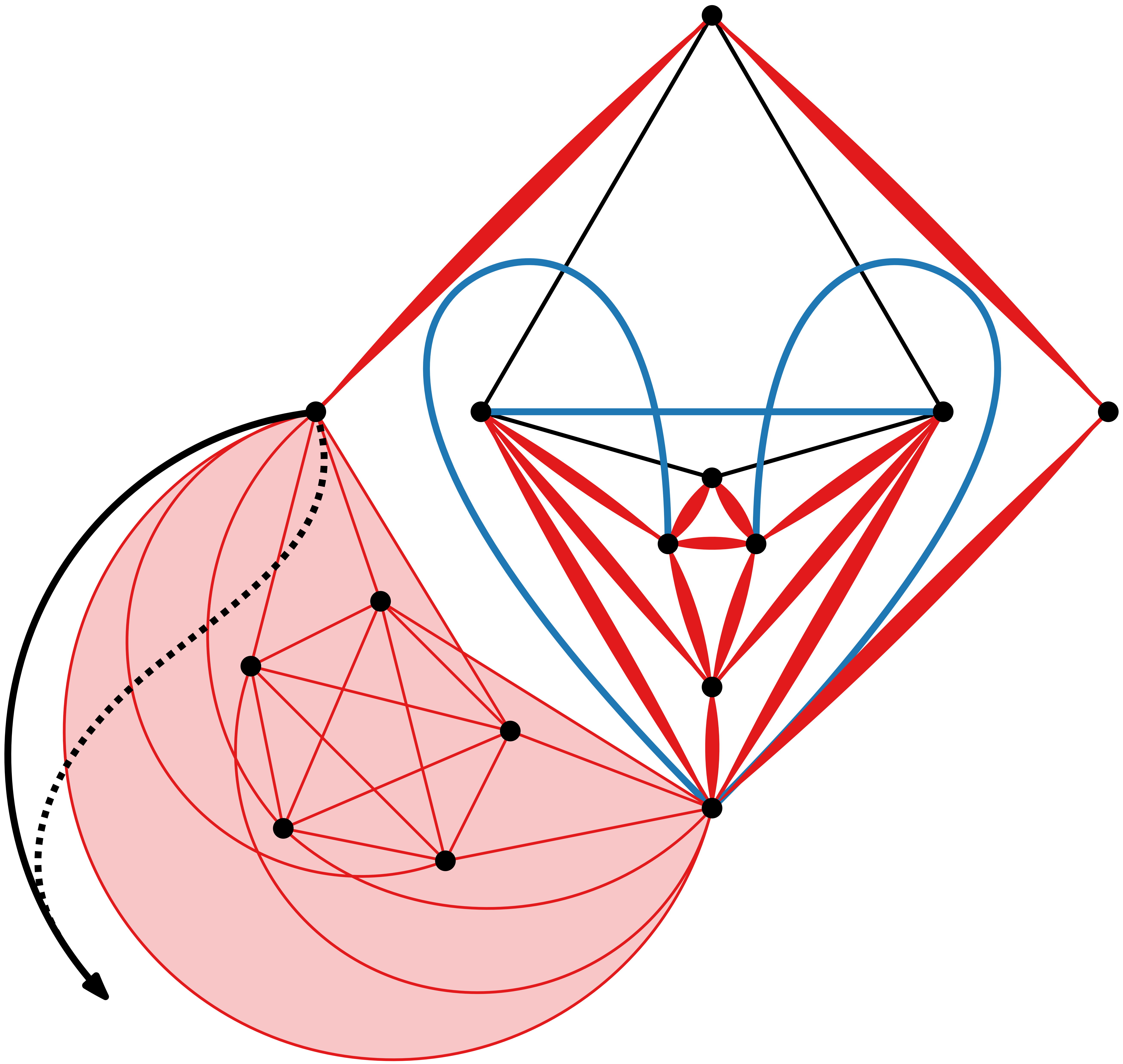

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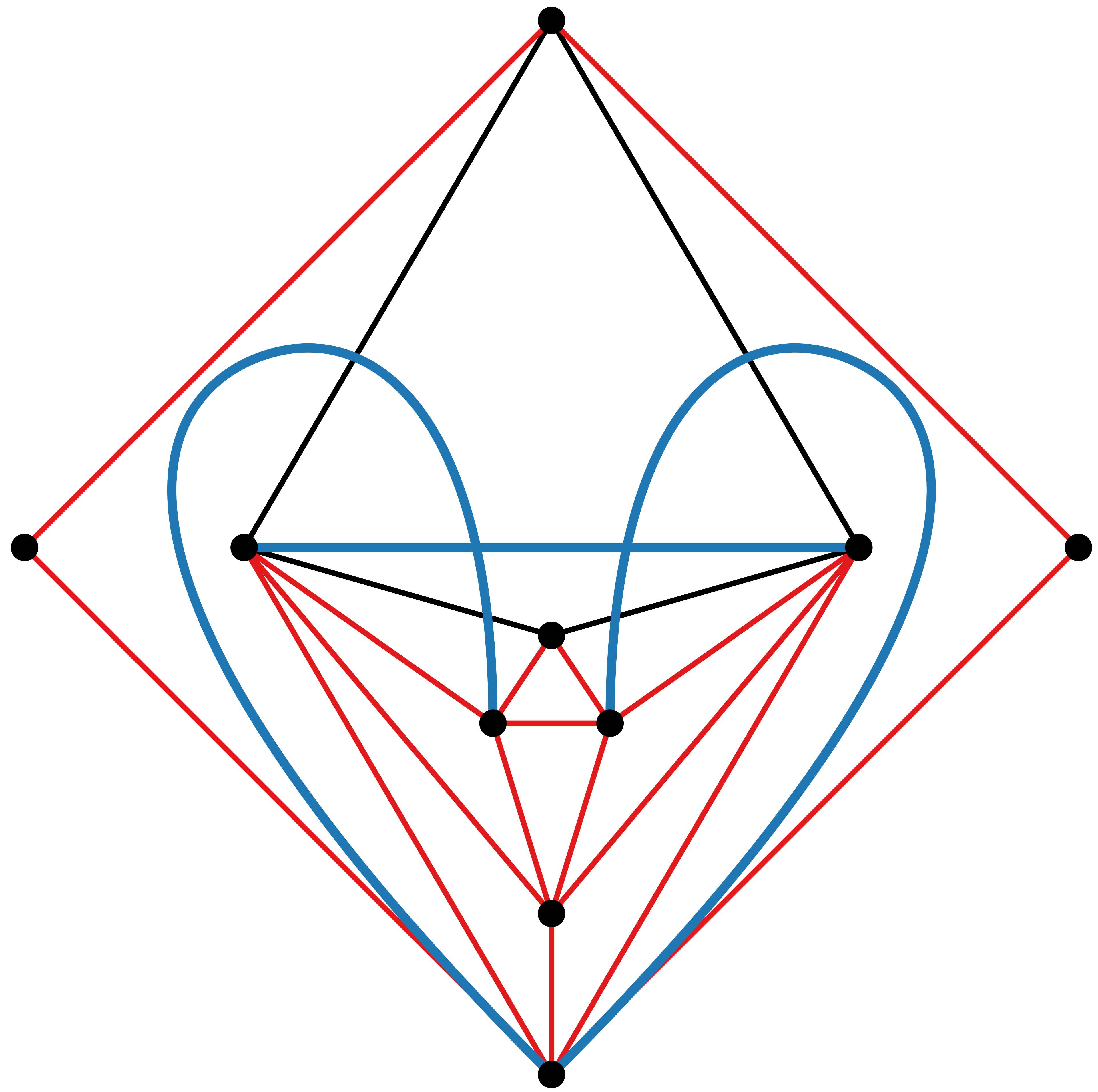

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... equivalent to a graph G' where
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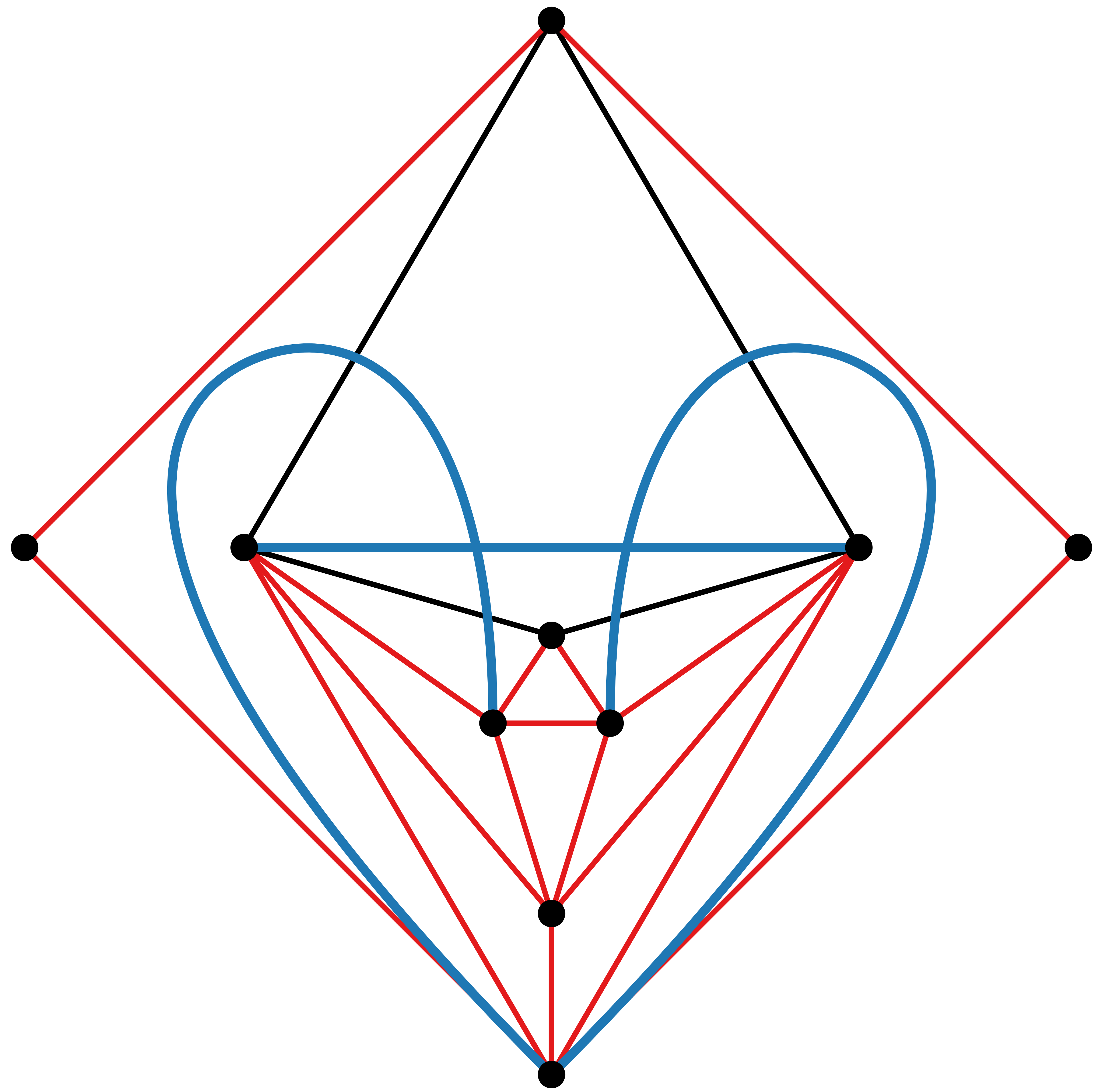

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We could still choose a
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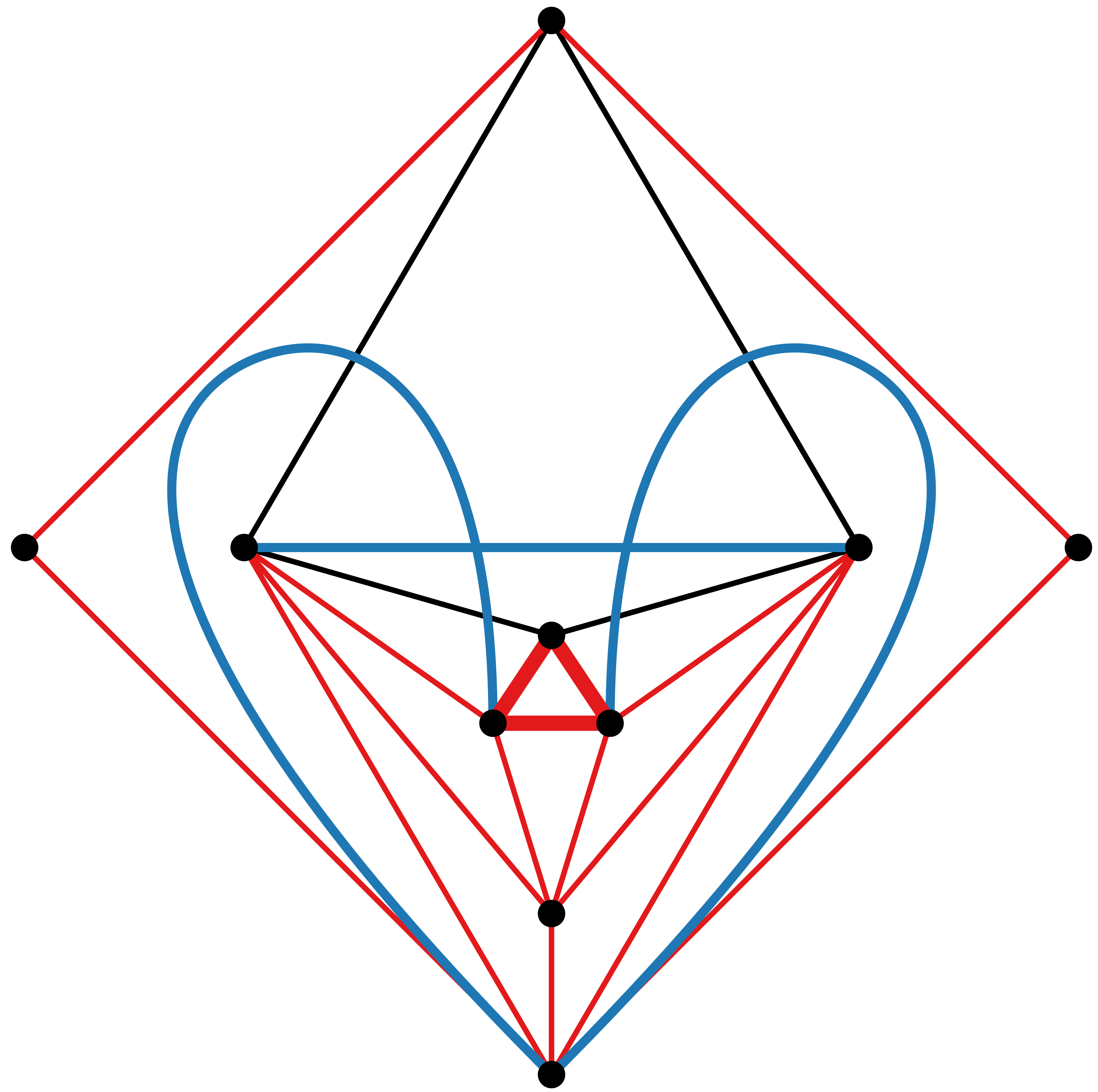

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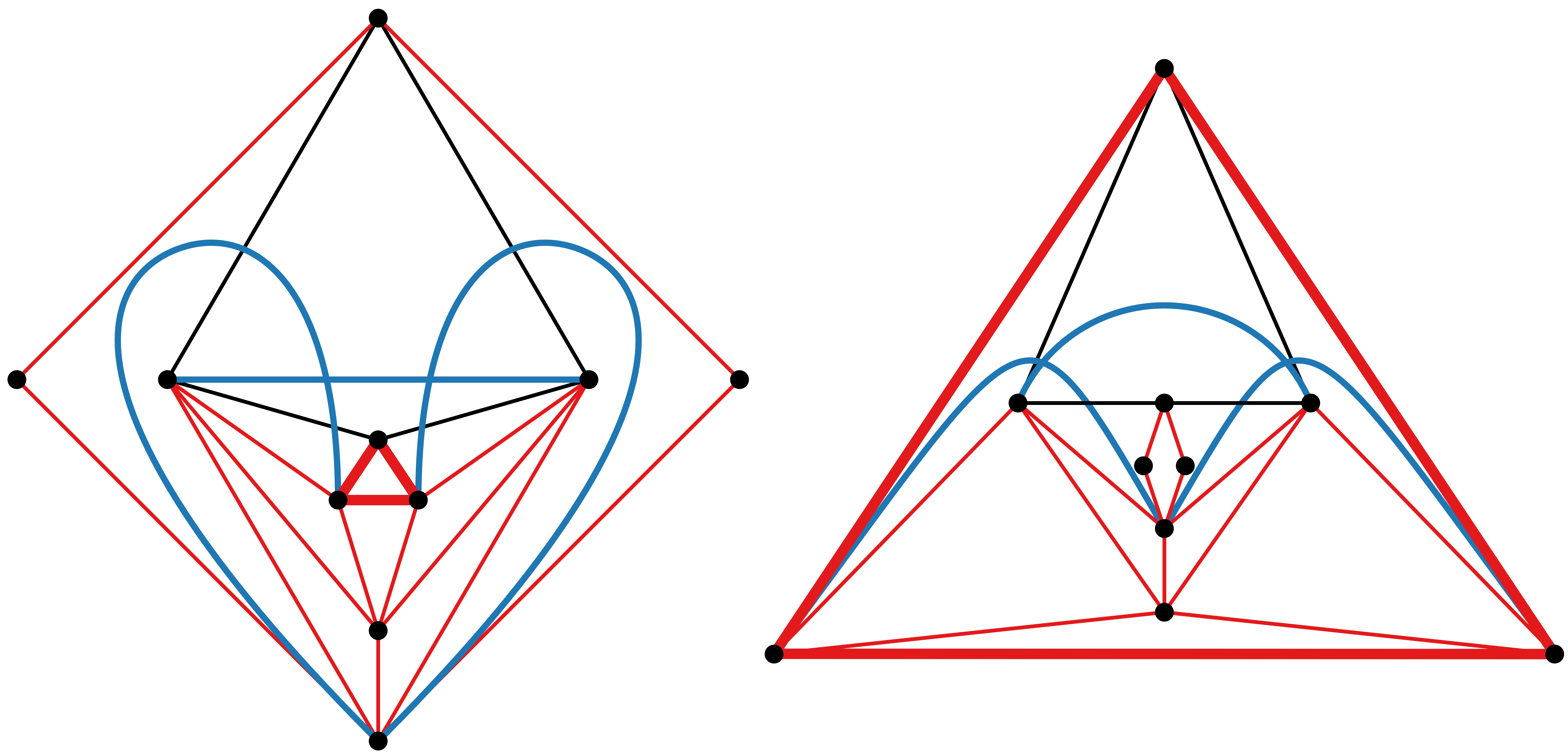

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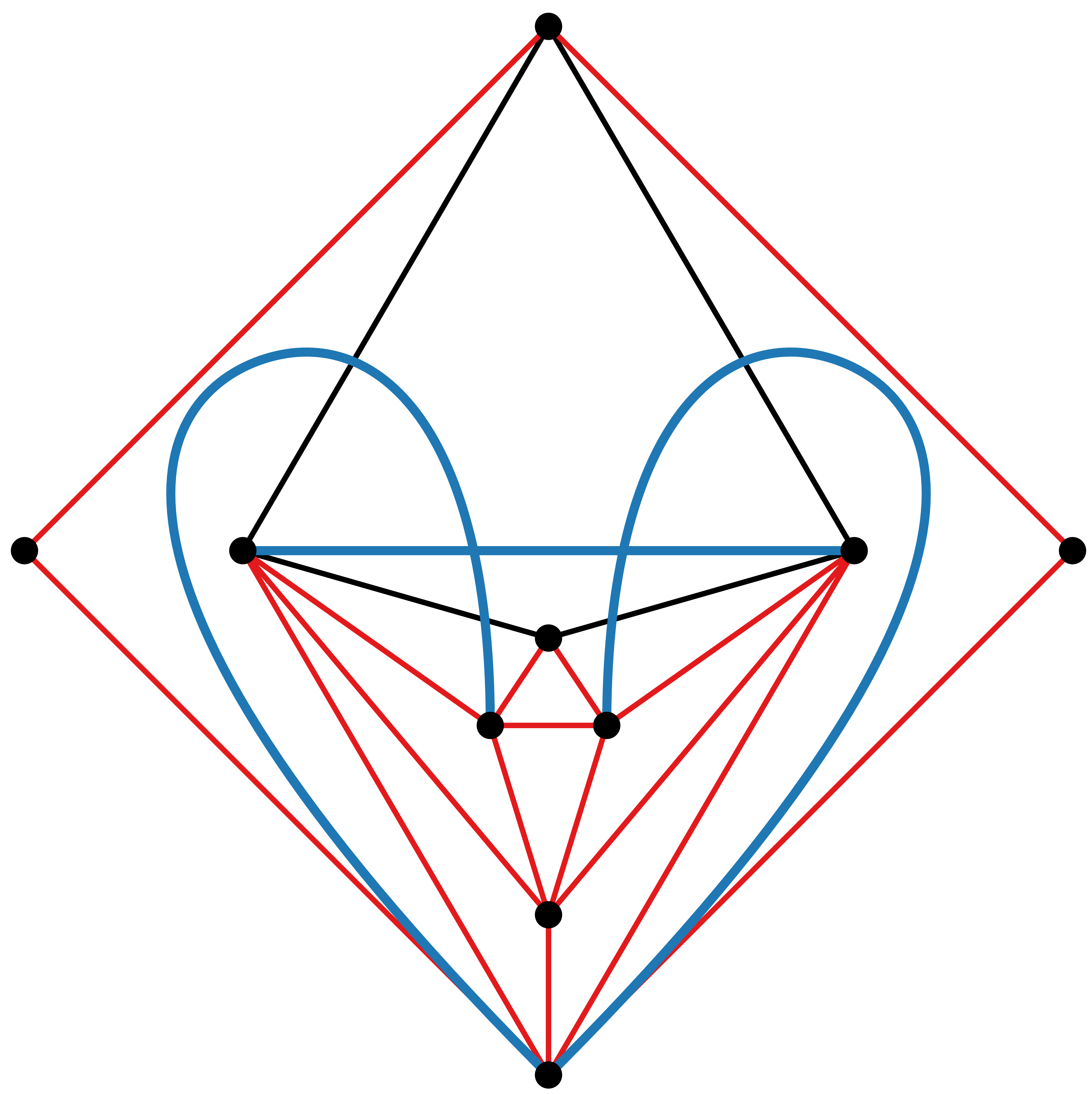
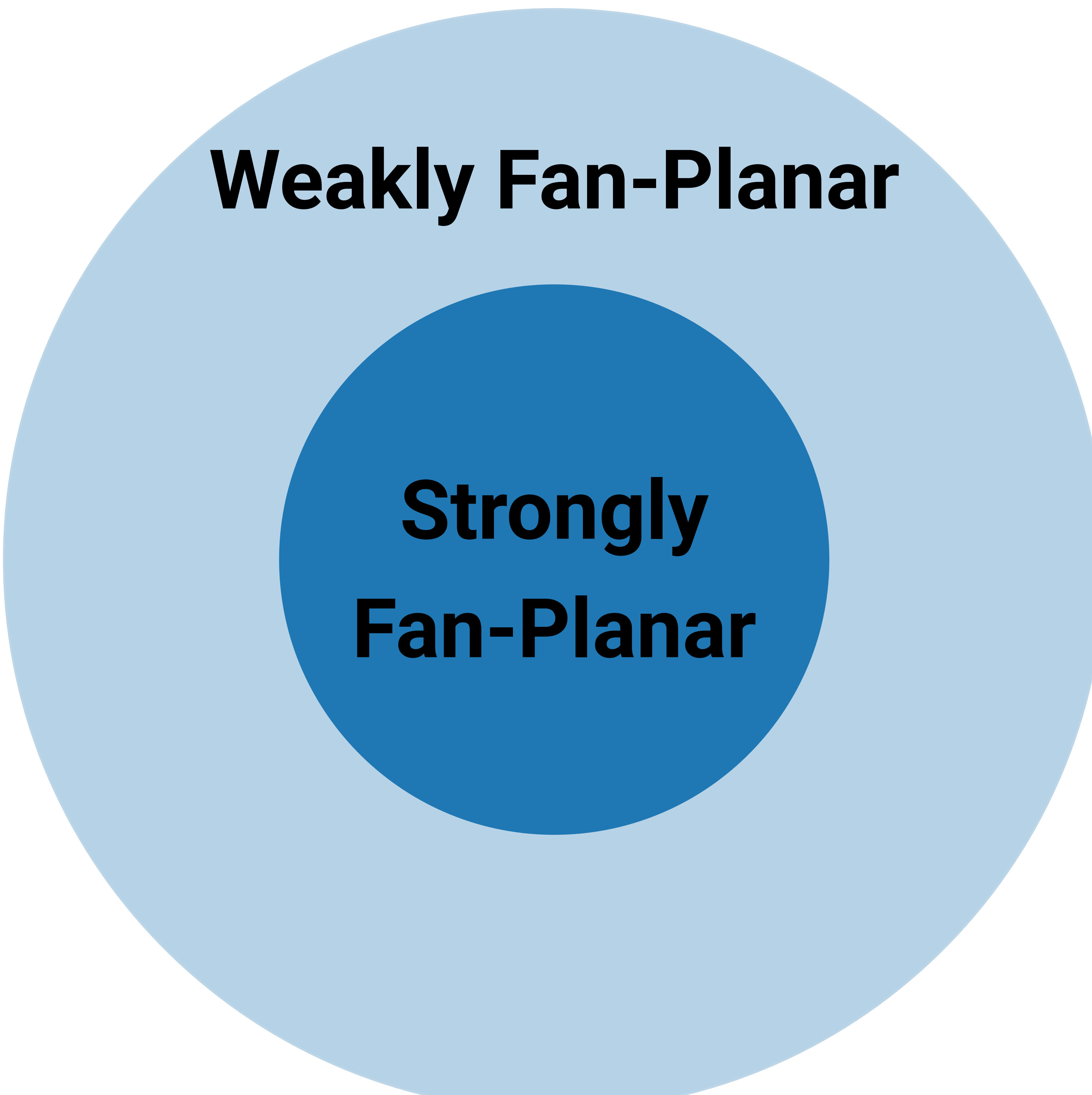


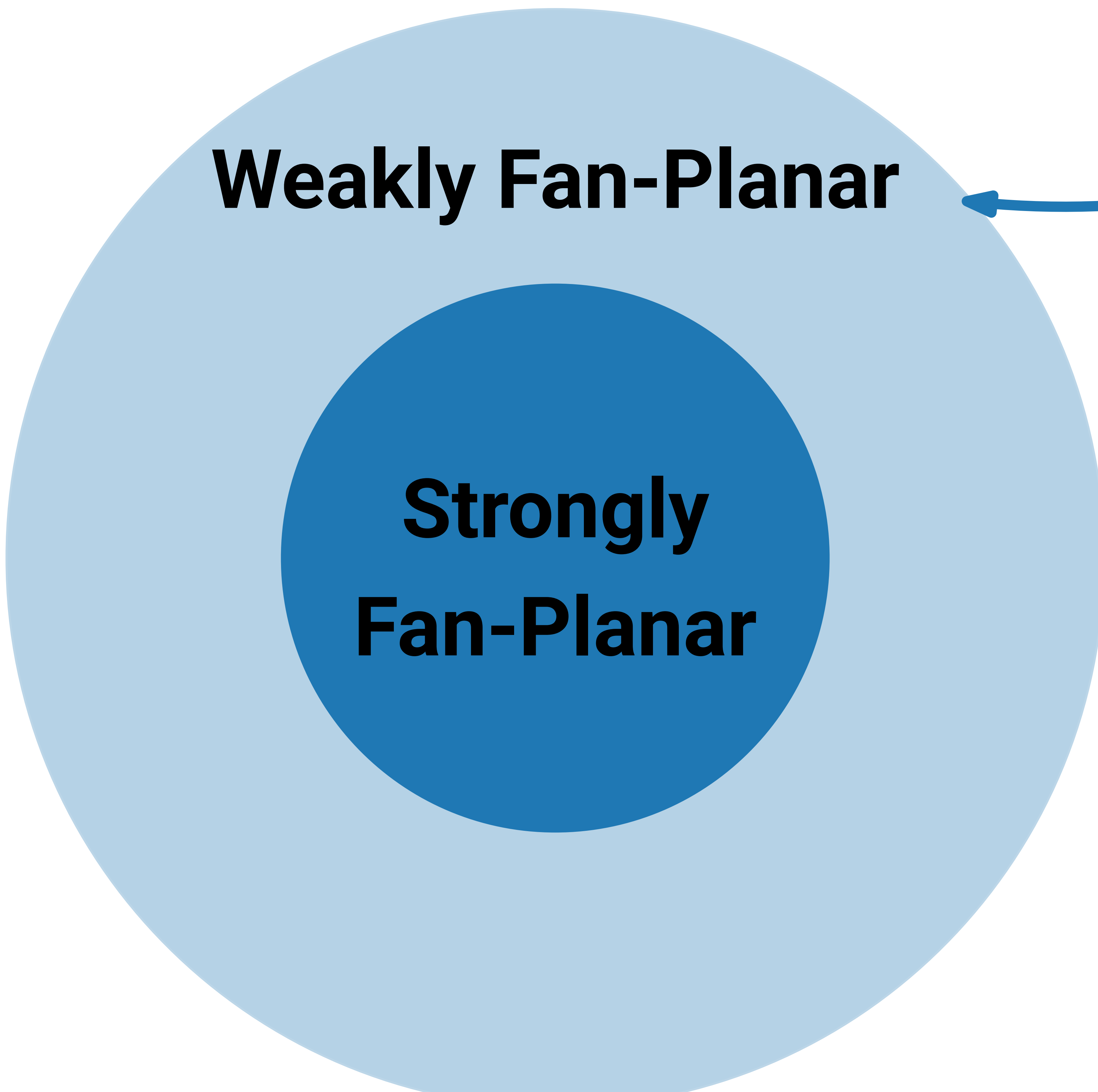

K_7
. . .

Weak vs. Strong Fan-Planarity



Theorem. There exists a weakly fan-planar graph that does not admit a strongly fan-planar drawing.

Weak vs. Strong Fan-Planarity



[Kaufmann and Ueckerdt, 2014]
Fan-planar graphs: $|E| \leq 5n - 10$

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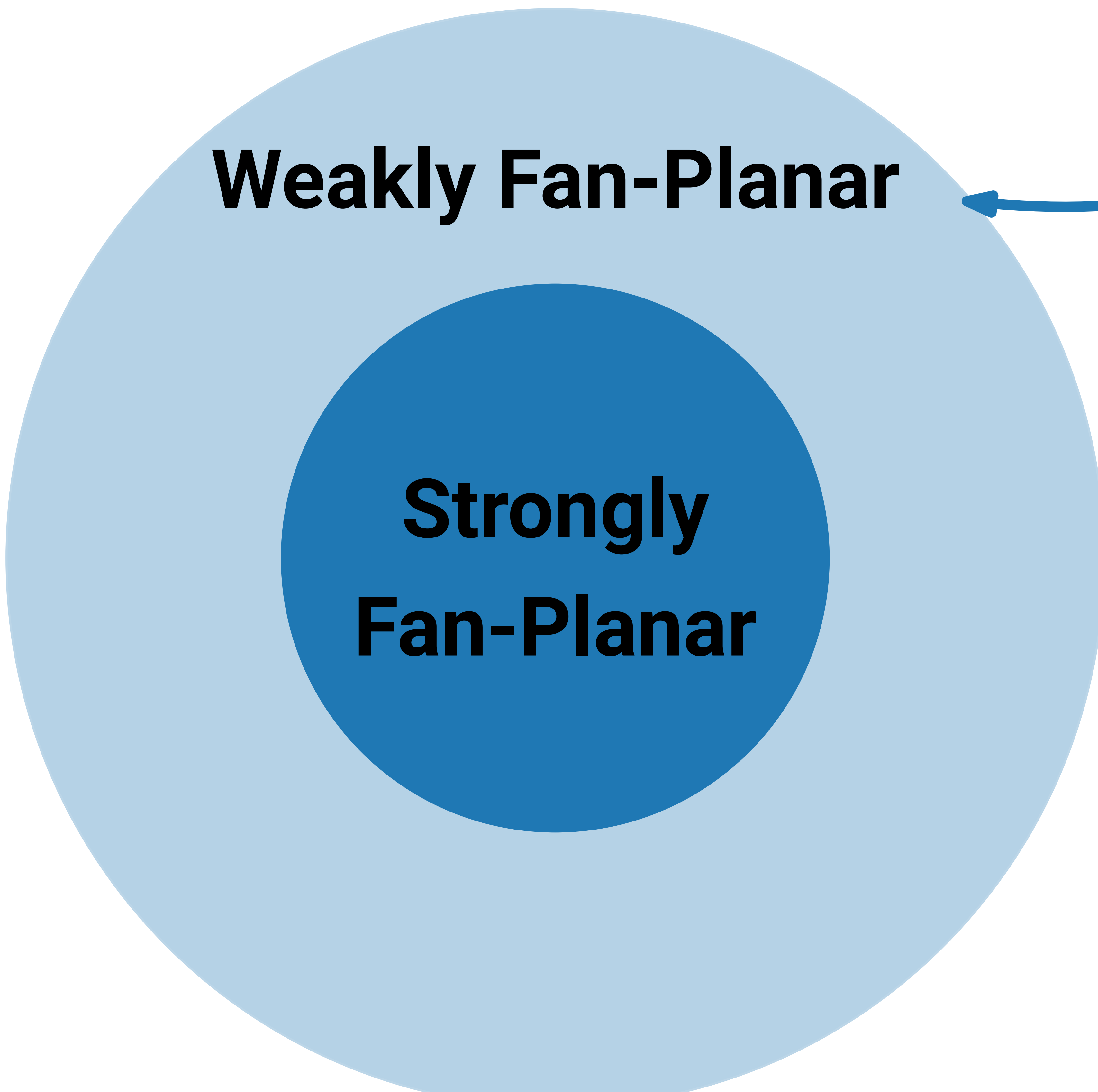

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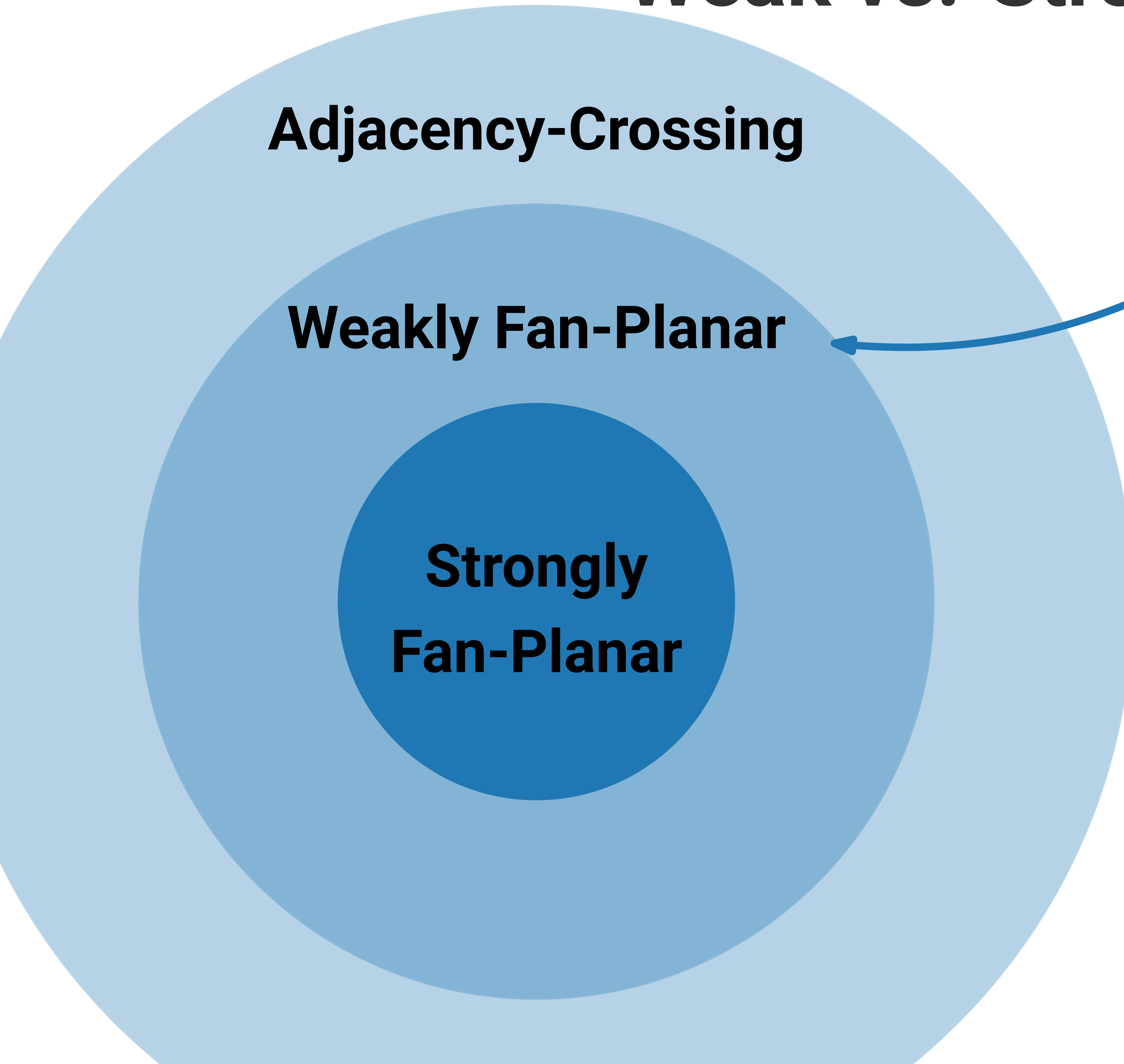

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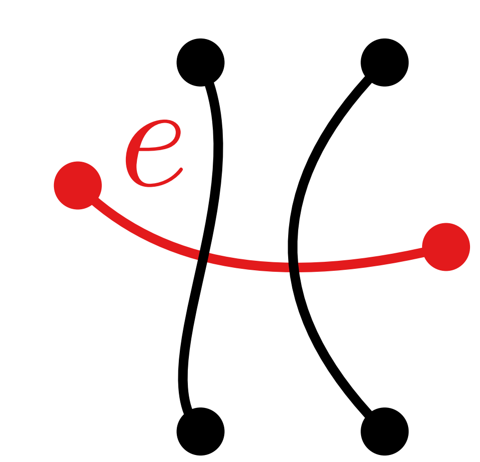

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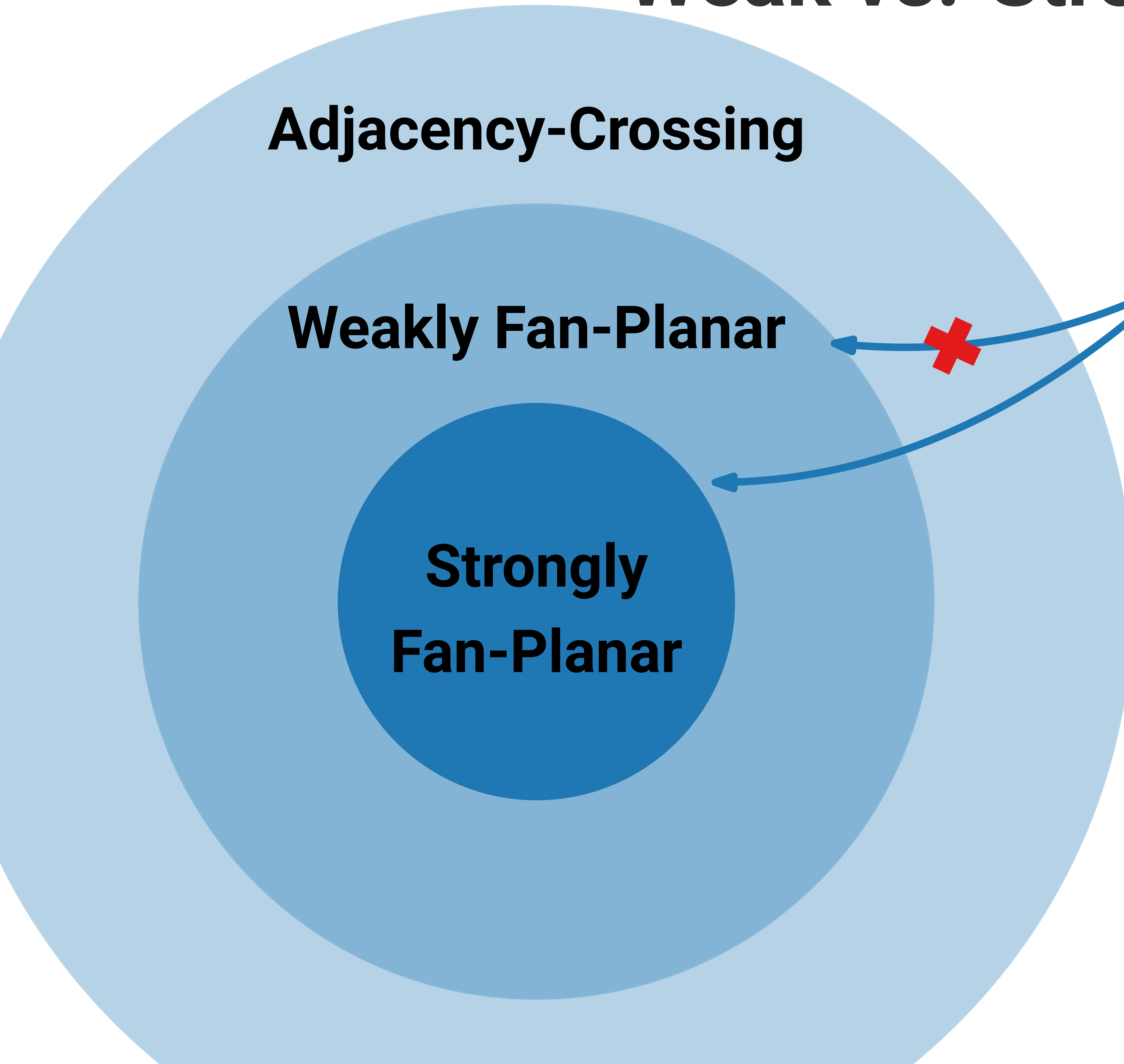

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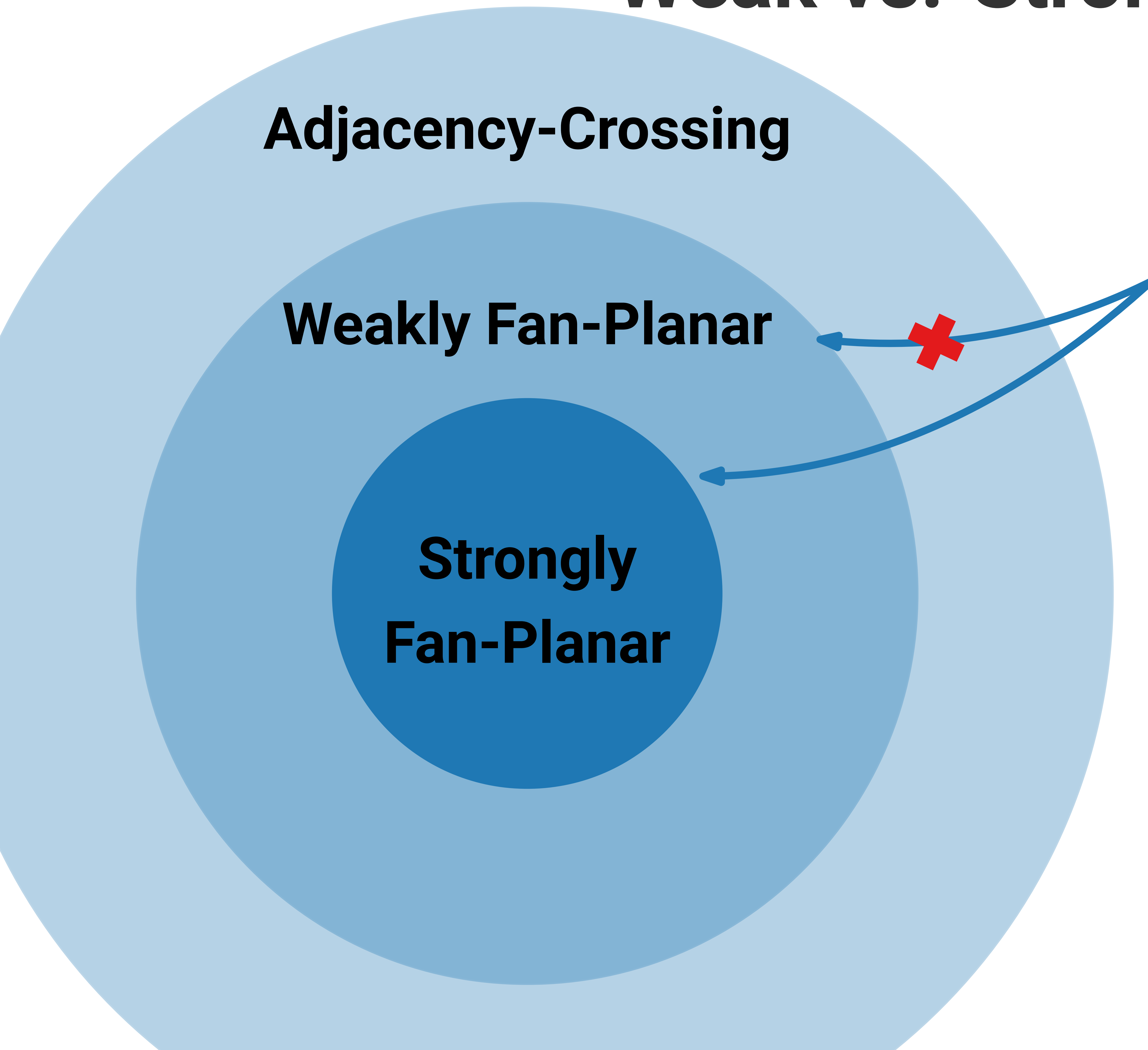

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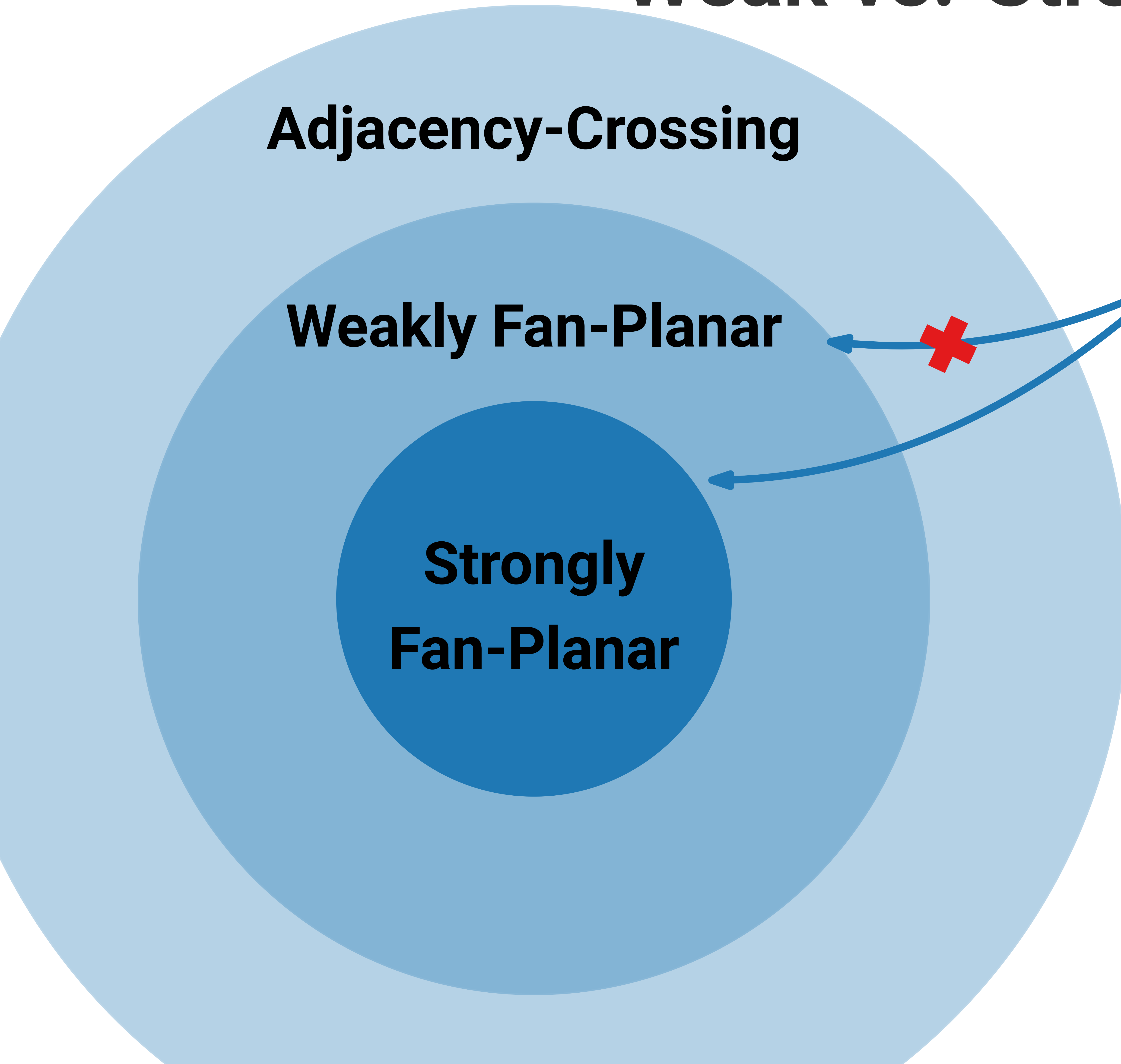

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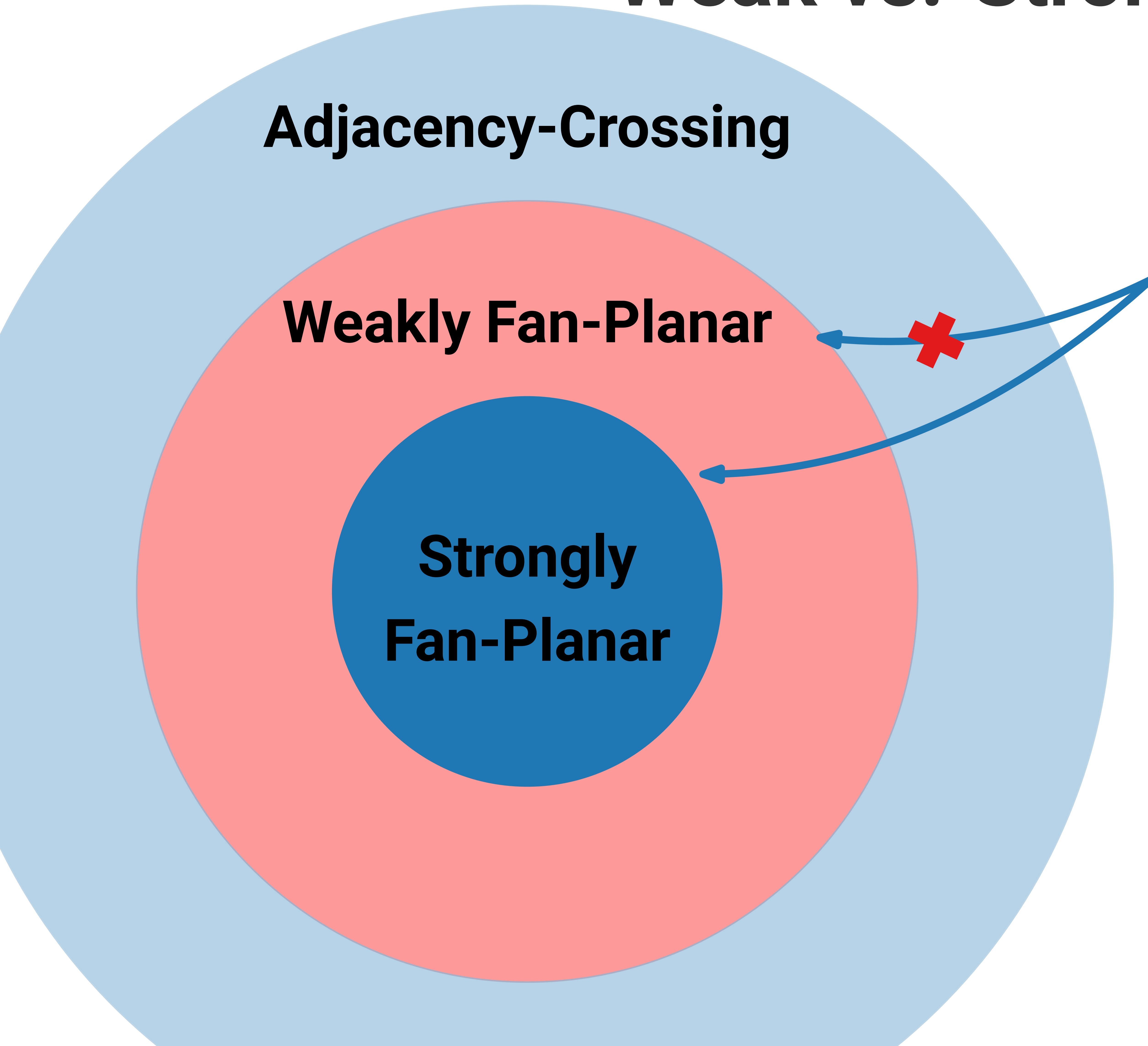

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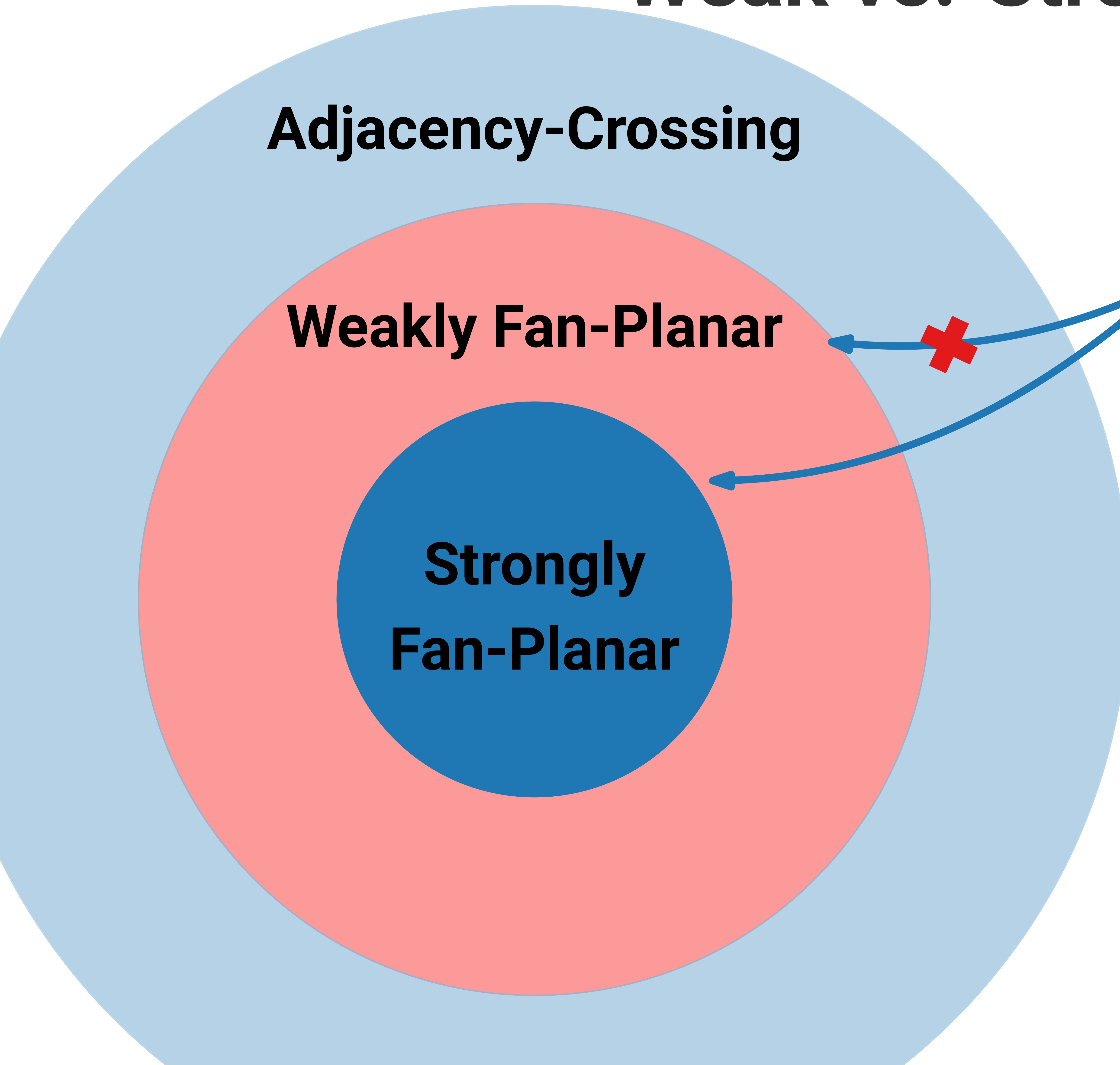

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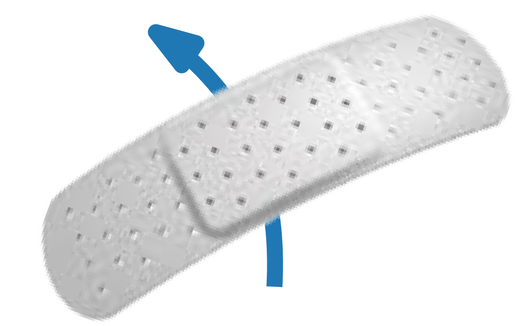

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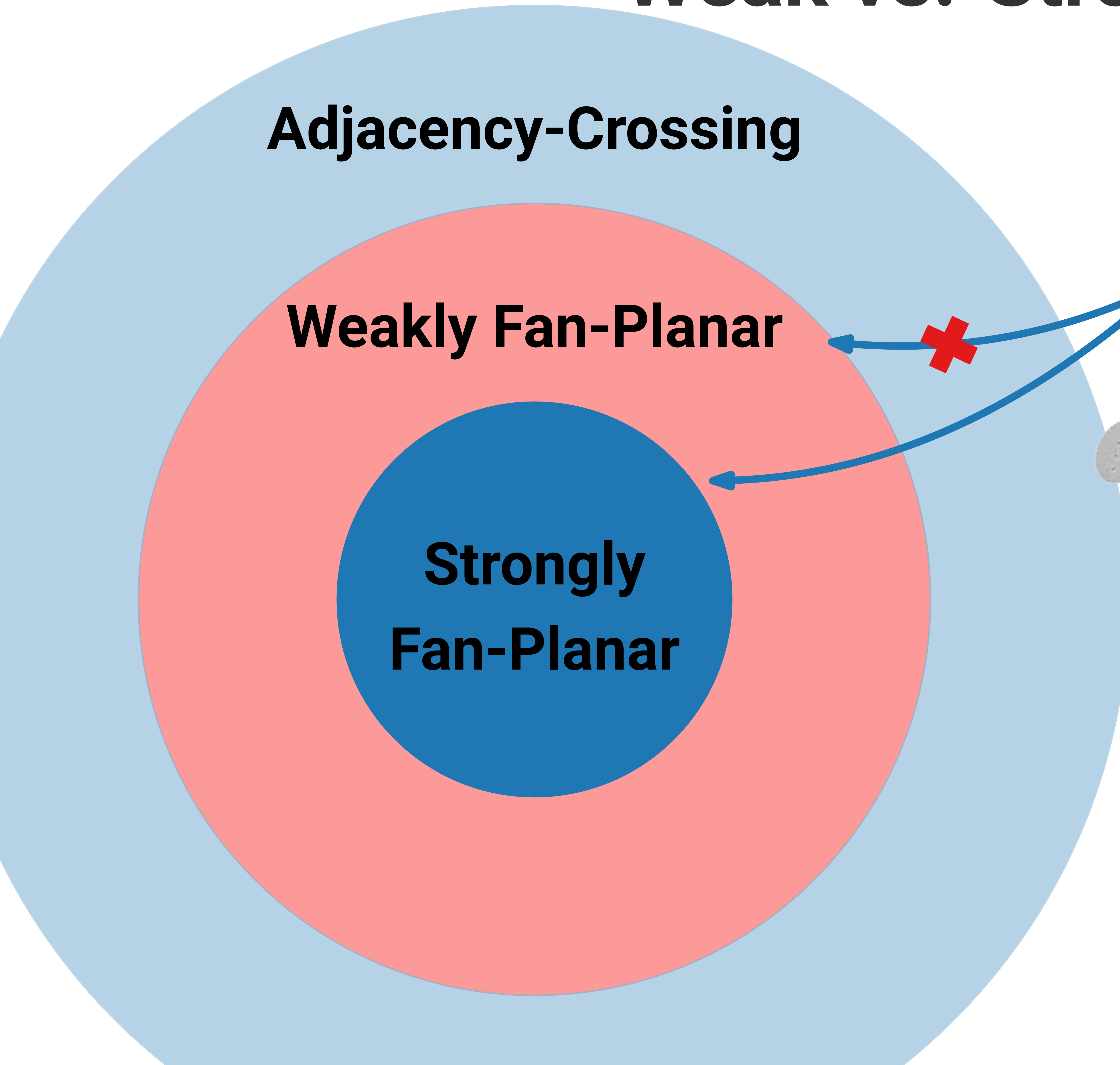

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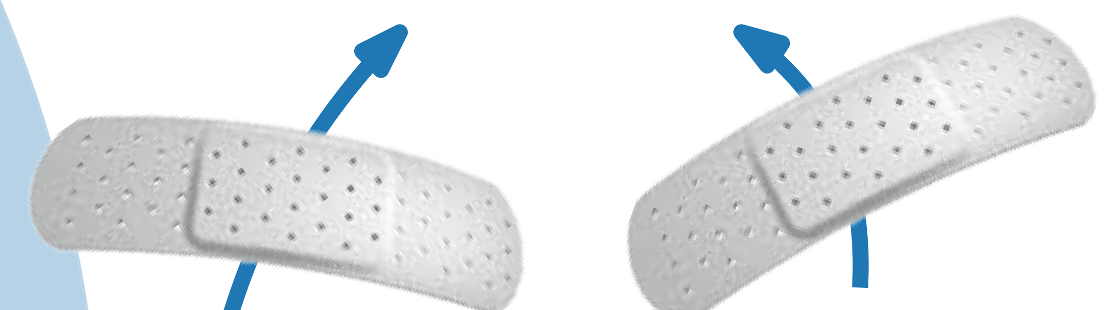

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Weak Fan-Planarity: Edge Density

Induction on the **# of edge triples forming pattern (III)**

Weak Fan-Planarity: Edge Density

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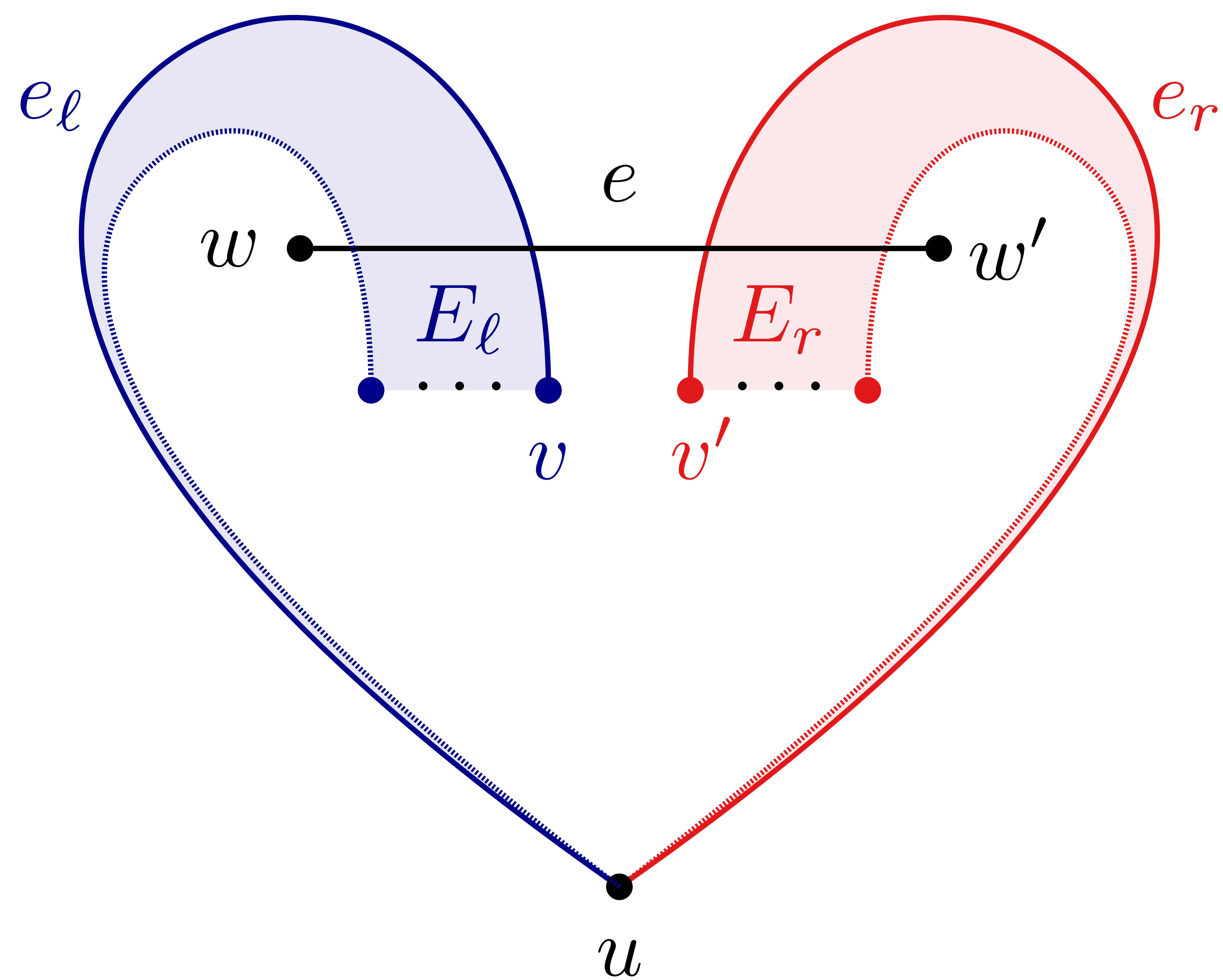
A **heart** formed by e_ℓ, e_r and e



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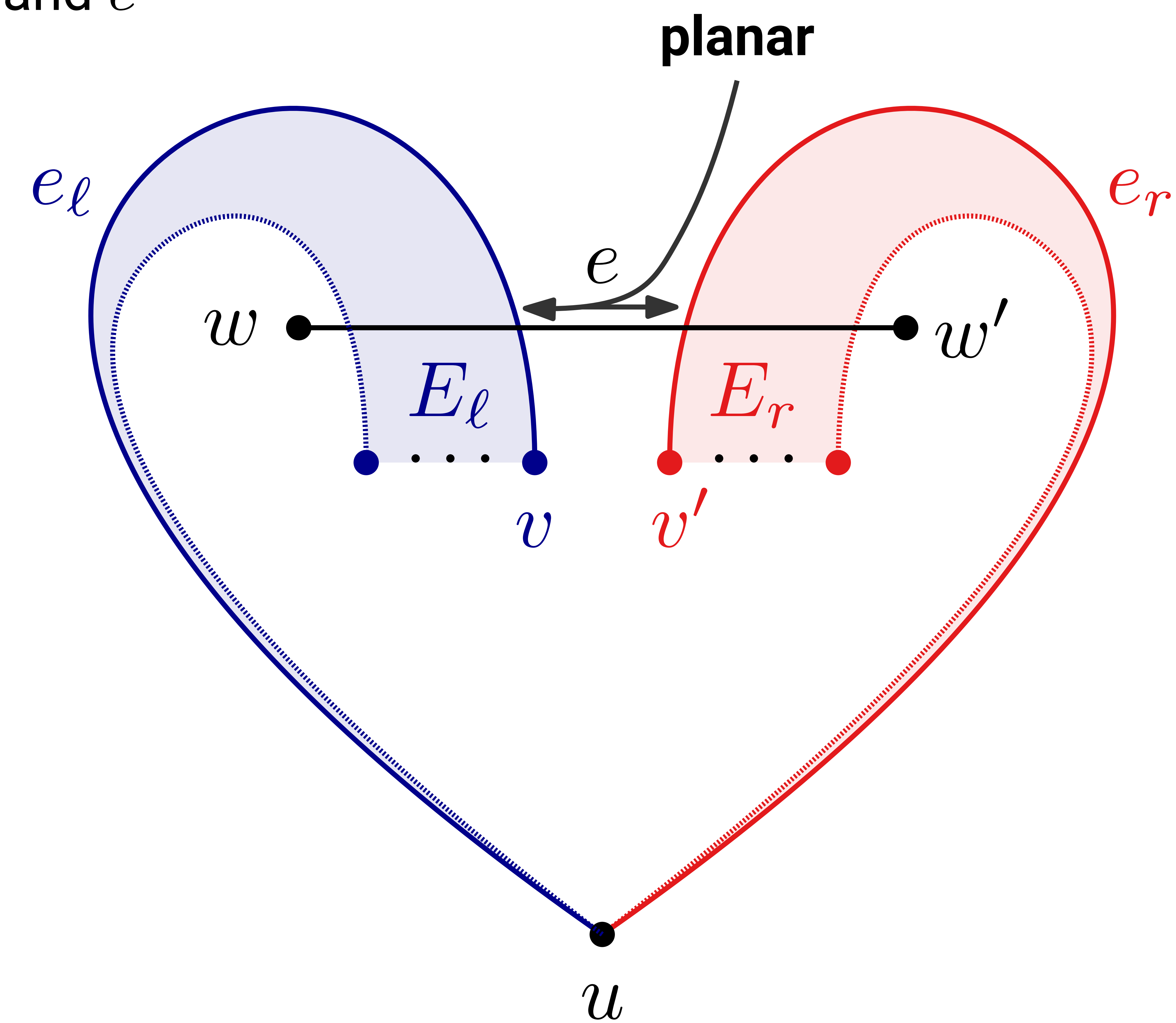

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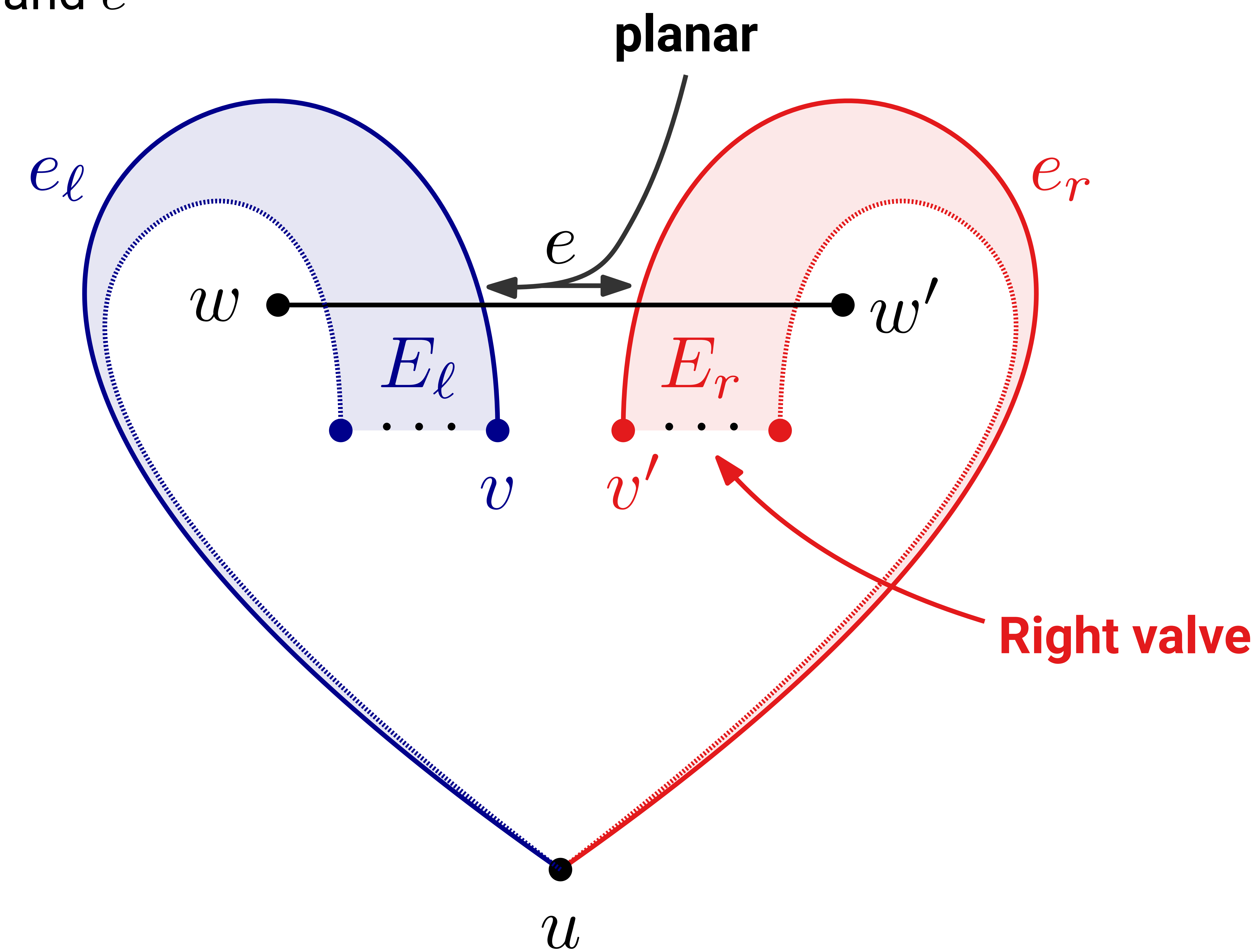

Weak Fan-Planarity: Edge Density

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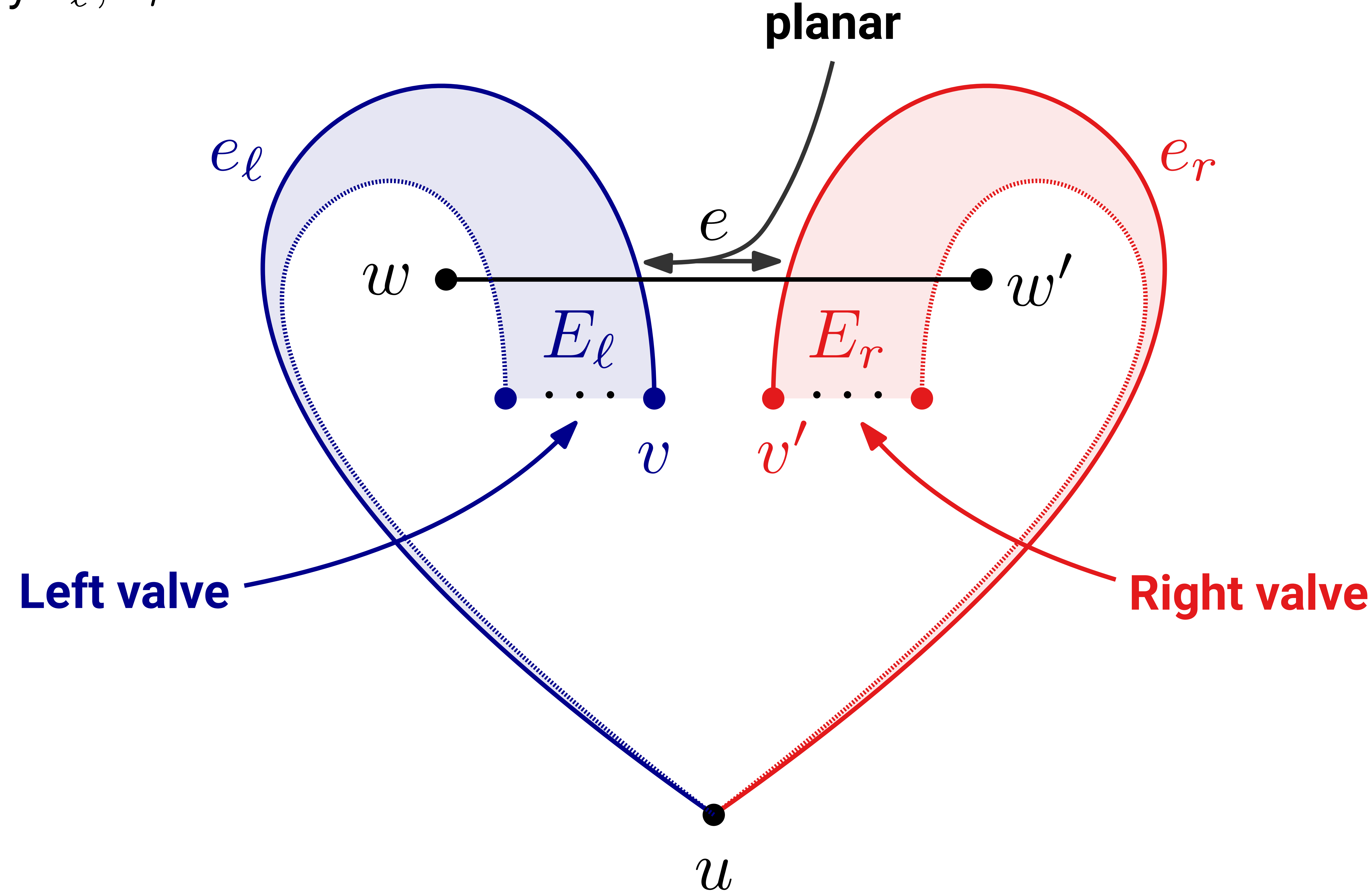

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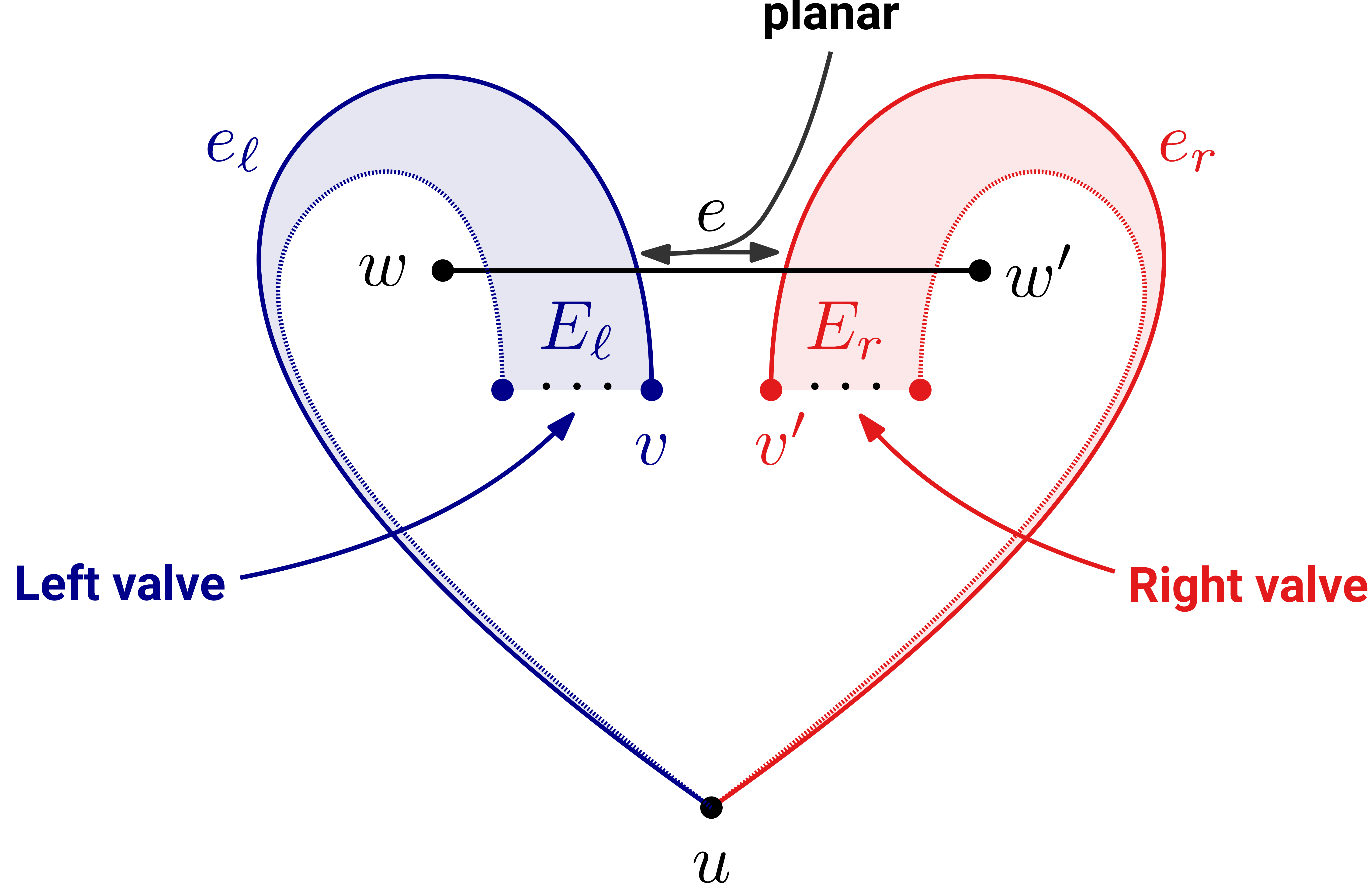

Weak Fan-Planarity: Edge Density

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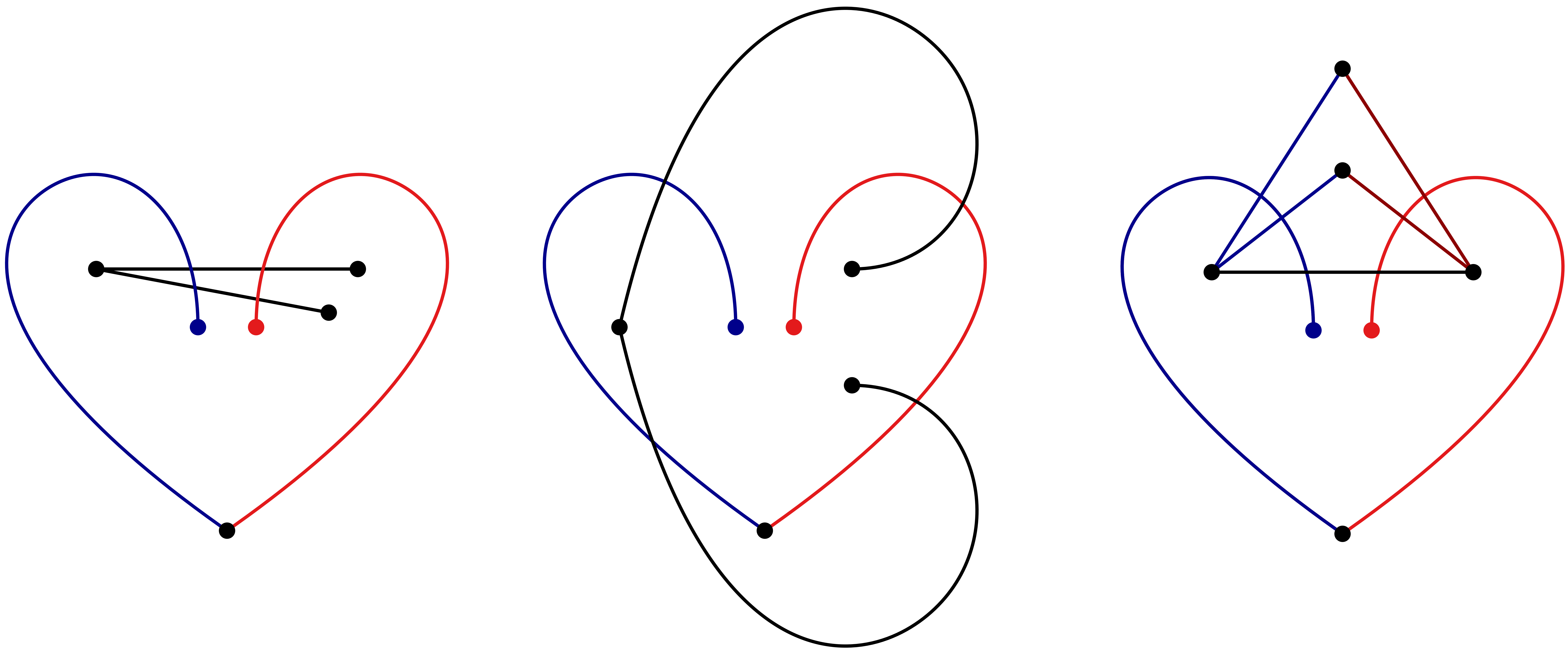
First step, we eliminate certain configurations:



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Input: Let $G = (V, E)$ be a (preprocessed) graph on n vertices;
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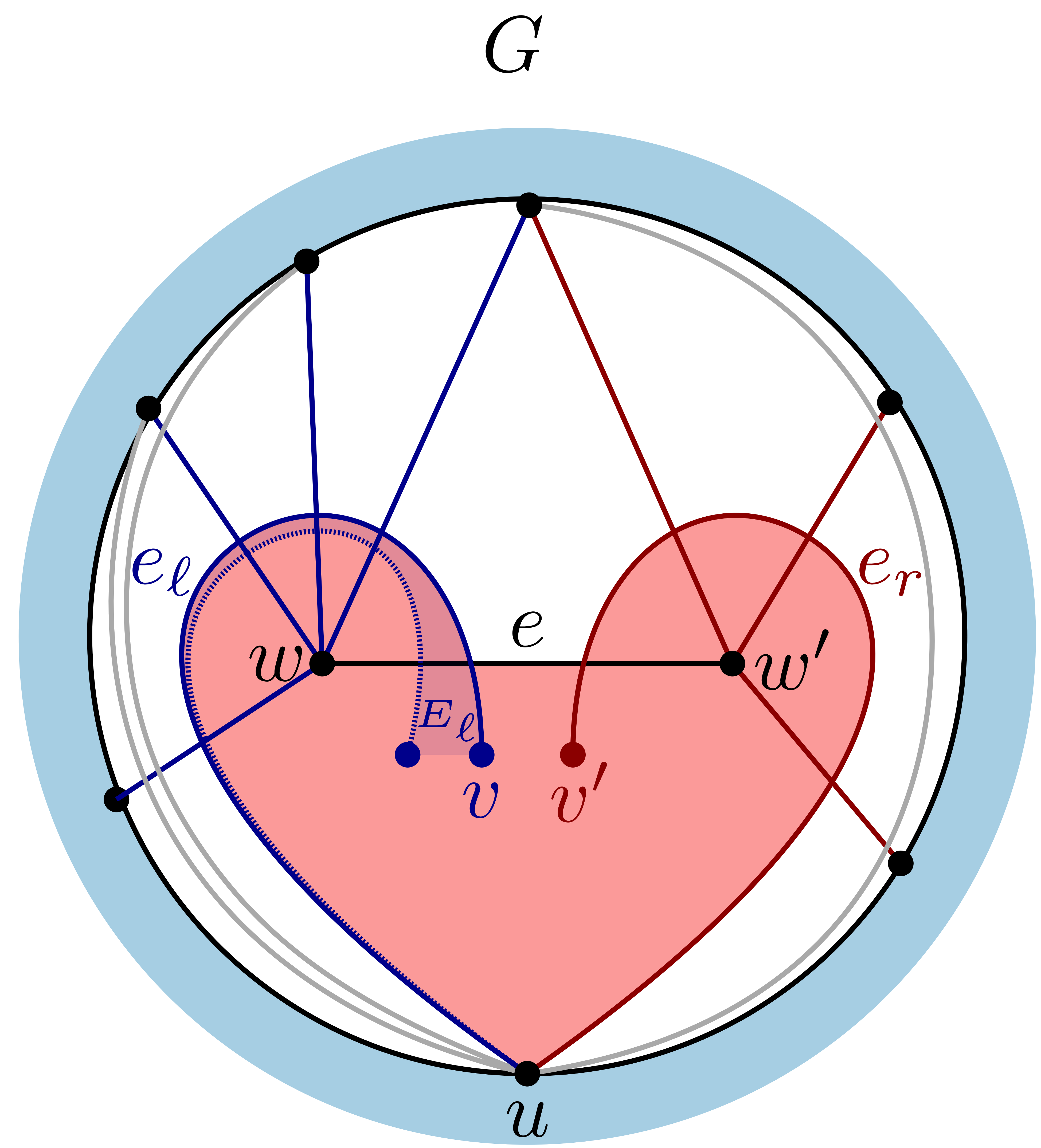

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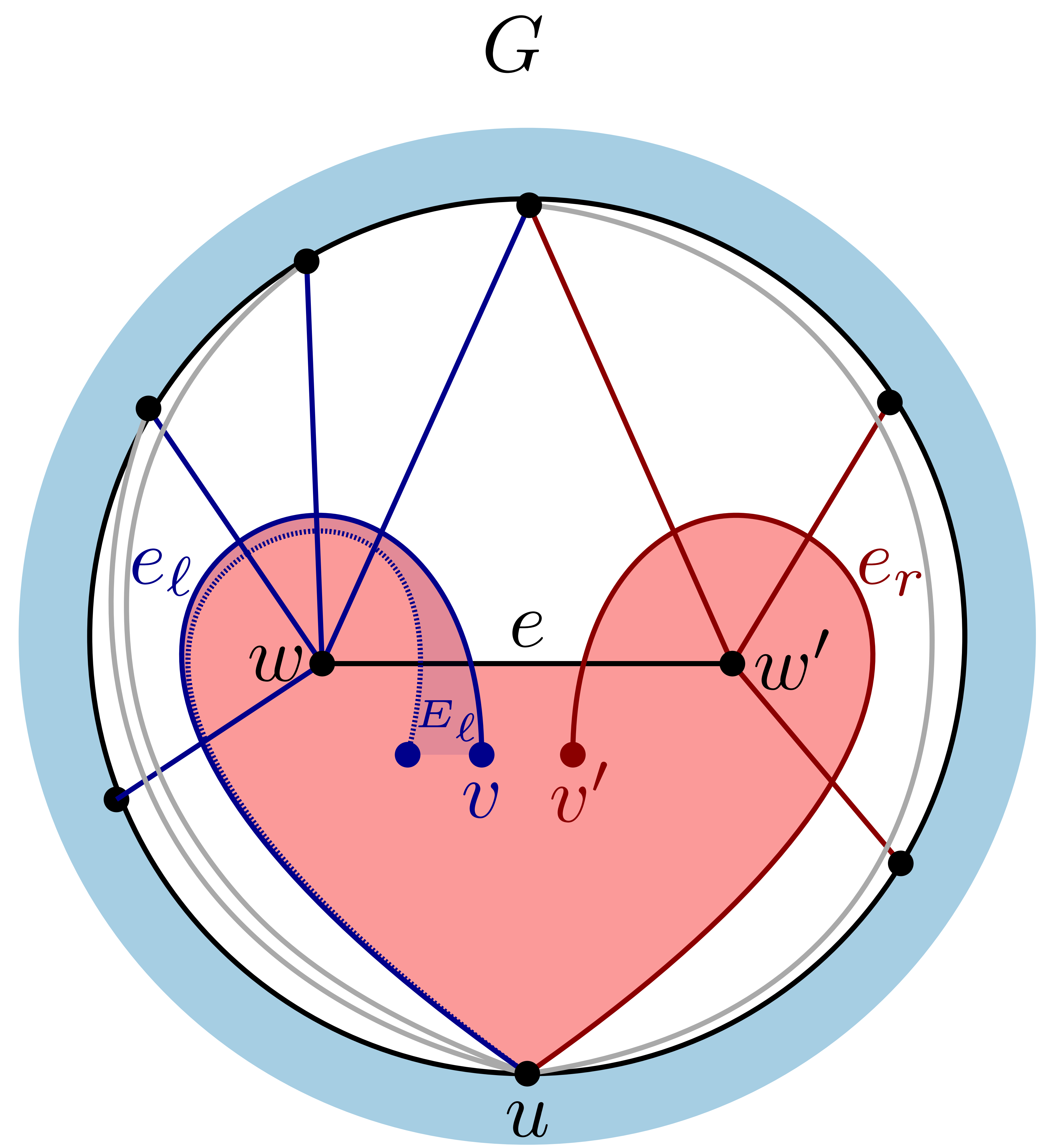

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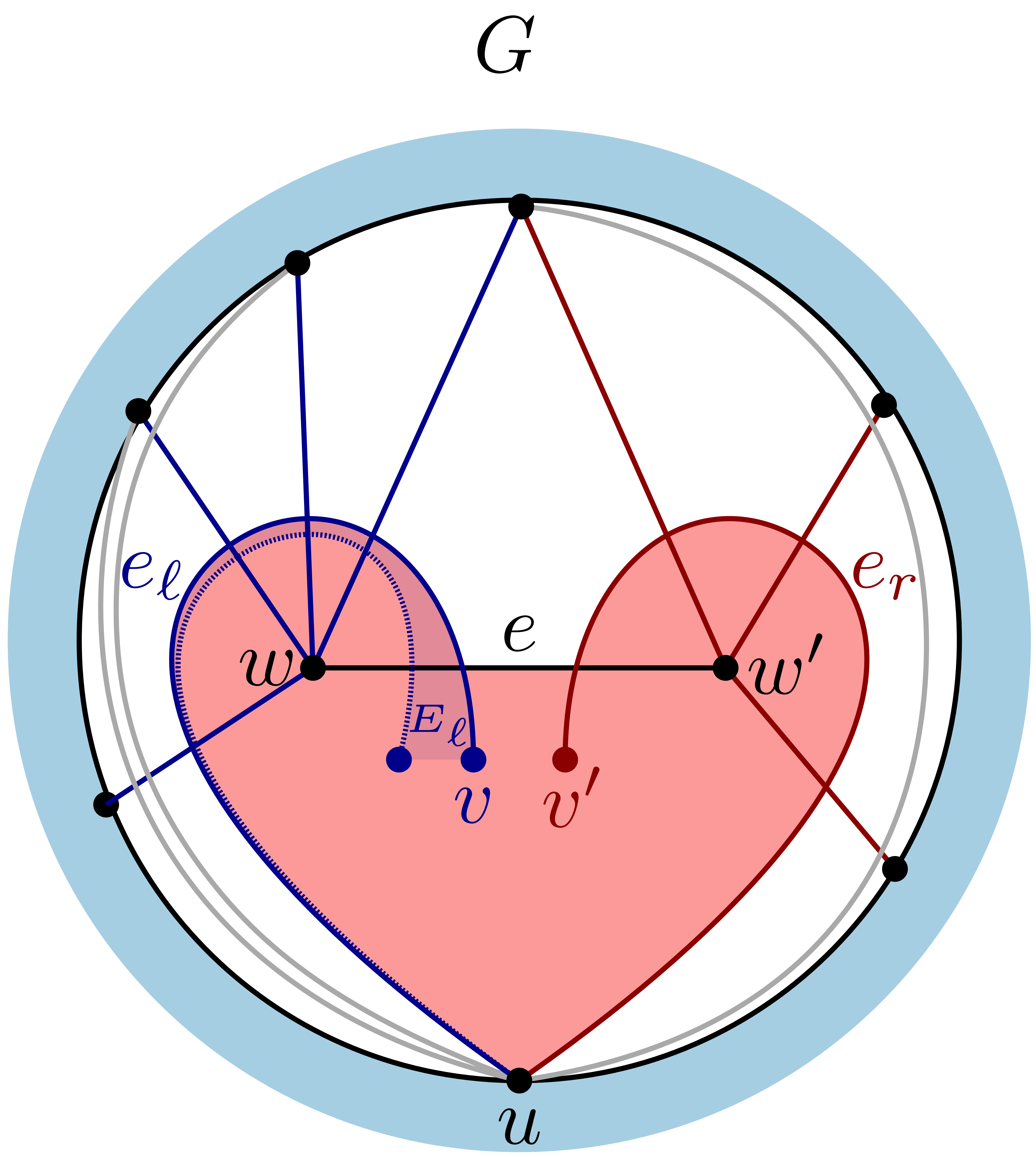

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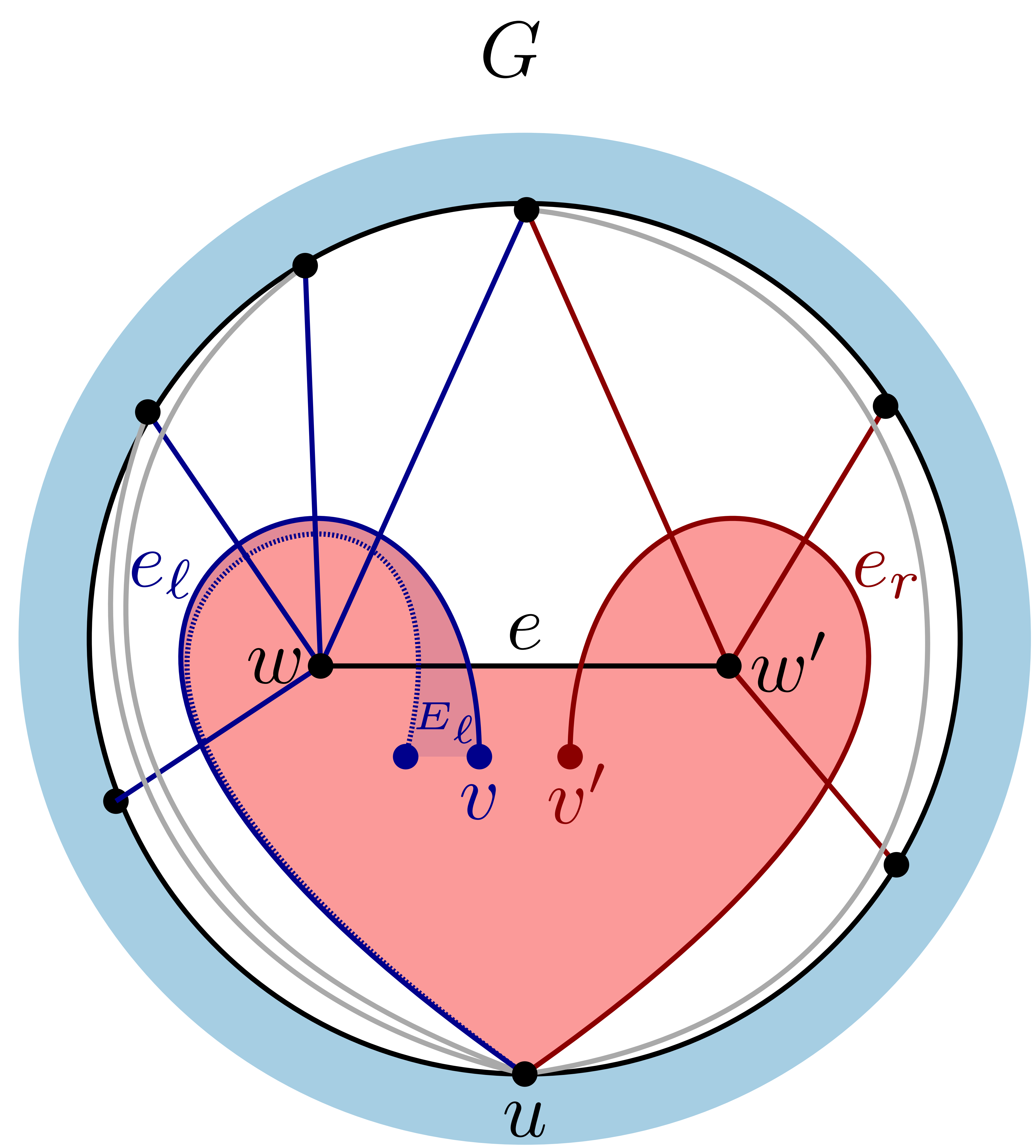
$|E(G)| = |E(G_1)| + |E(G_2)|$



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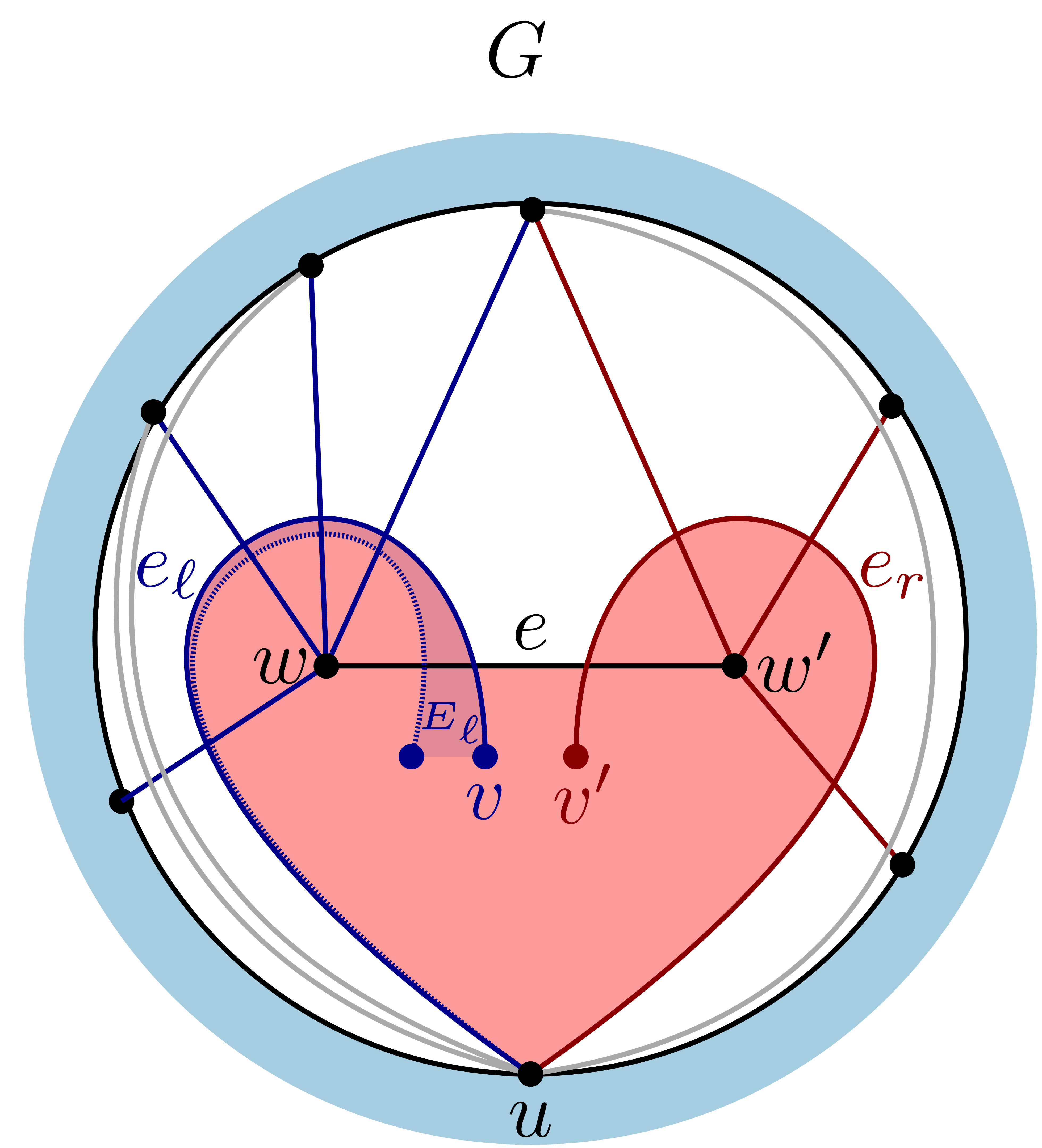

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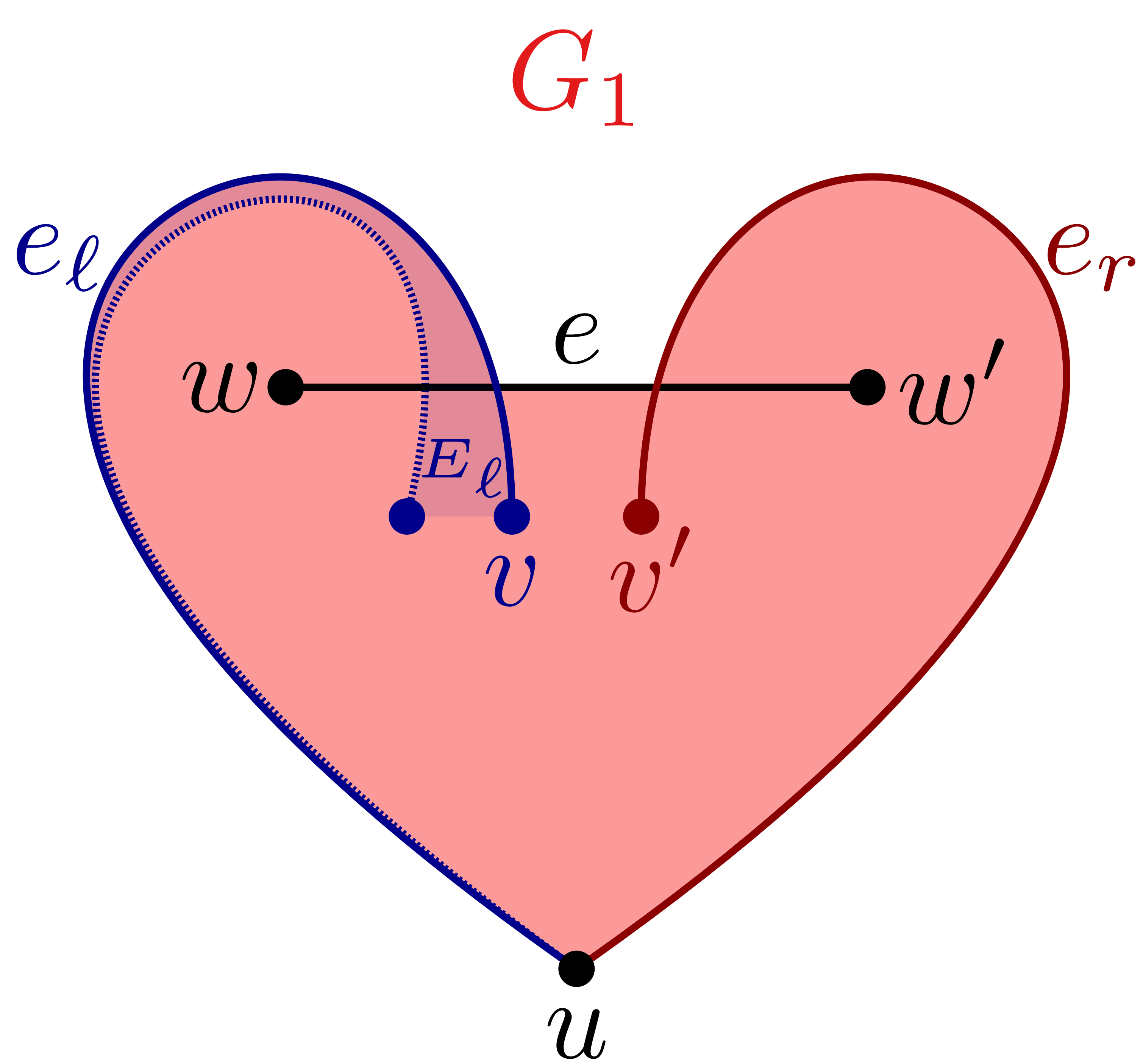

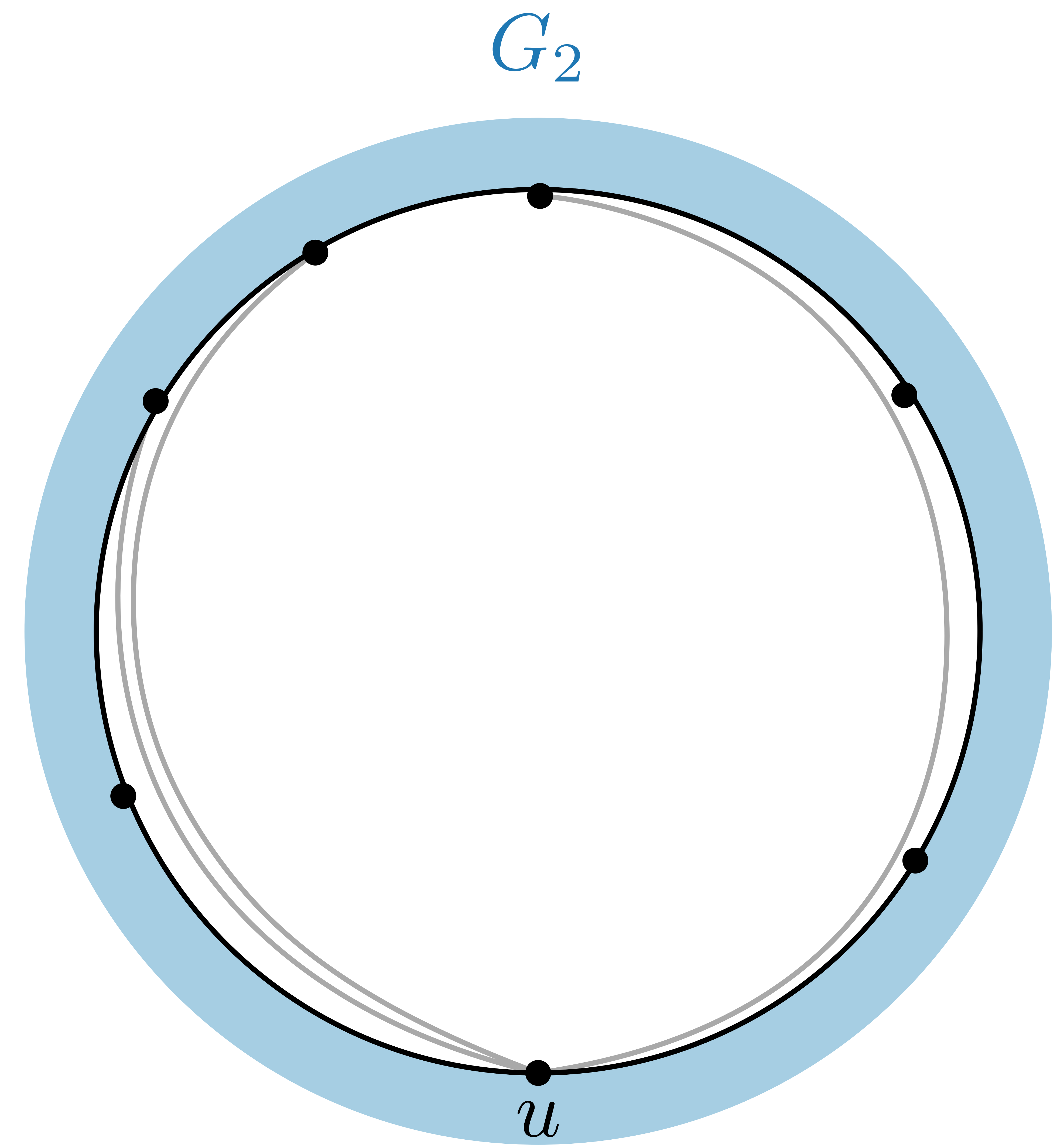

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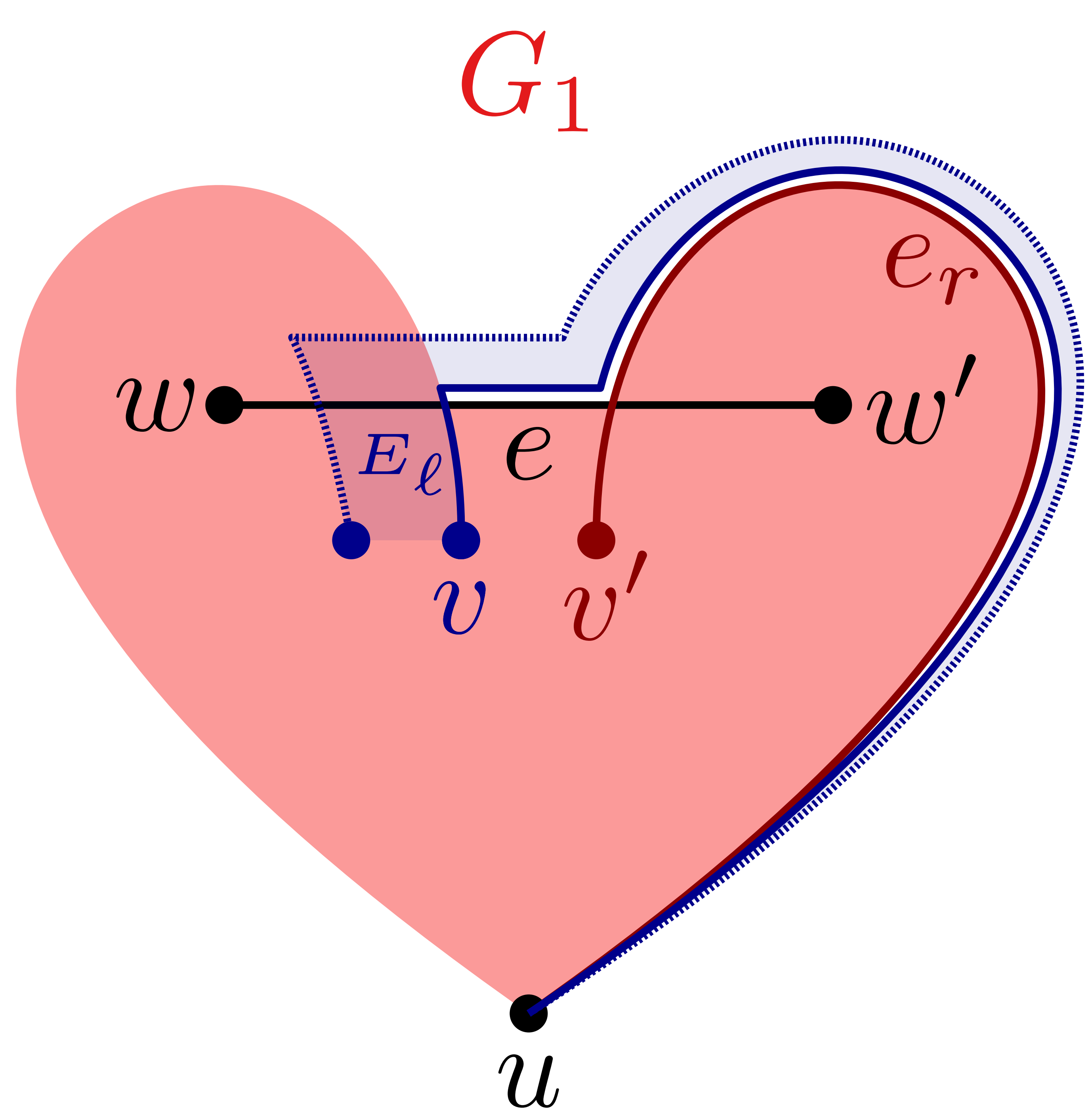

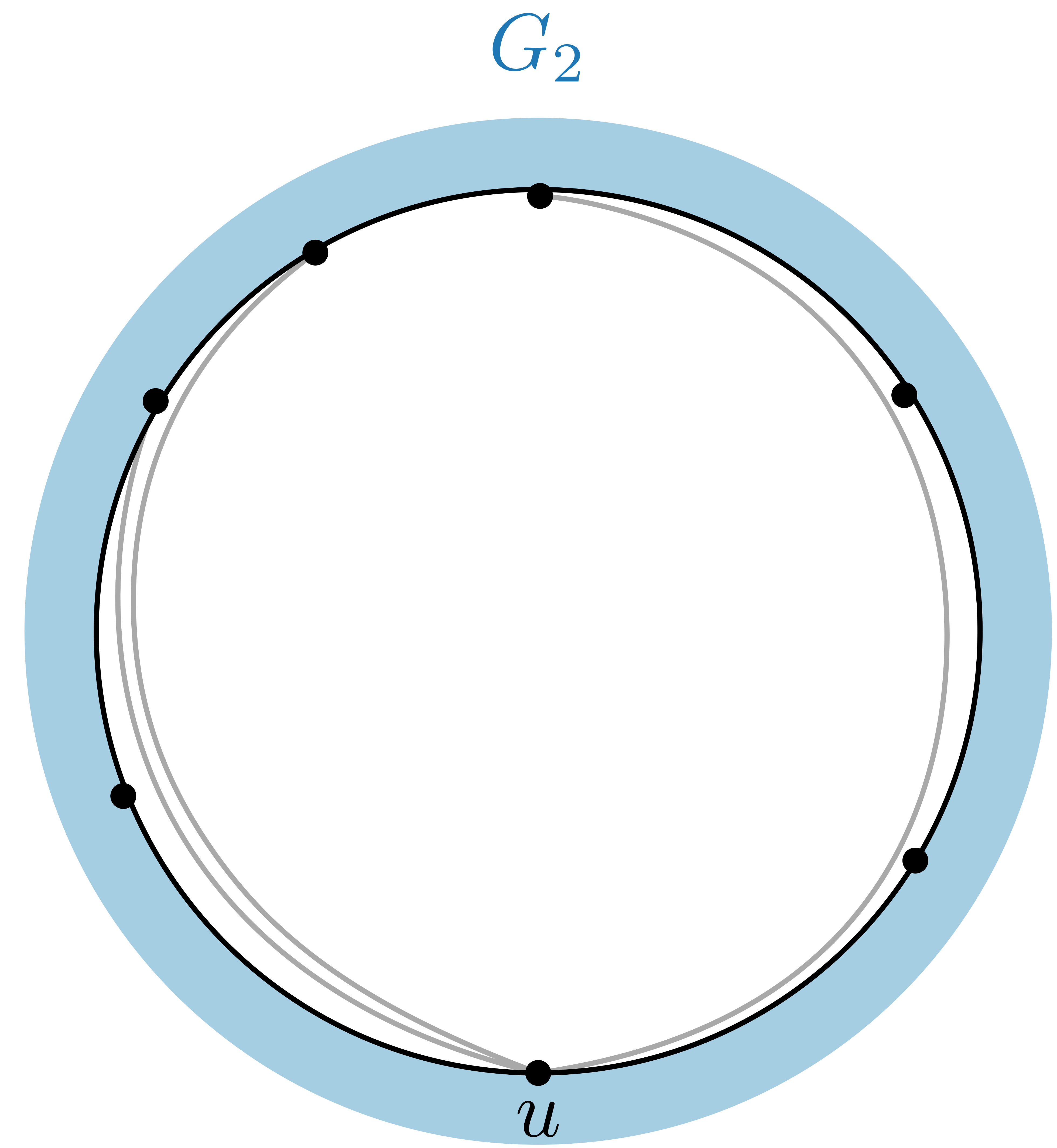

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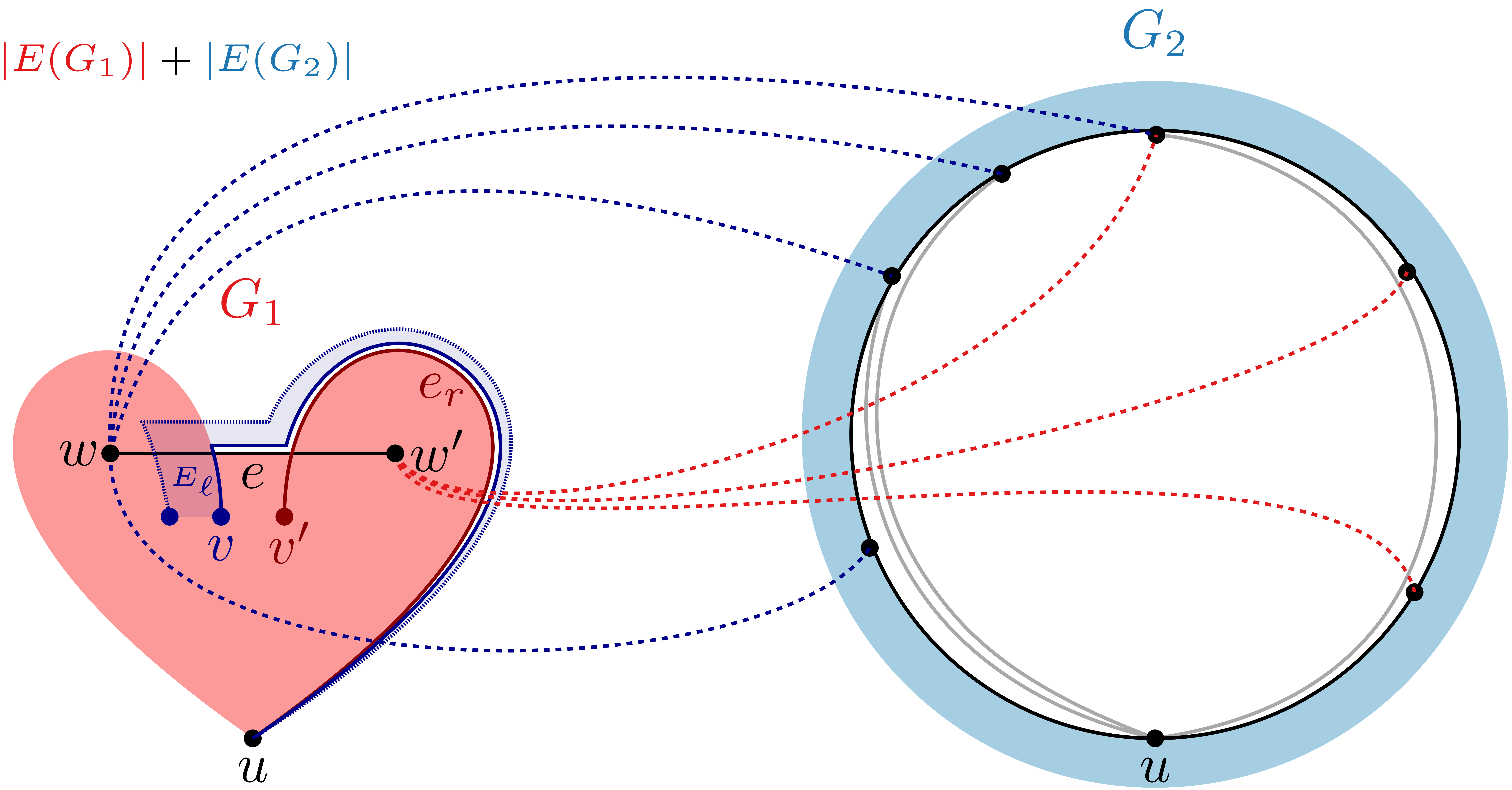

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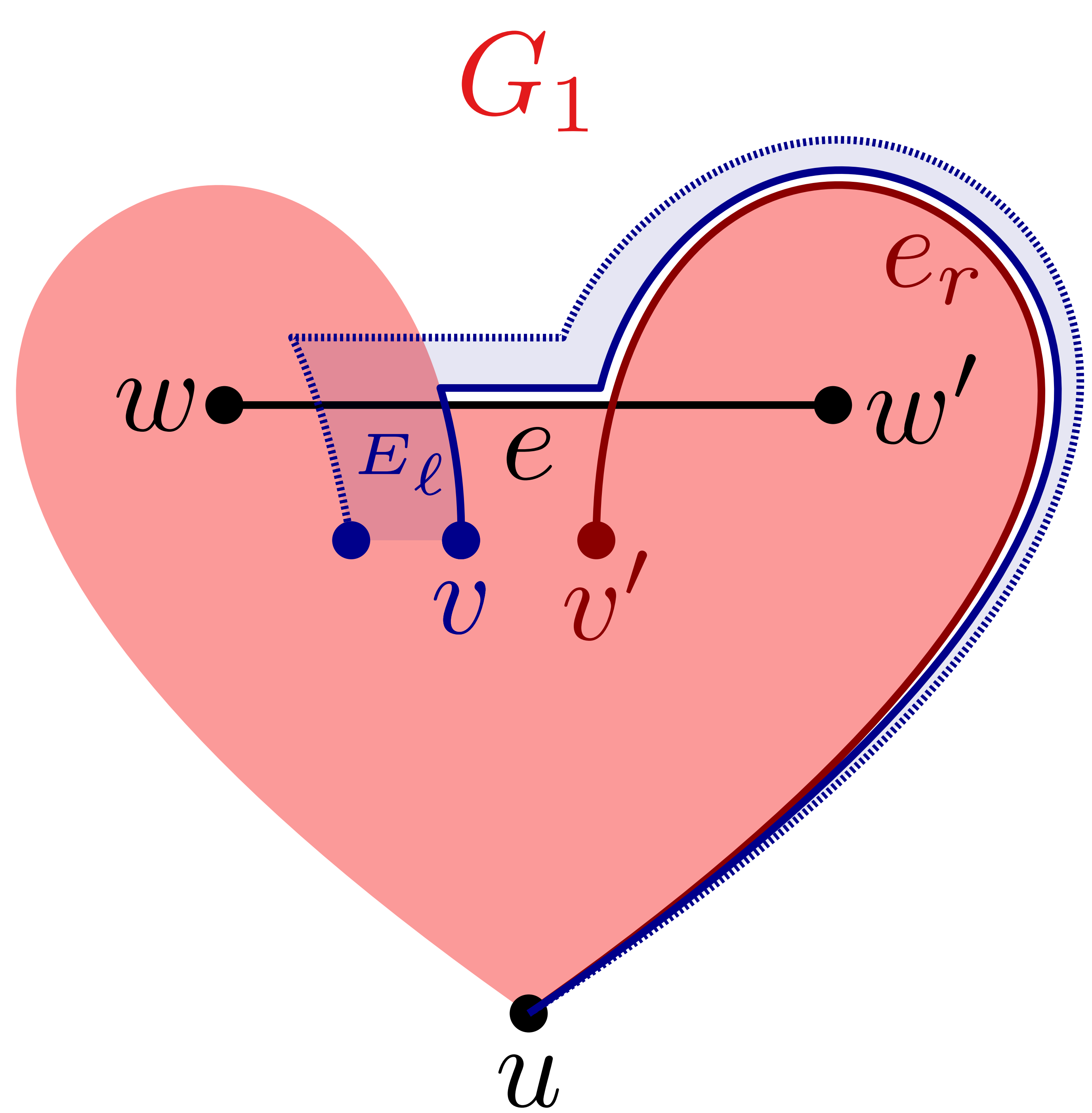
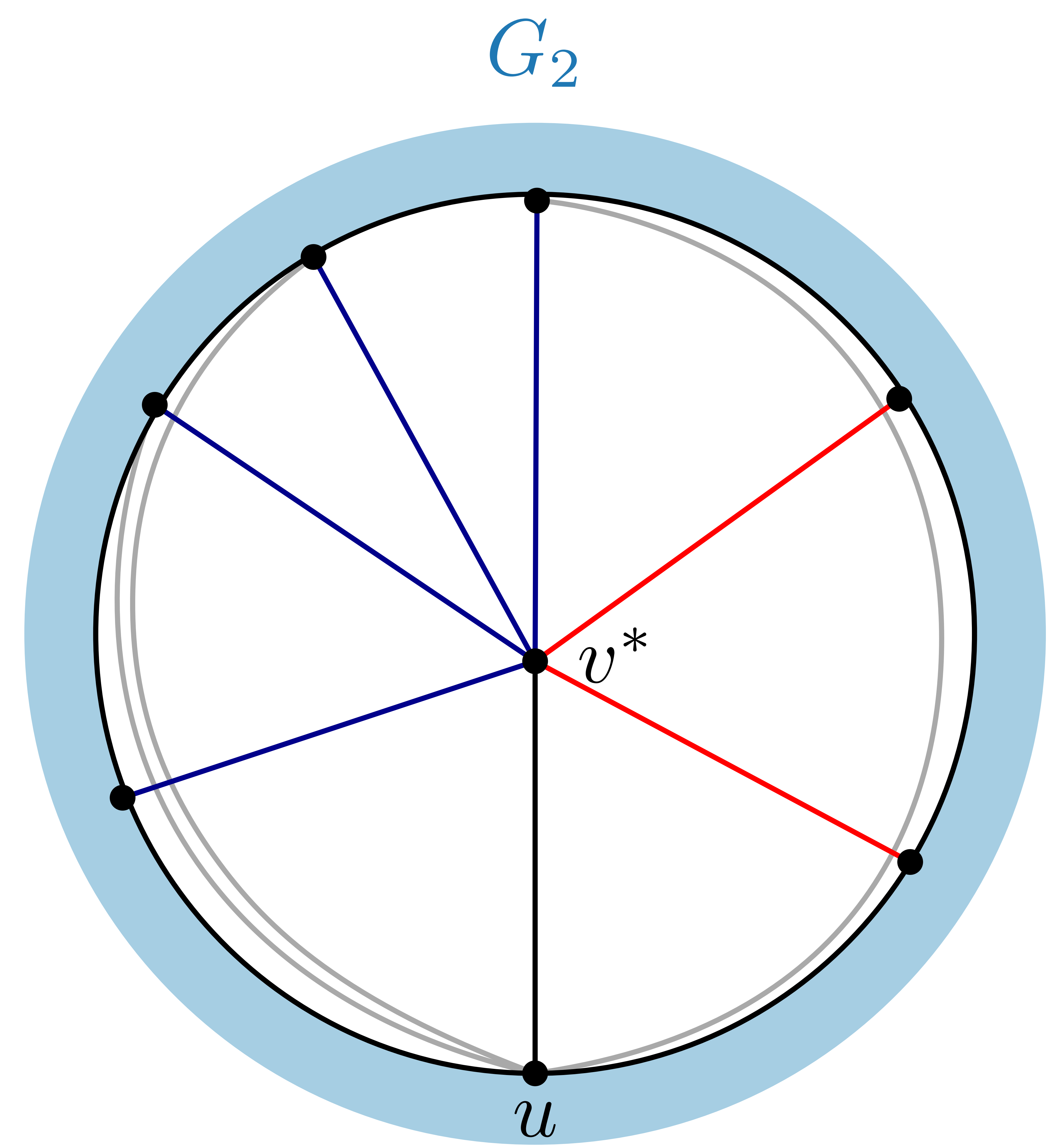
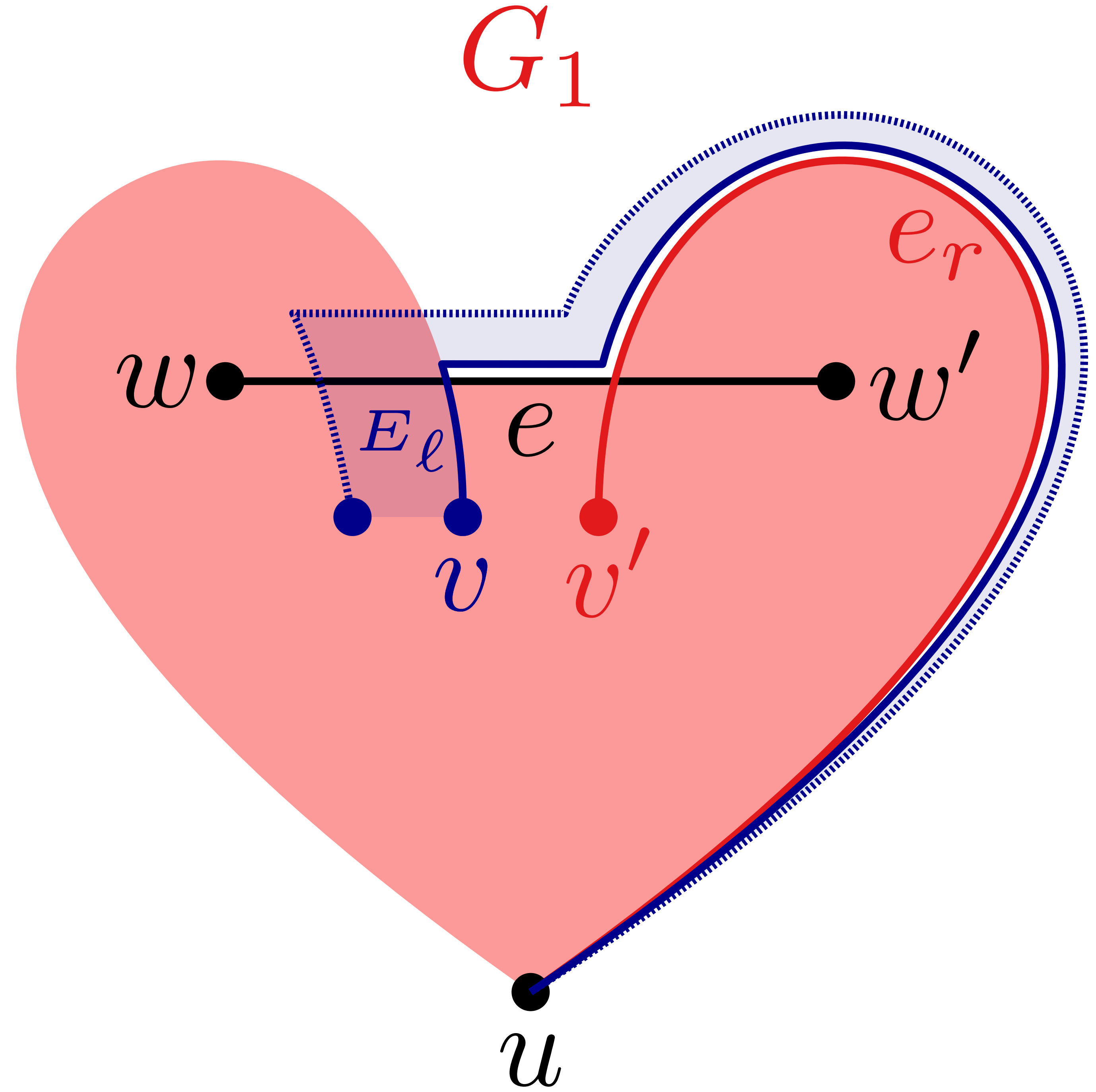


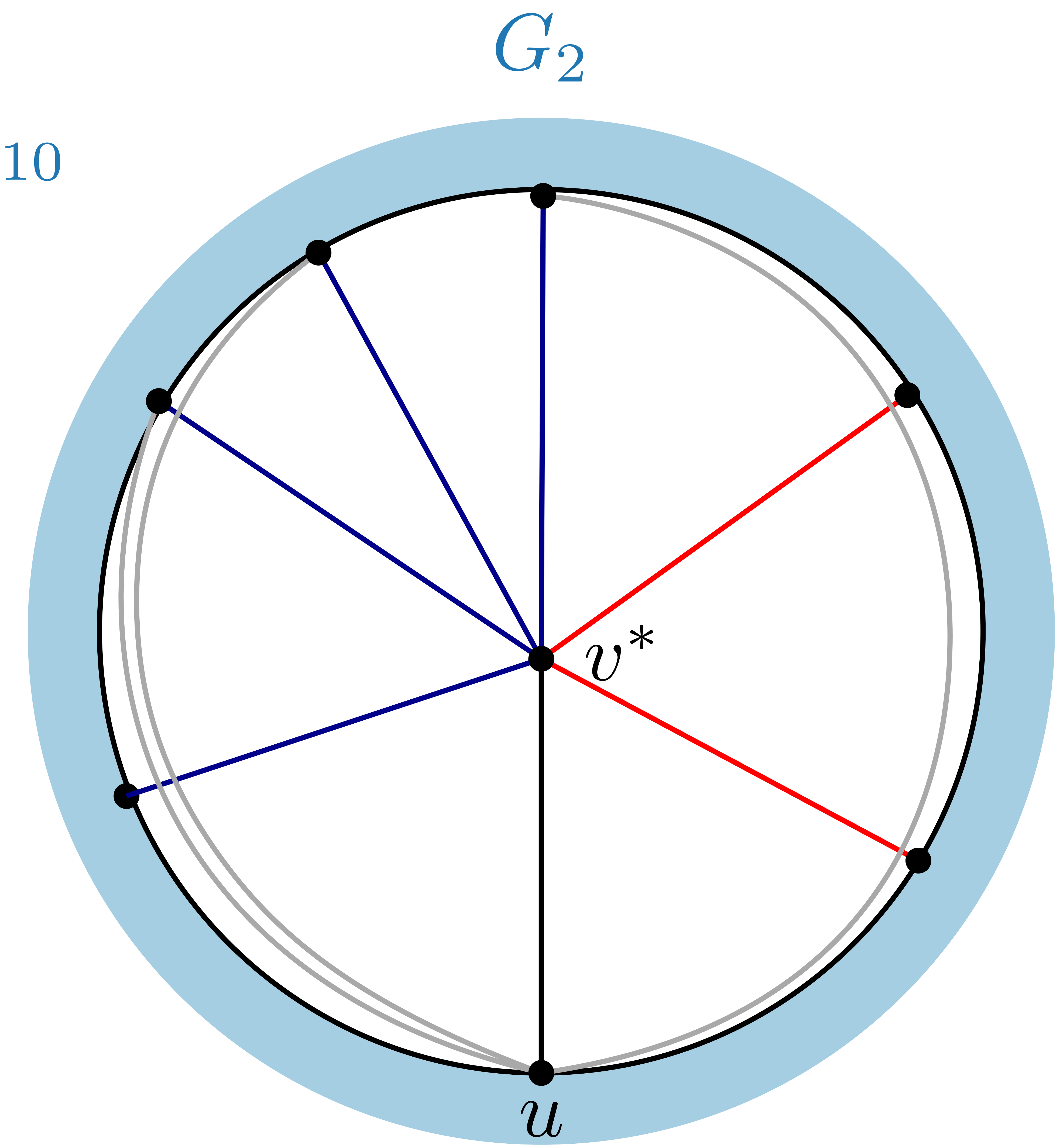


Weak Fan-Planarity: Edge Density

Induction on the **# of edge triples forming pattern (III)**

$$\begin{aligned}|E(G)| &= |E(G_1)| + |E(G_2)| \\ &\leq 5(n - |V(G_2)| + 2) - 10 + 5(|V(G_2)|) - 10\end{aligned}$$





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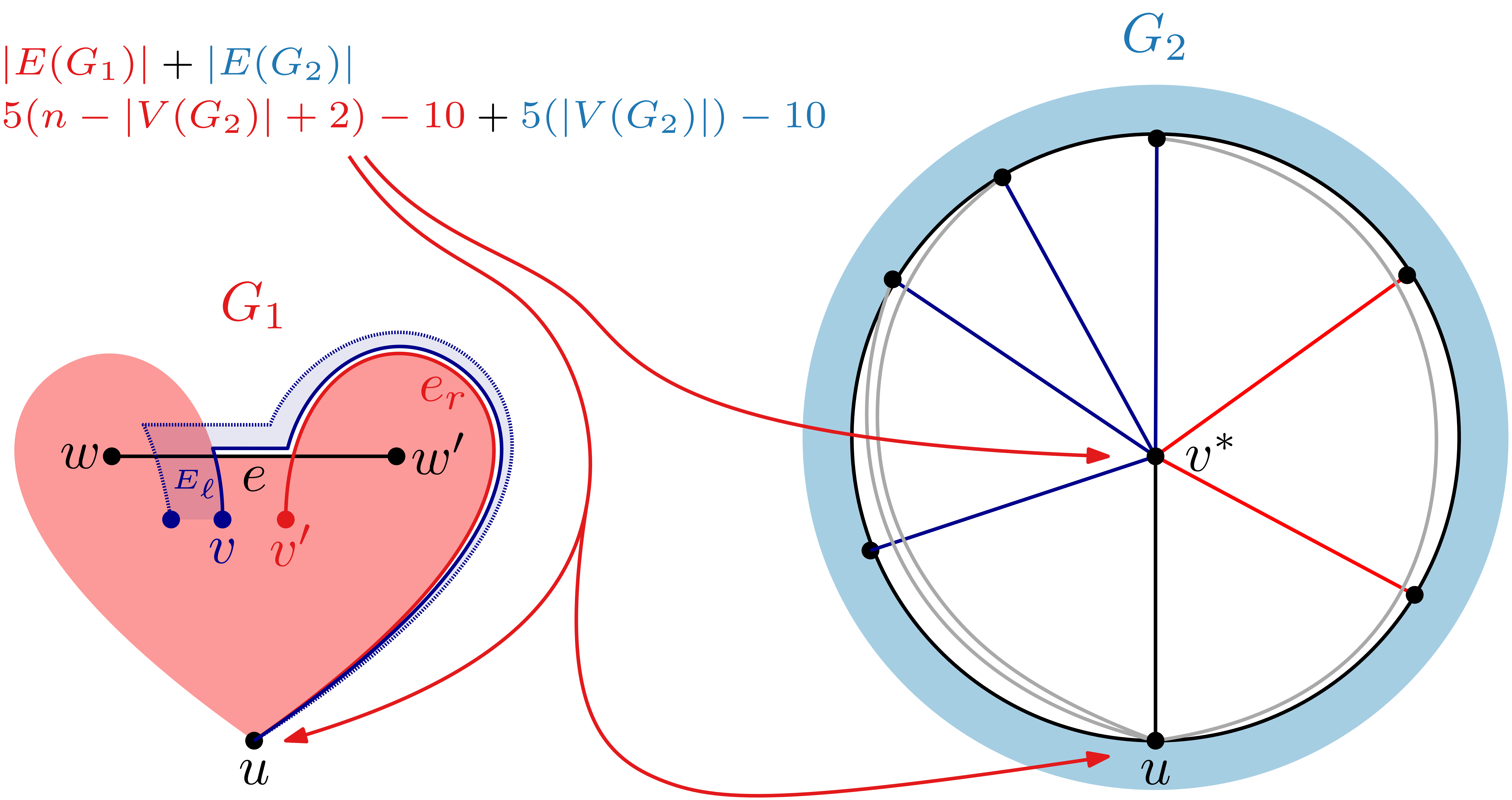

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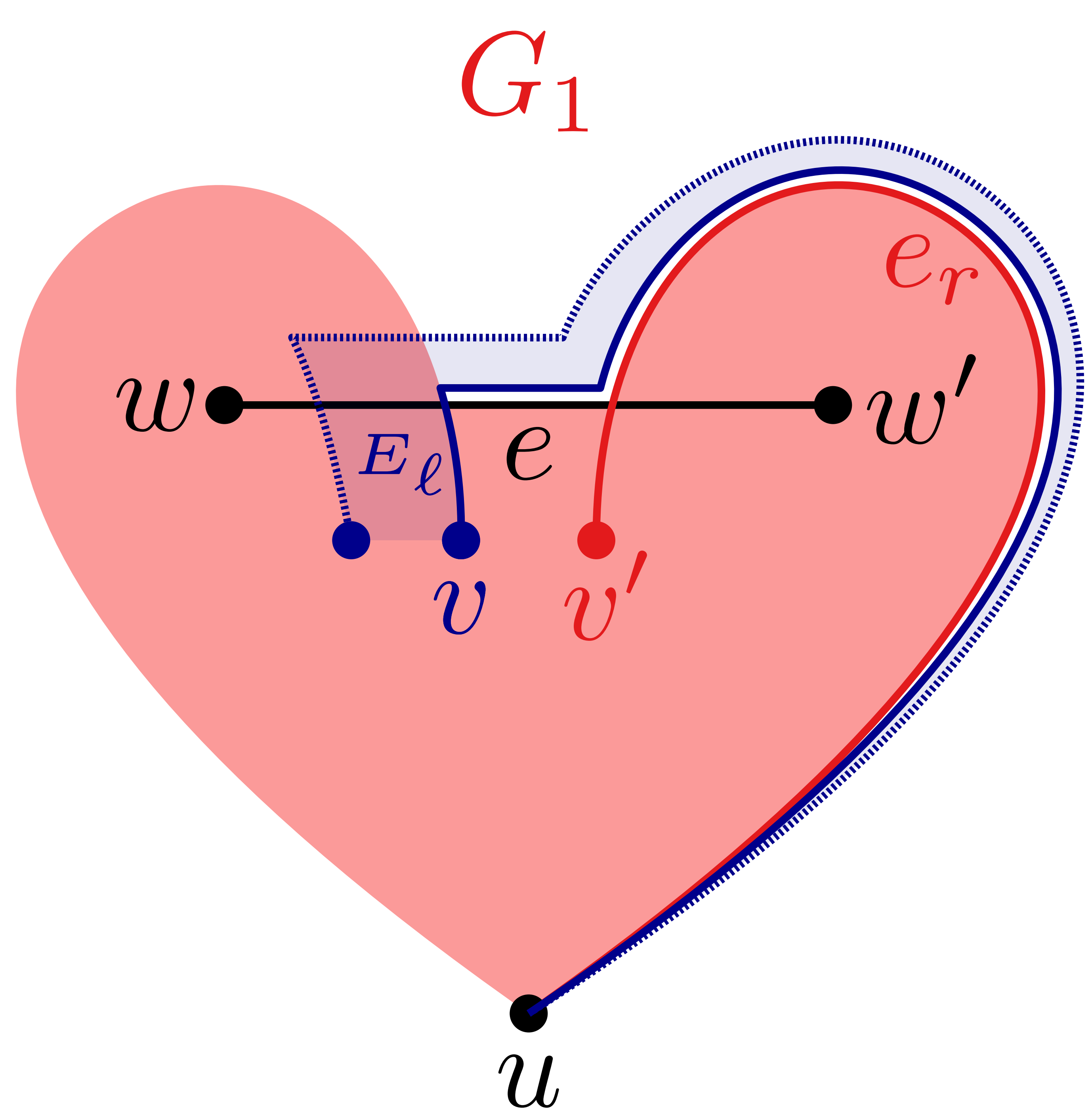
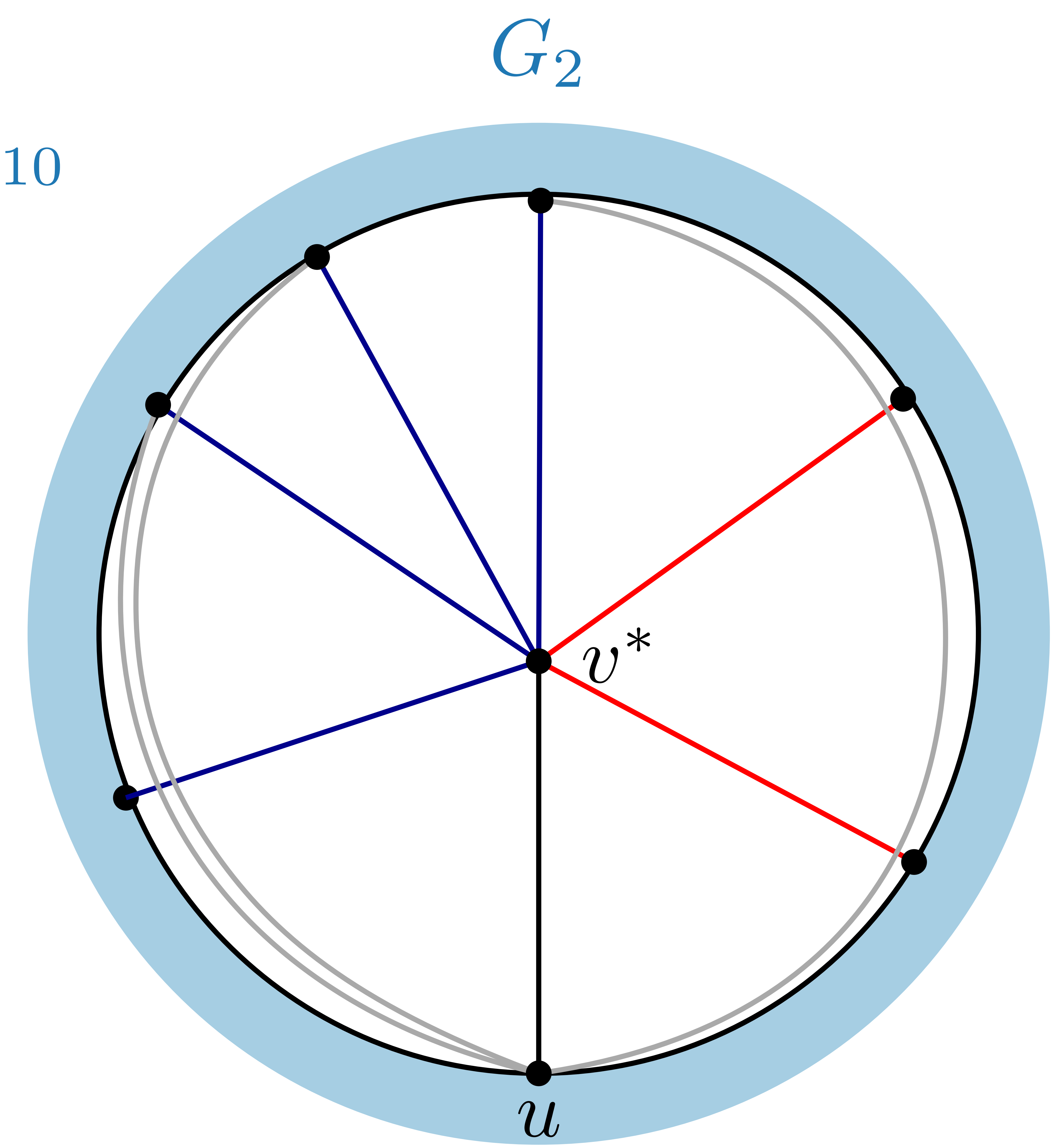
$$\begin{aligned}|E(G)| &= |E(G_1)| + |E(G_2)| \\ &\leq 5(n - |V(G_2)| + 2) - 10 + 5(|V(G_2)|) - 10 \\ &= 5n - 10\end{aligned}$$





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What has been...

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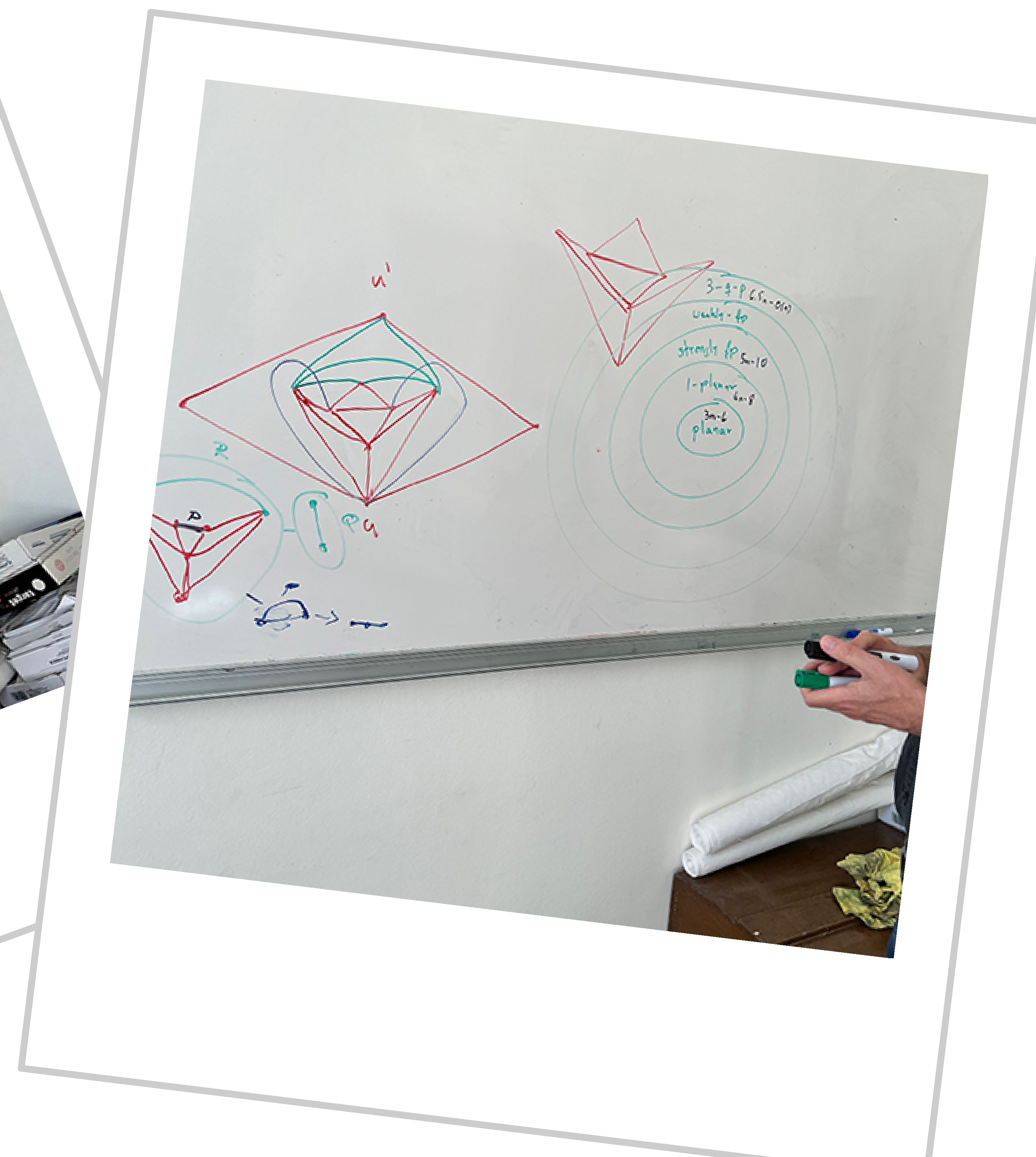

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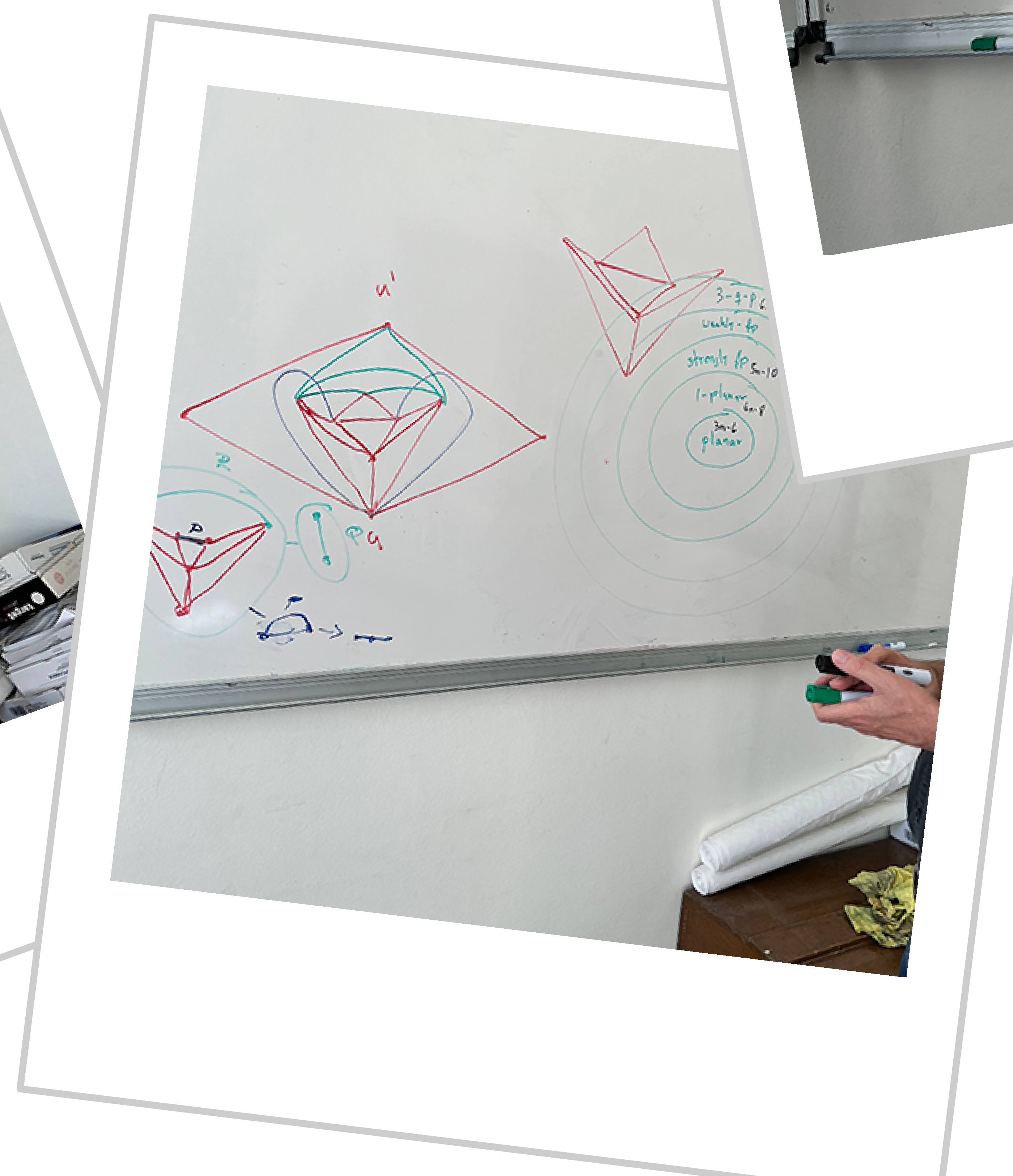

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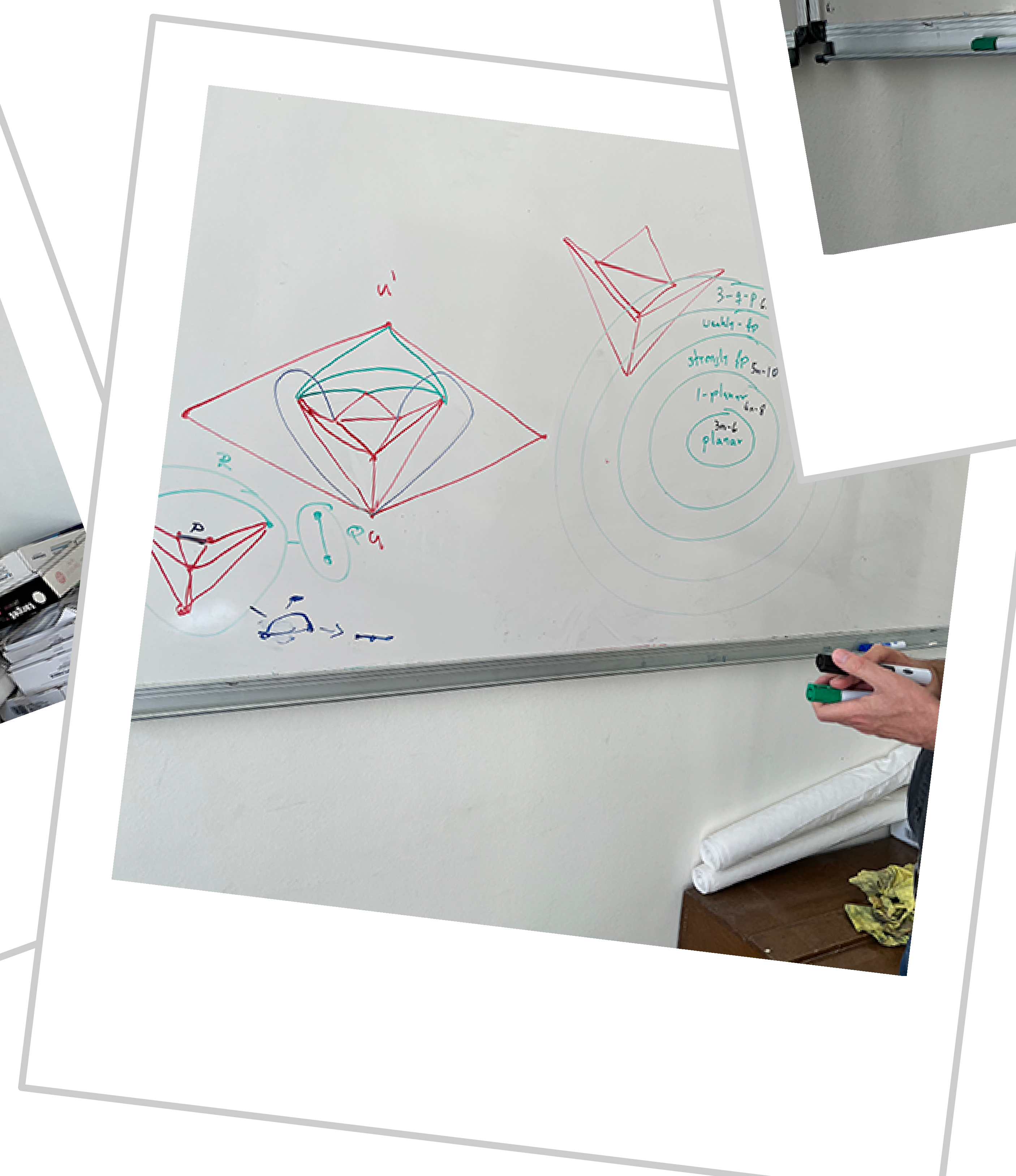

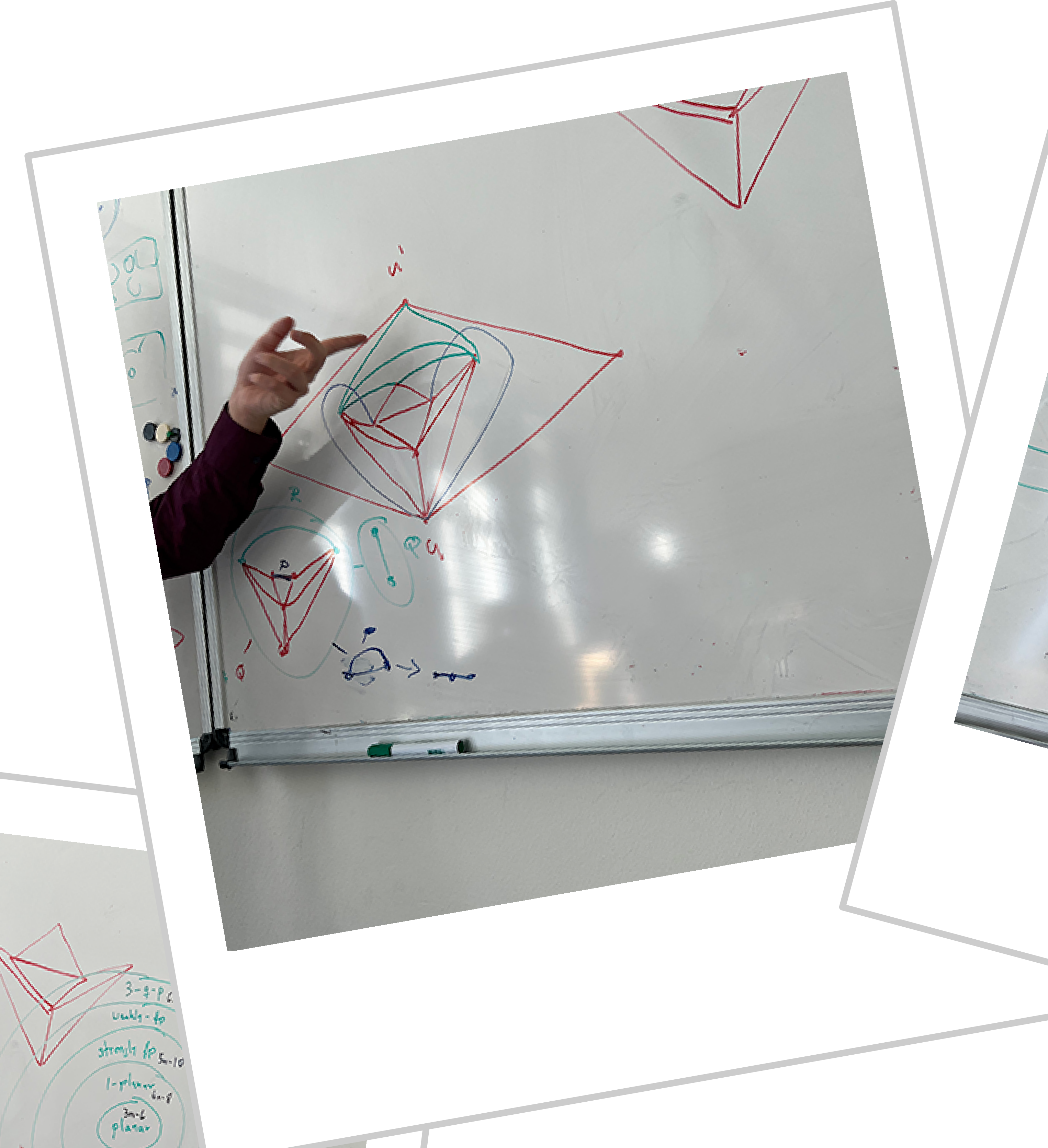

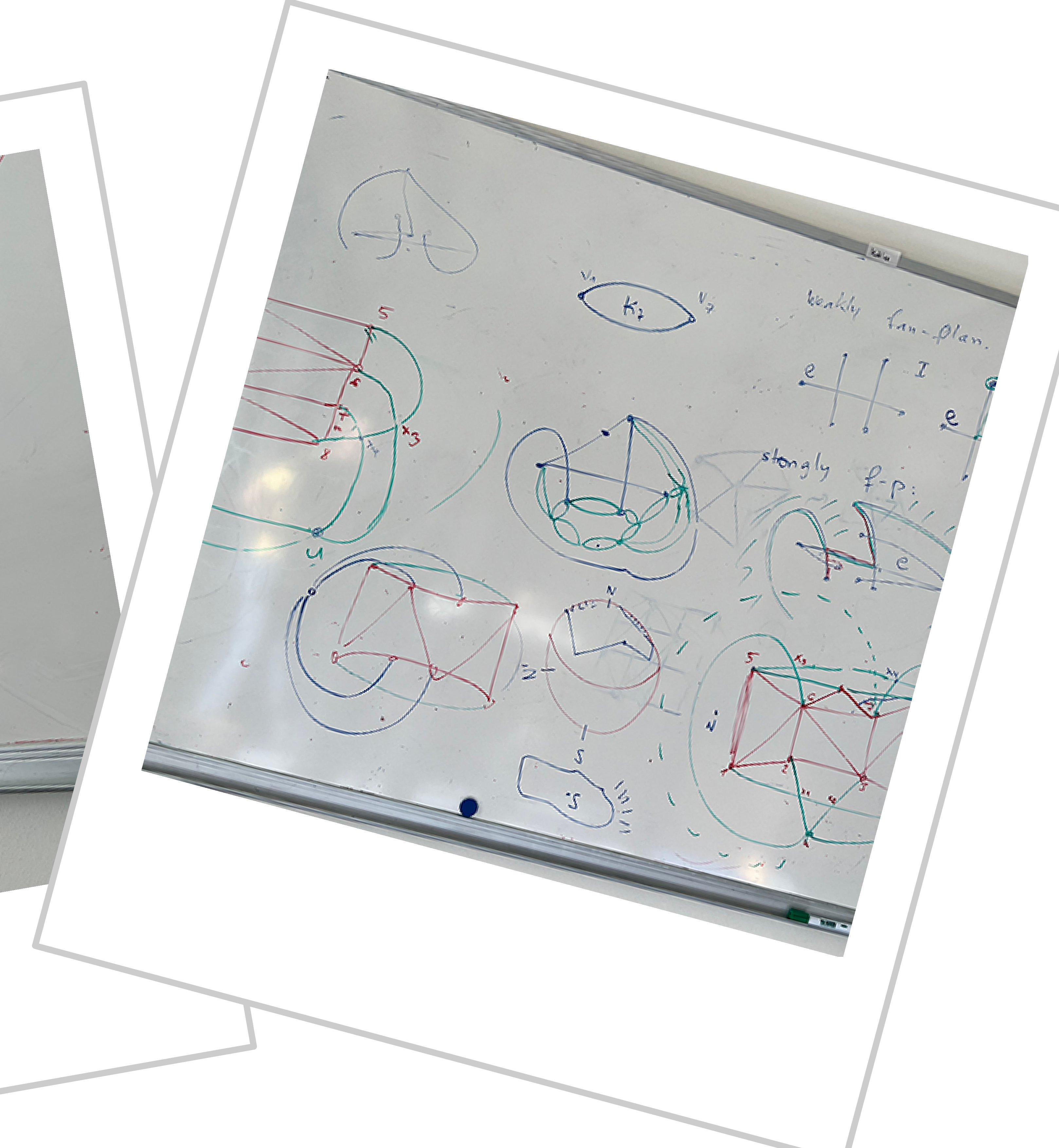

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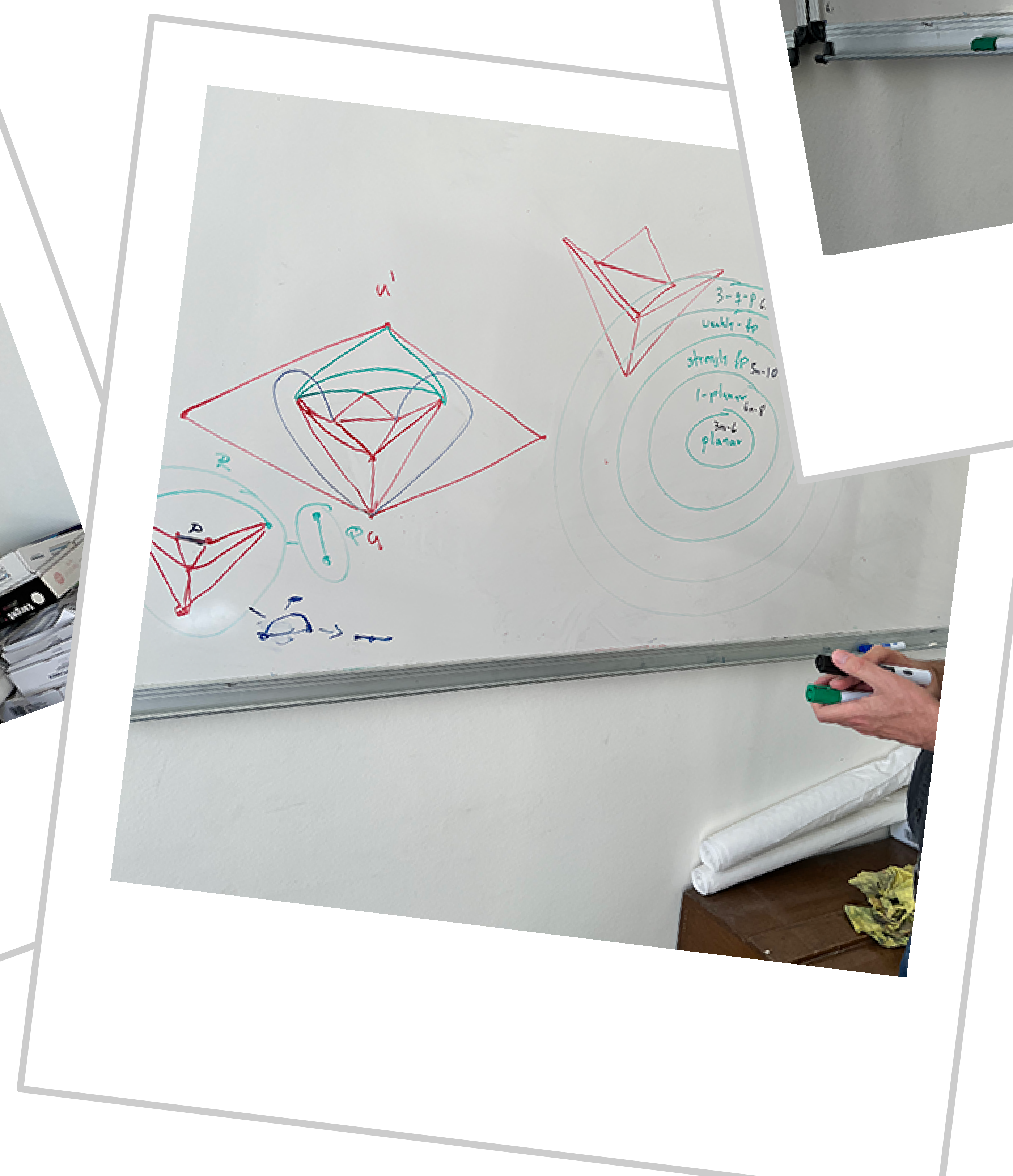

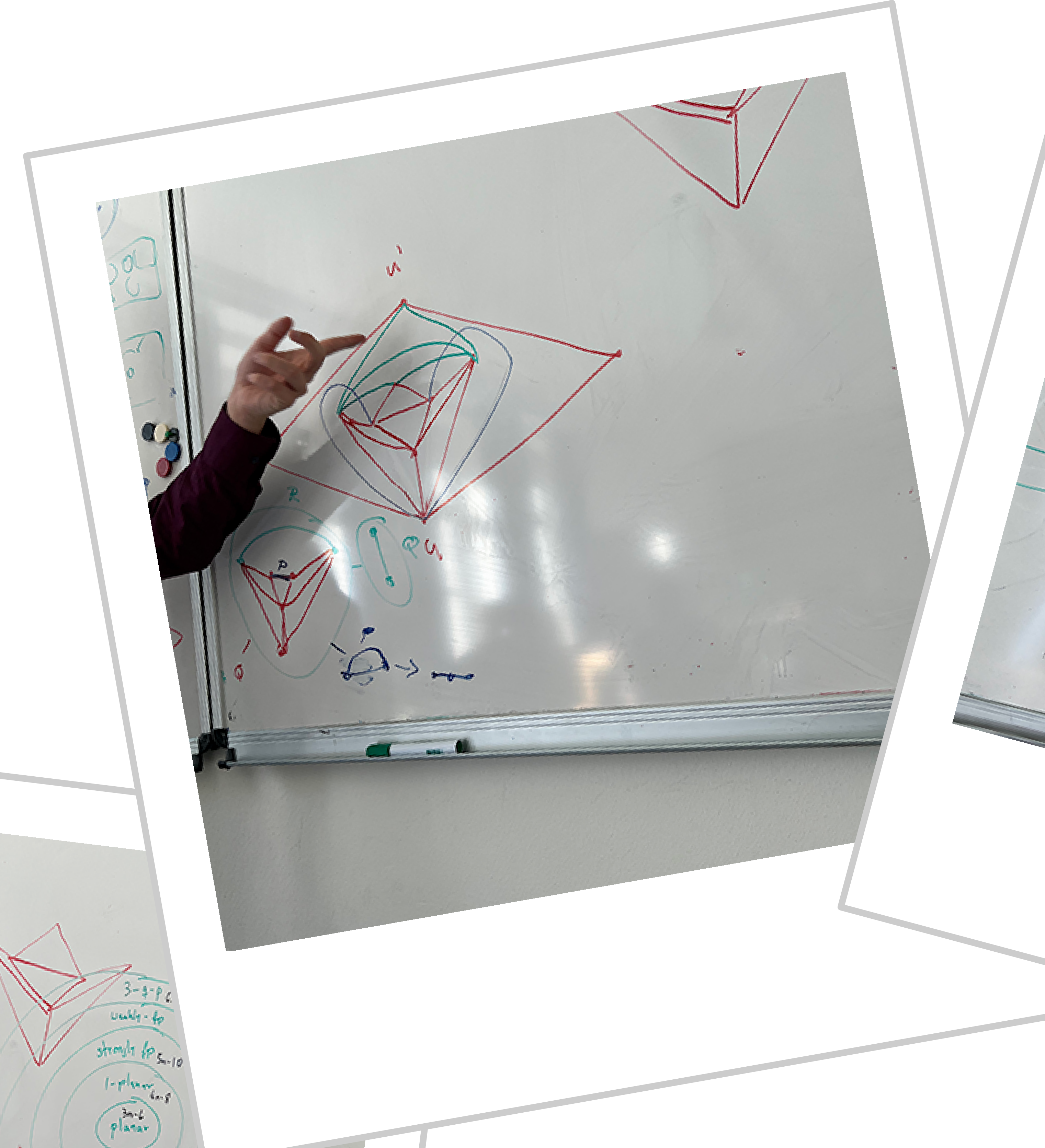

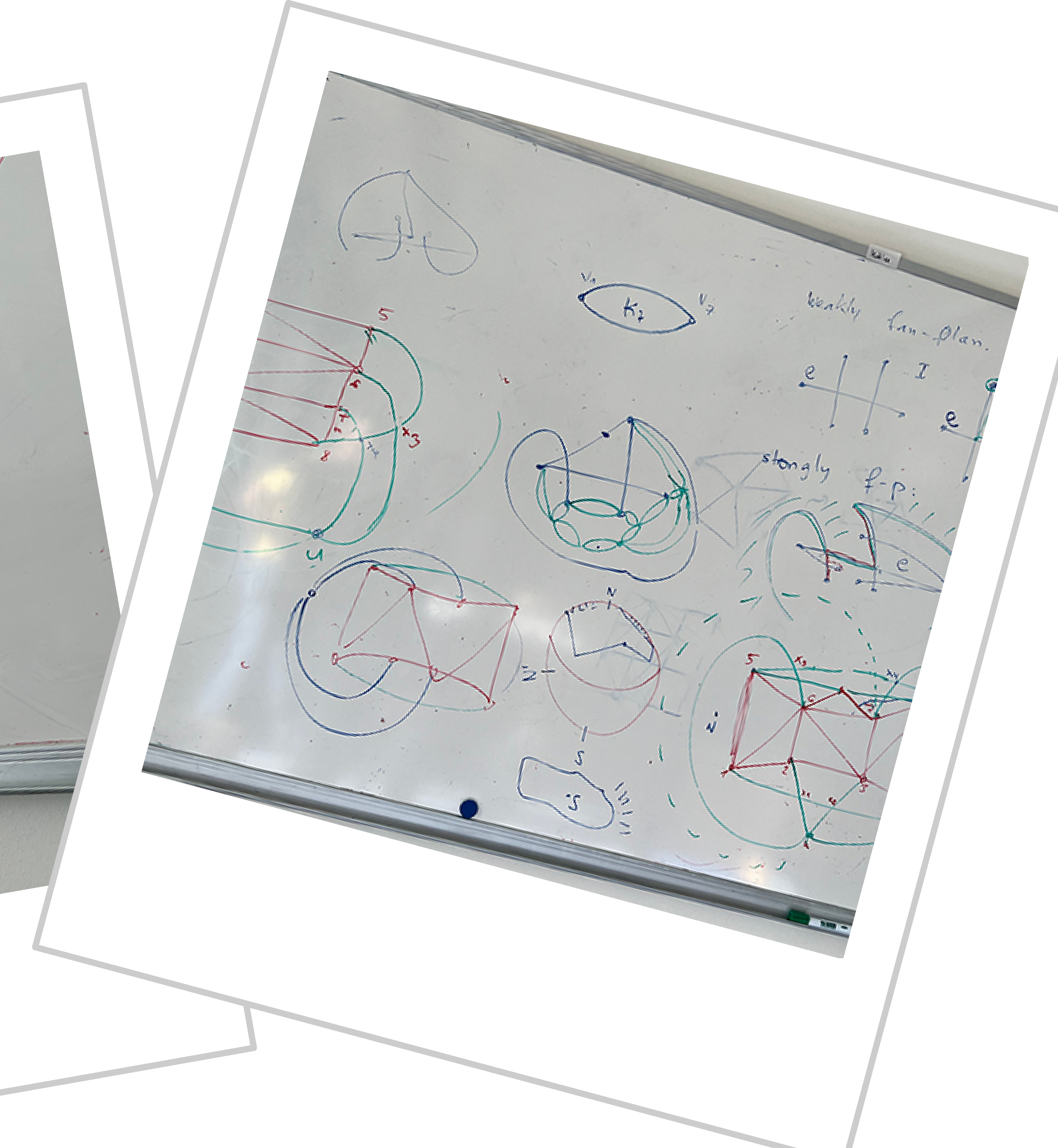

...What’s next

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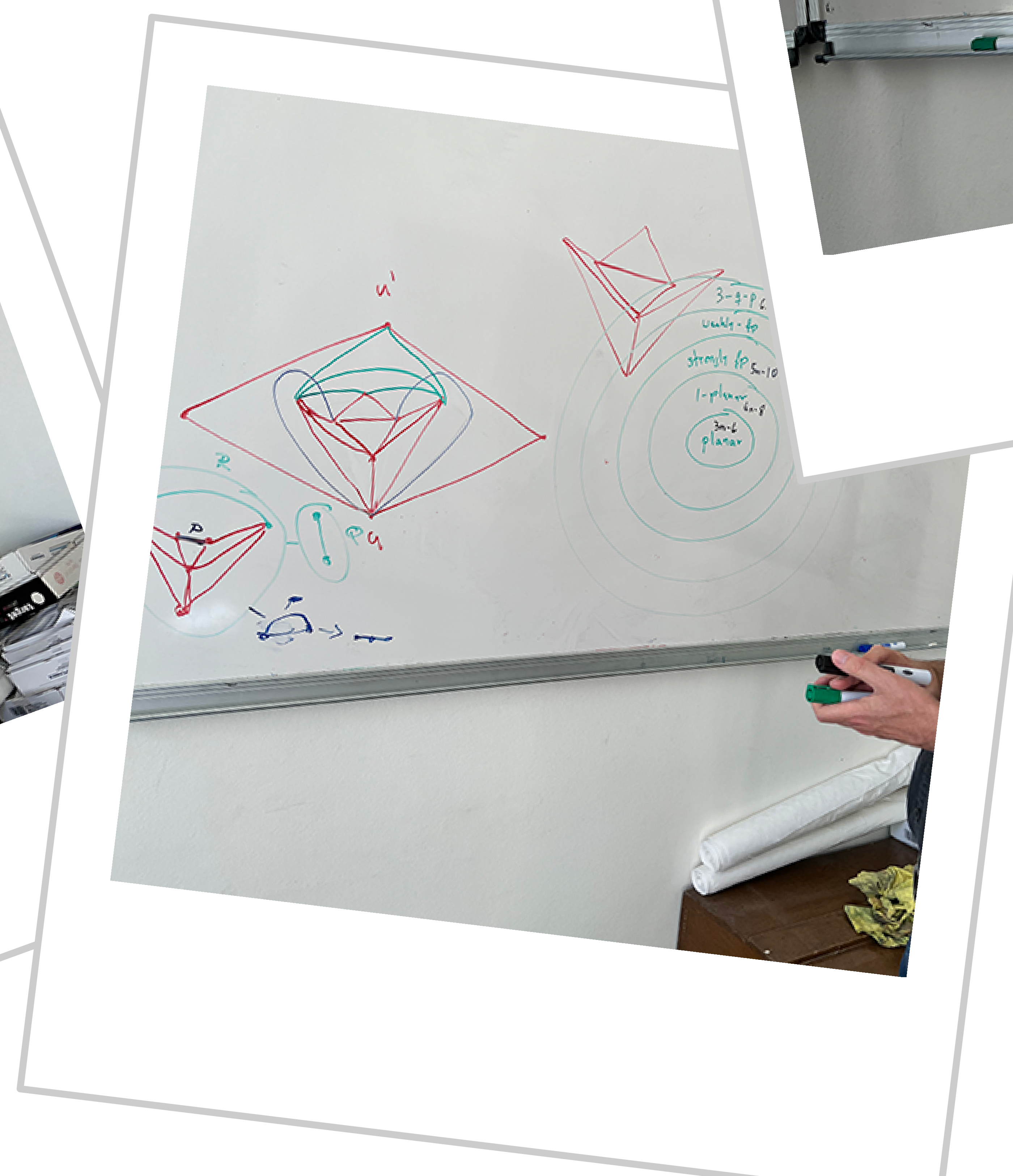

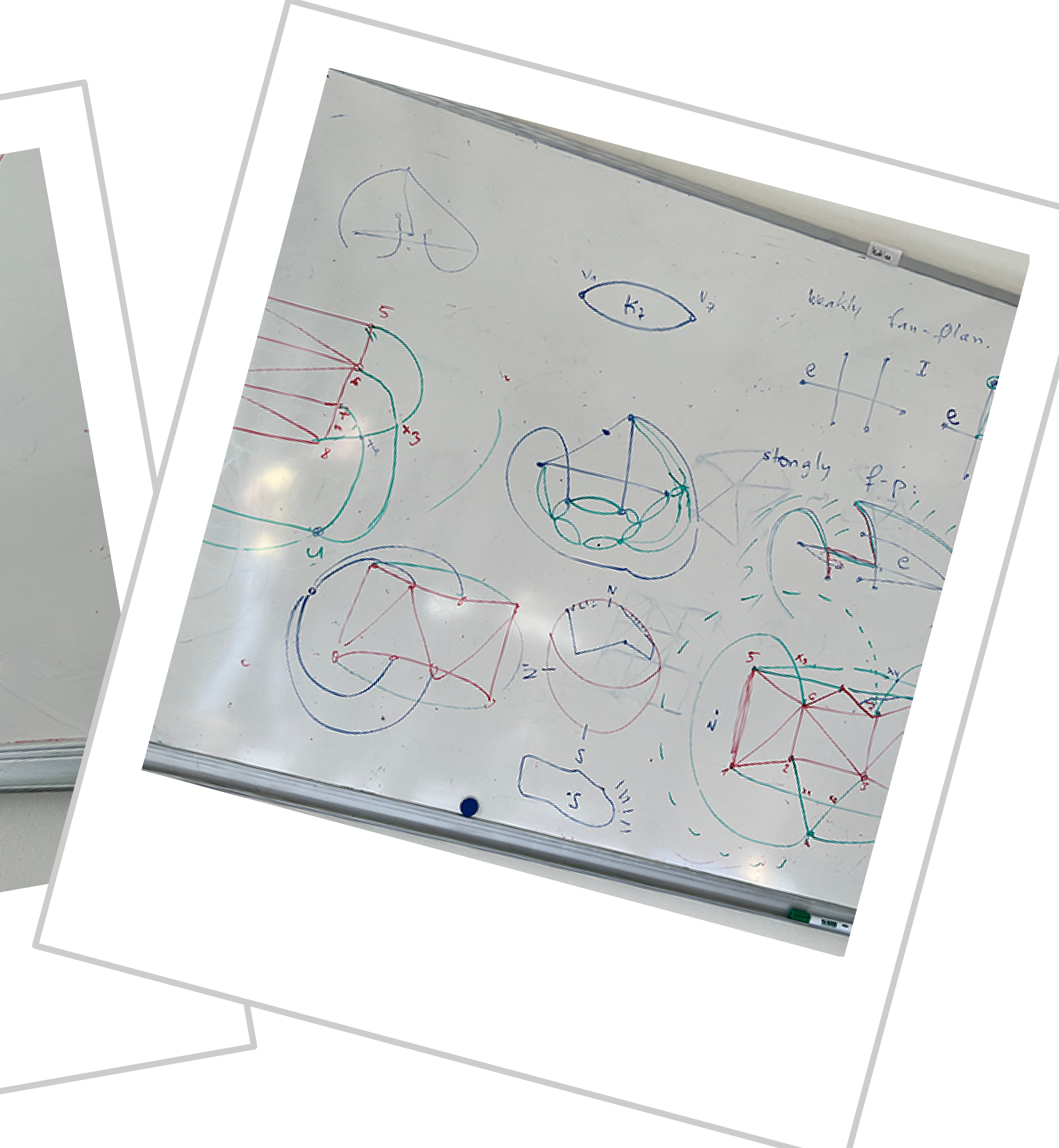

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Survey on weak and strong fan-planarity

Thank you!

Happy to take your **questions**