# On RAC Drawings of Graphs with Two Bends per Edge

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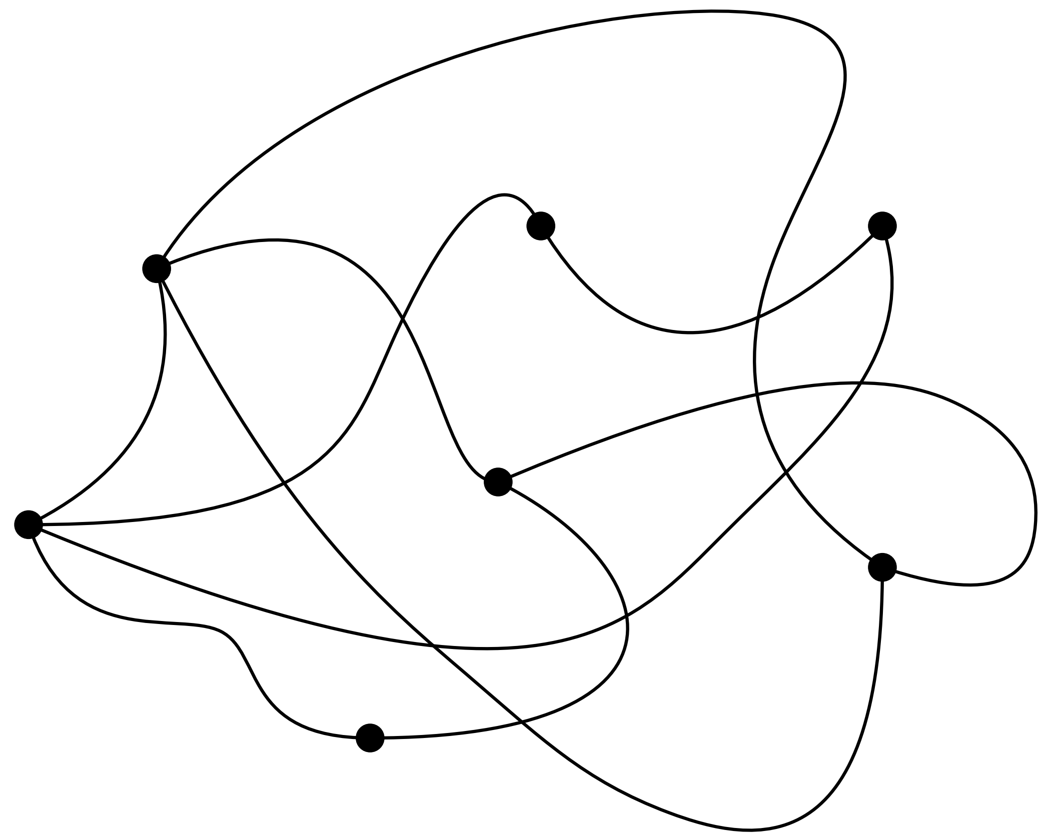
# Right Angle Crossing (RAC) Drawings

In a **topological graph** $G = (V, E)$, the vertices are distinct points in the plane, and every edge is a Jordan arc between two vertices (that does not pass through any vertices).



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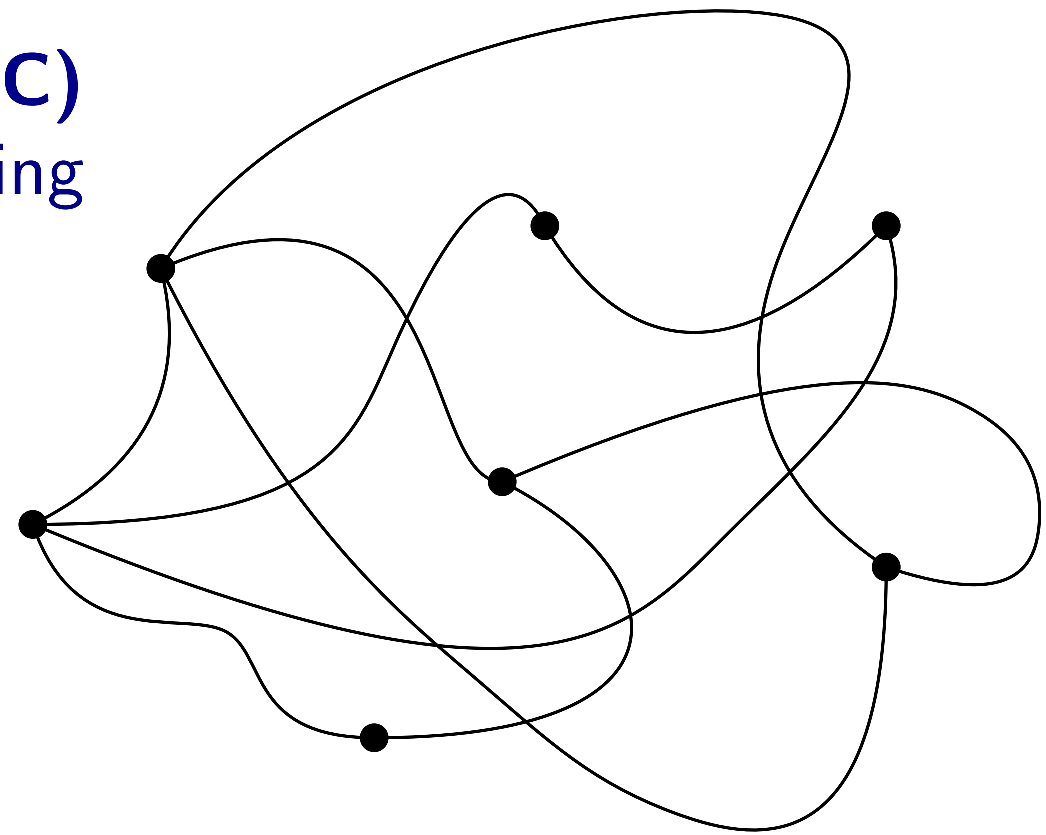

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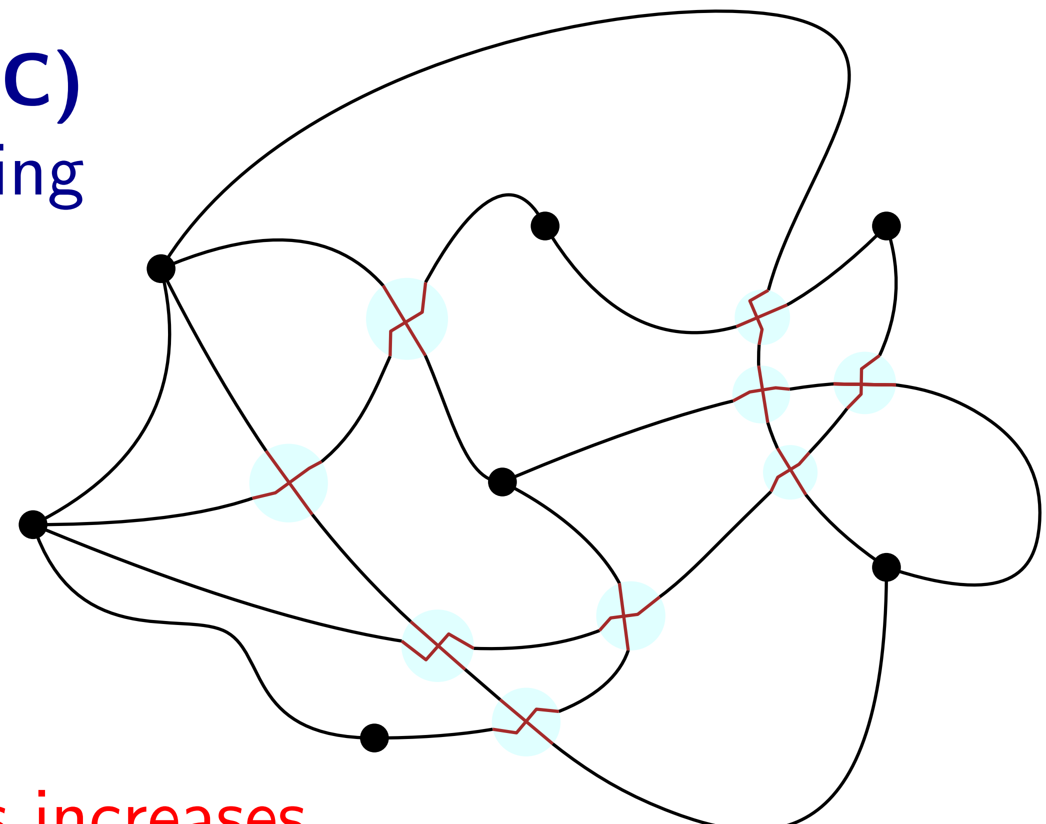
**Observation.** Every good drawing can be perturbed into a RAC drawing, by modiyfying the edges in the neighborhood of crossings.



...the complexity of the edges increases...

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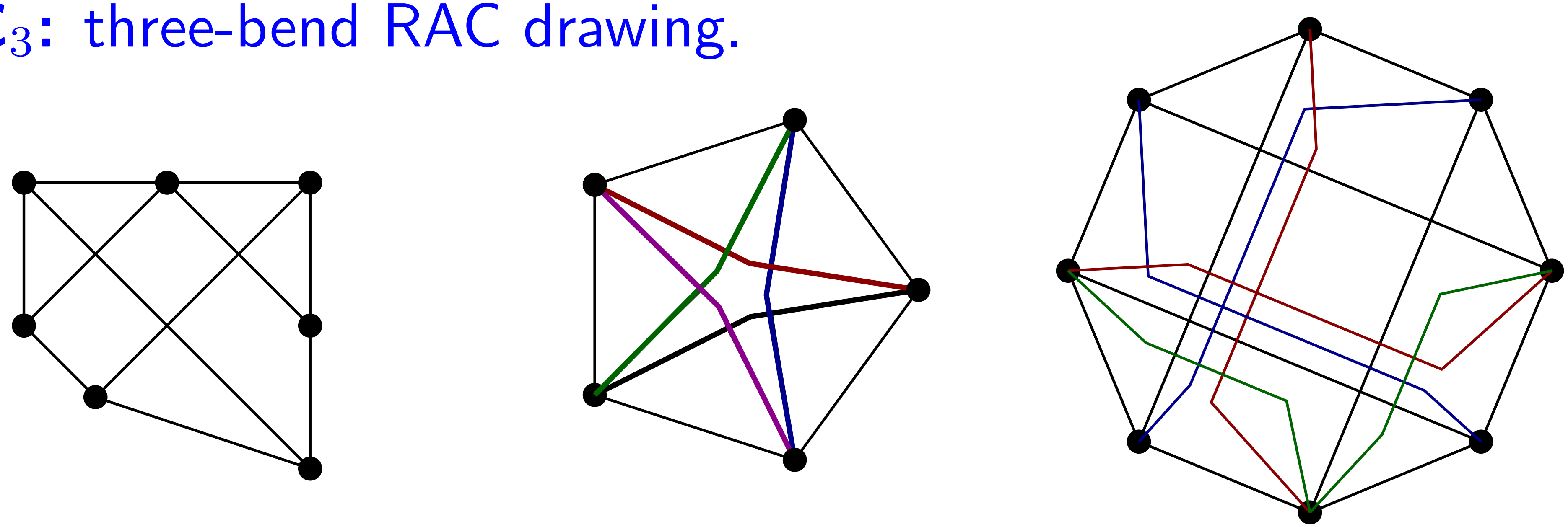
# Right Angle Crossing (RAC) Drawings

For $b \geq 0$, a **RAC$_b$ drawing** is a RAC drawing in which every edge is a polygonal path with at most $b$ interior vetices (bends).

**RAC$_0$:** straight-line RAC drawing,
**RAC$_1$:** one-bend RAC drawing,
**RAC$_2$:** two-bend RAC drawing,
**RAC$_3$:** three-bend RAC drawing.

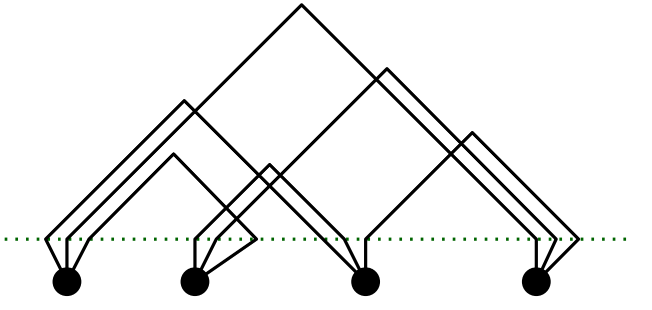


An (abstract) graph is a **RAC$_b$** graph if it admits a RAC$_b$ drawing.

# Right Angle Crossing (RAC) Drawings

How many edges can an $n$-vertex $RAC_b$ graph have?

**Theorem (Didimo, Eades, and Liotta, 2009).**
Every graph is $RAC_3$. This yields an upper bound of $\binom{n}{2}$.



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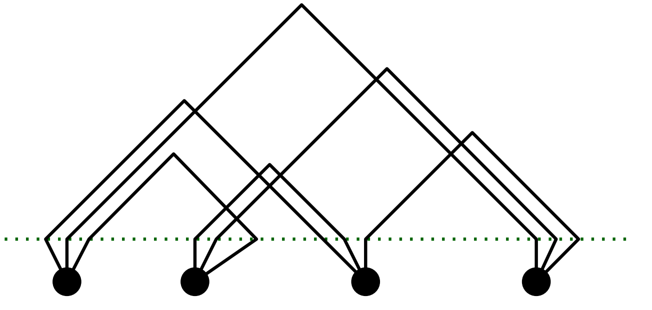

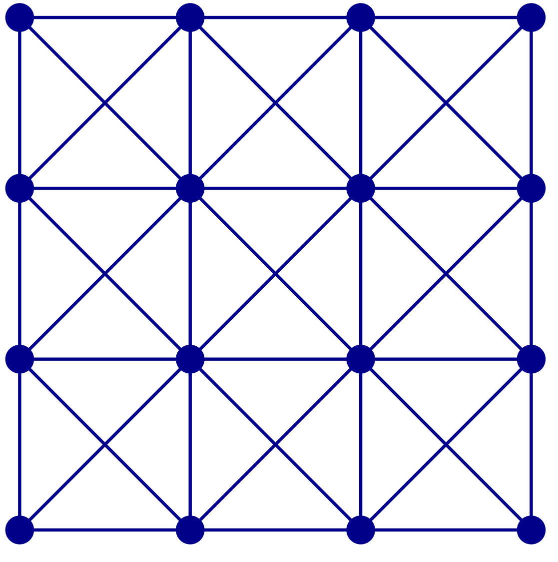

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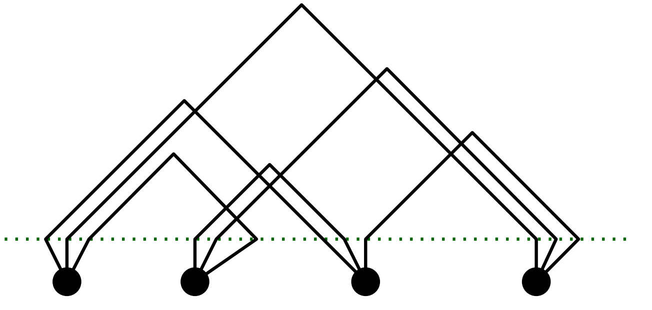

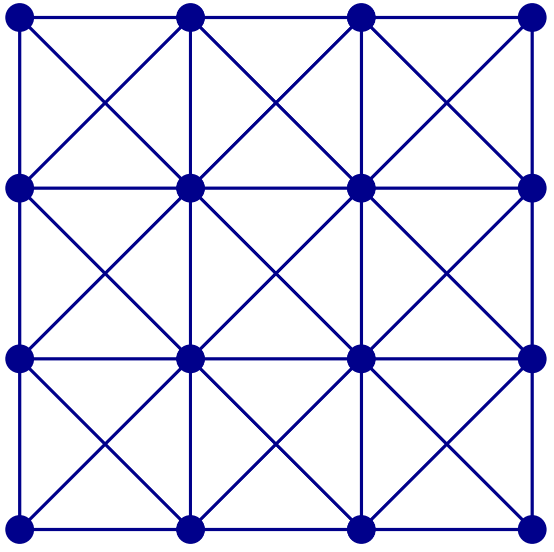

**Theorem (Angelini, Bekos, Förster, and Kaufmann, 2020).**
Every $RAC_1$ graph on $n$ vertices has at most $5.5n - O(1)$ edges, and this bound is the best possible.

# Right Angle Crossing (RAC) Drawings

How many edges can an $n$-vertex $RAC_2$ graph have?

$\leq 74.2n$ (Arikushi, Fulek, Keszegh, Moric, and Tóth, 2010)

$\geq 7.83n - O(\sqrt{n})$ (Arikushi, Fulek, Keszegh, Moric, Tóth, 2010)

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**Theorem.** Every $RAC_2$ graph on $n \geq 3$ vertices has at most $24n - 26$ edges.

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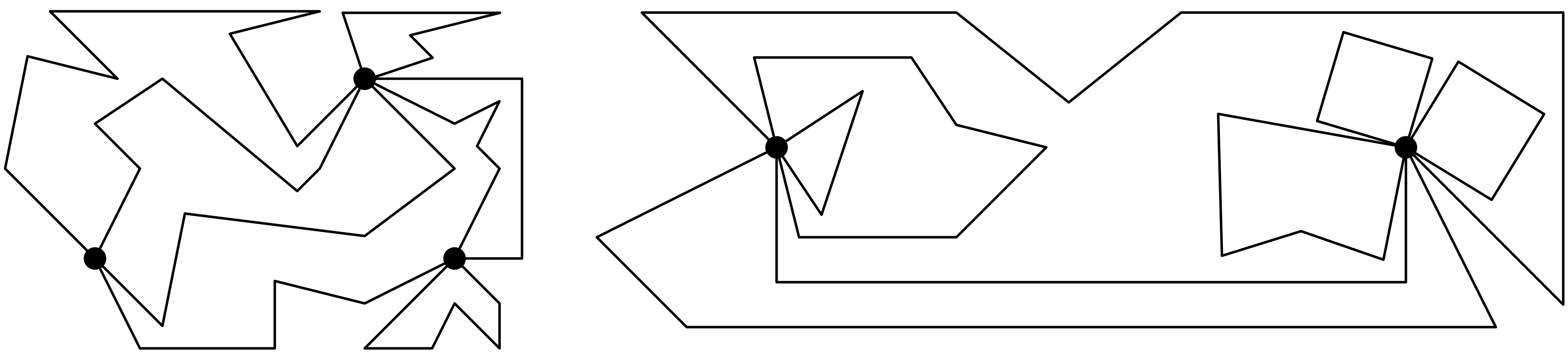
**Theorem.** Every $RAC_2$ graph on $n \geq 3$ vertices has at most ~~$24n - 26$~~ $20n - 24$ edges.

# Multigraphs with Angle-Constrained End Segments

In a **plane multigraph**, the vertices are distinct points, and the edges are Jordan arcs between the corresponding vertices (not passing through any other vertex), and any pair of edges may intersect only at vertices.

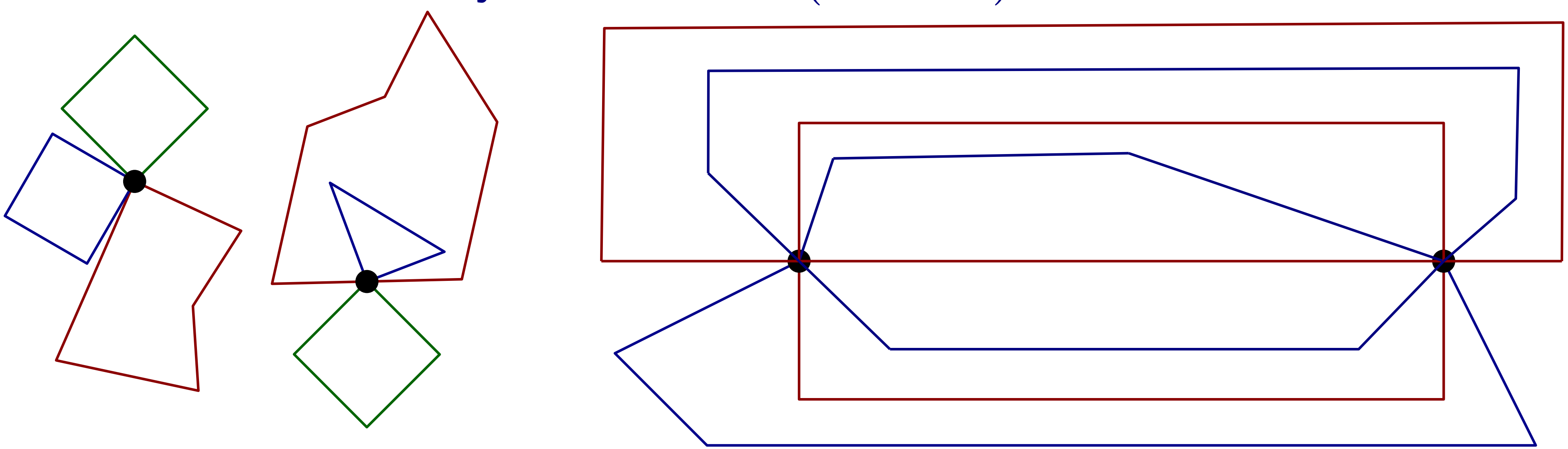


A **plane ortho-fin multigraph** is a plane multigraph such that every edge is a polygonal path $(p_0, p_1, \ldots, p_k)$ where the first and last edge segments are either parallel or orthogonal, that is, $p_0p_1 \| p_{k-1}p_k$ or $p_0p_1 \perp p_{k-1}p_k$.

# Multigraphs with Angle-Constrained End Segments

How many edges can an $n$-vertex plane ortho-fin multigraph have?

- Every vertex has at most 3 loops.
- Every edge $uv$ has multiplicity at most 8.
- Euler's formula yields $3n + 8(3n-6) = 27n - 48$ for $n \geq 3$.



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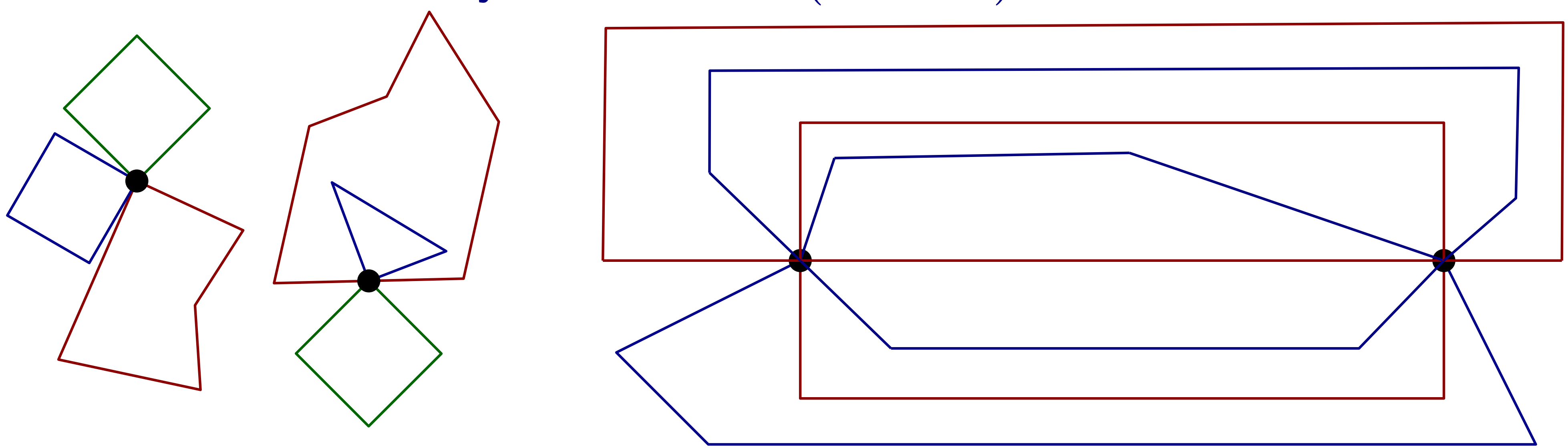
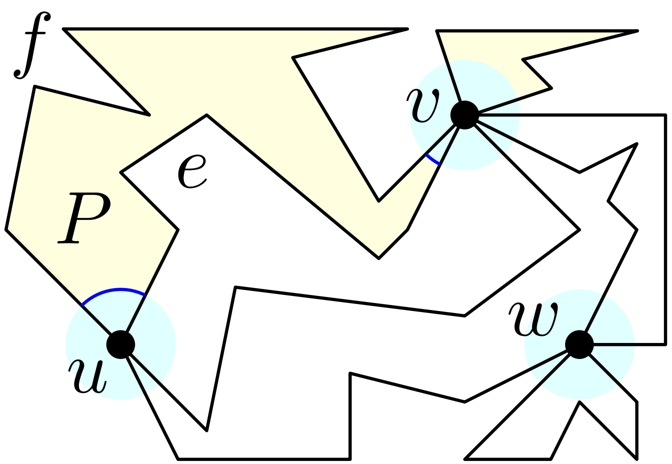


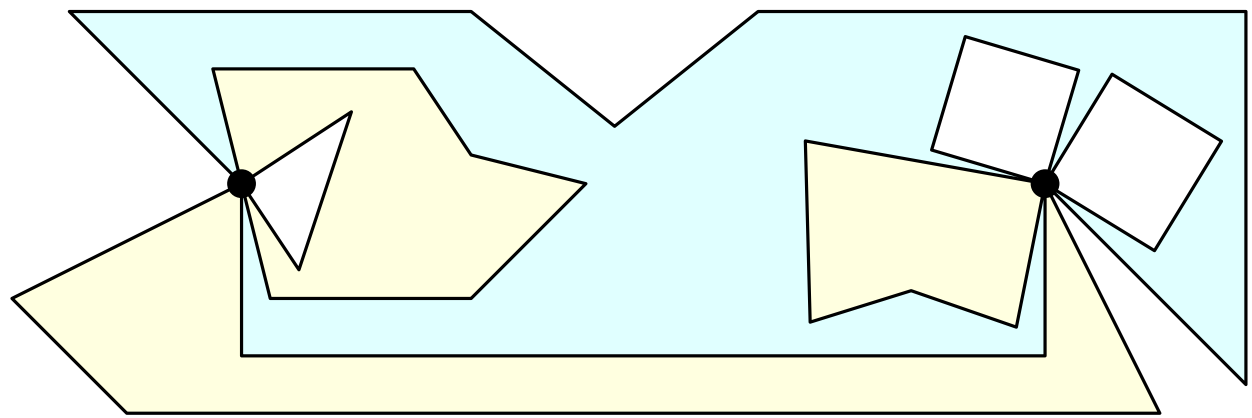
We can improve this bound to $5n - 2$ for all $n \geq 1$.

# Multigraphs with Angle-Constrained End Segments

- The **potential** $\Phi(P)$ of a face $P$ is the sum of interior angles of $P$ over all vertices in $V$ incident to $P$.

**Lemma.** For every face $P$ of an ortho-fin multigraph, $\Phi(P)$ is a multiple of $\pi/2$, in particuar, $\Phi(P) \geq \pi/2$.



**Theorem.** A weak ortho-fin multigraph has at most $5n-2$ edges, and this bound is the best possible.

Summation over all faces yields $\sum_P \Phi(P) \leq 2\pi \cdot n$.

Consequently, the number of faces is at most $f \leq \frac{2\pi \cdot n}{\pi/2} = 4n$.

Combined with Euler's formua, $m = n + f - 2 \Rightarrow m \leq 5n - 2$.

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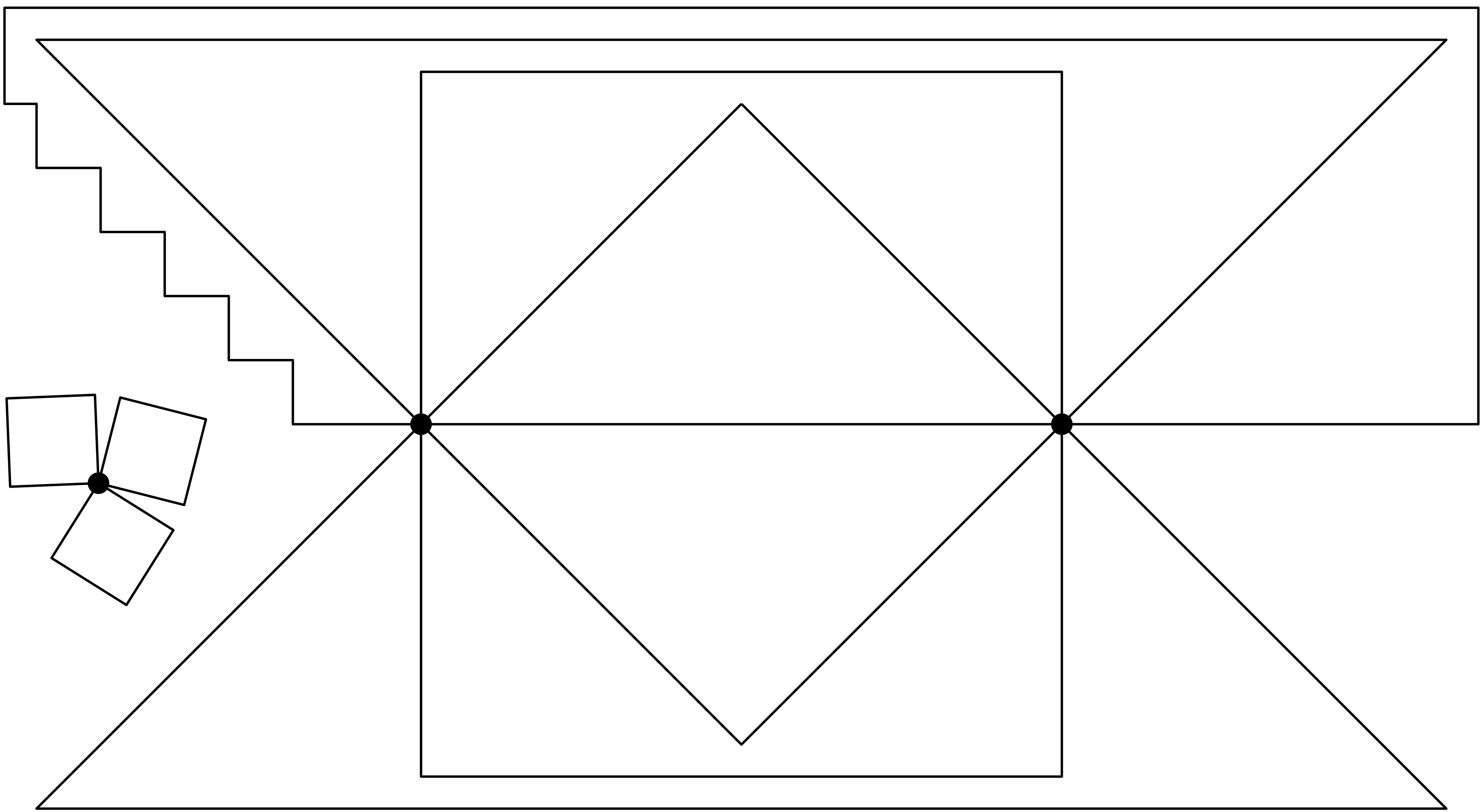

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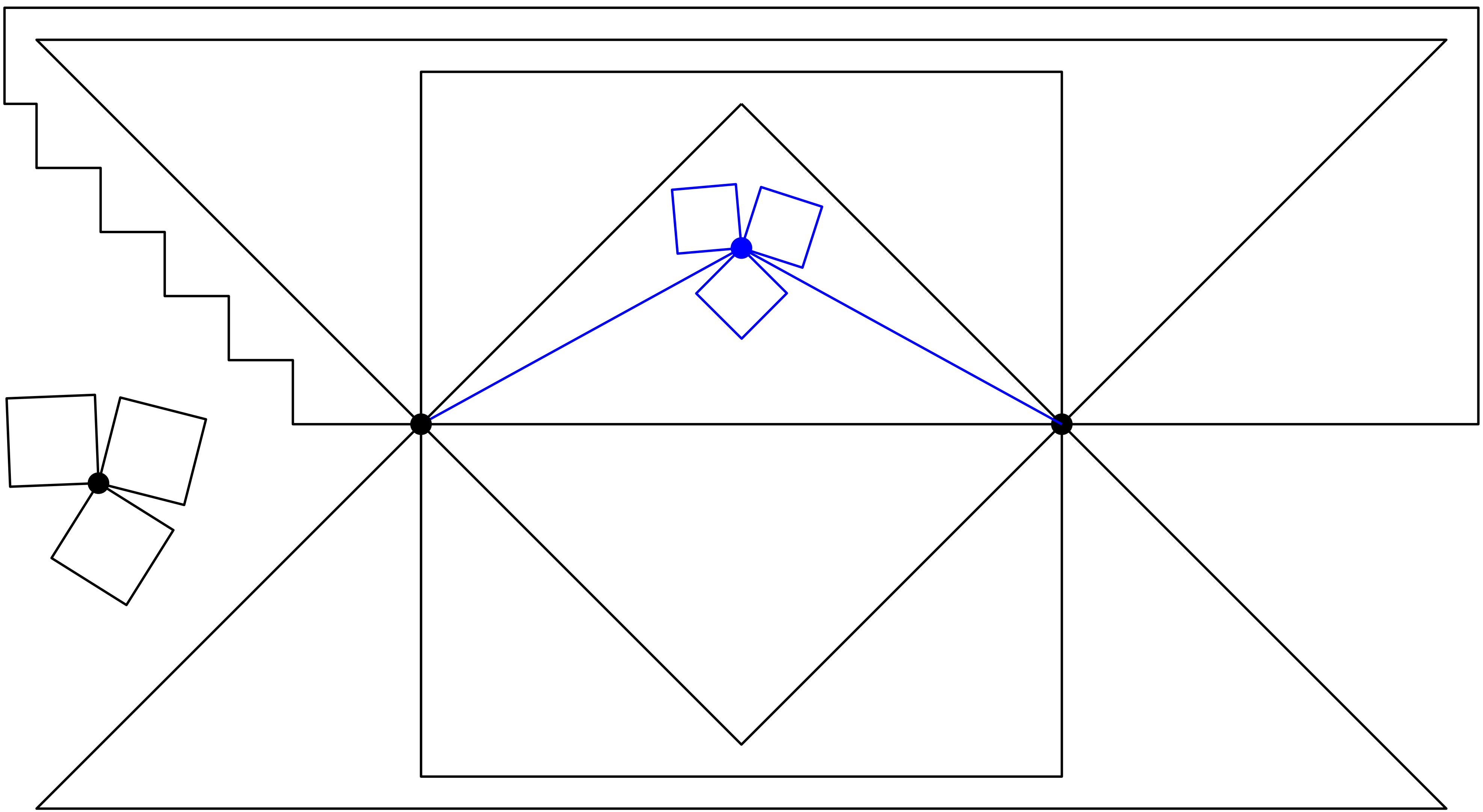

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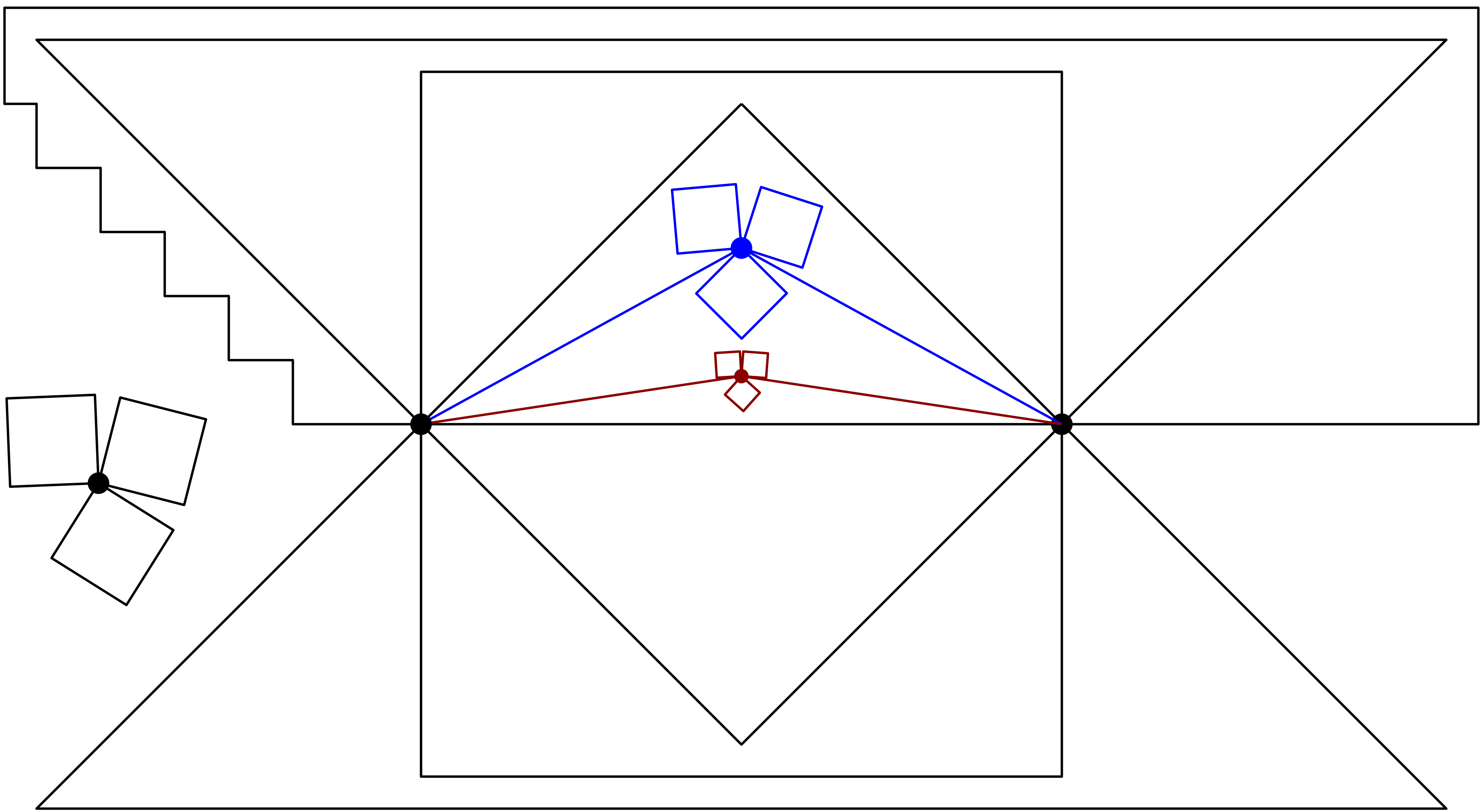
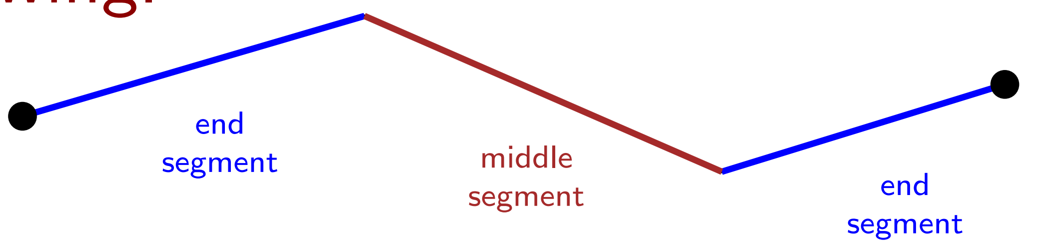
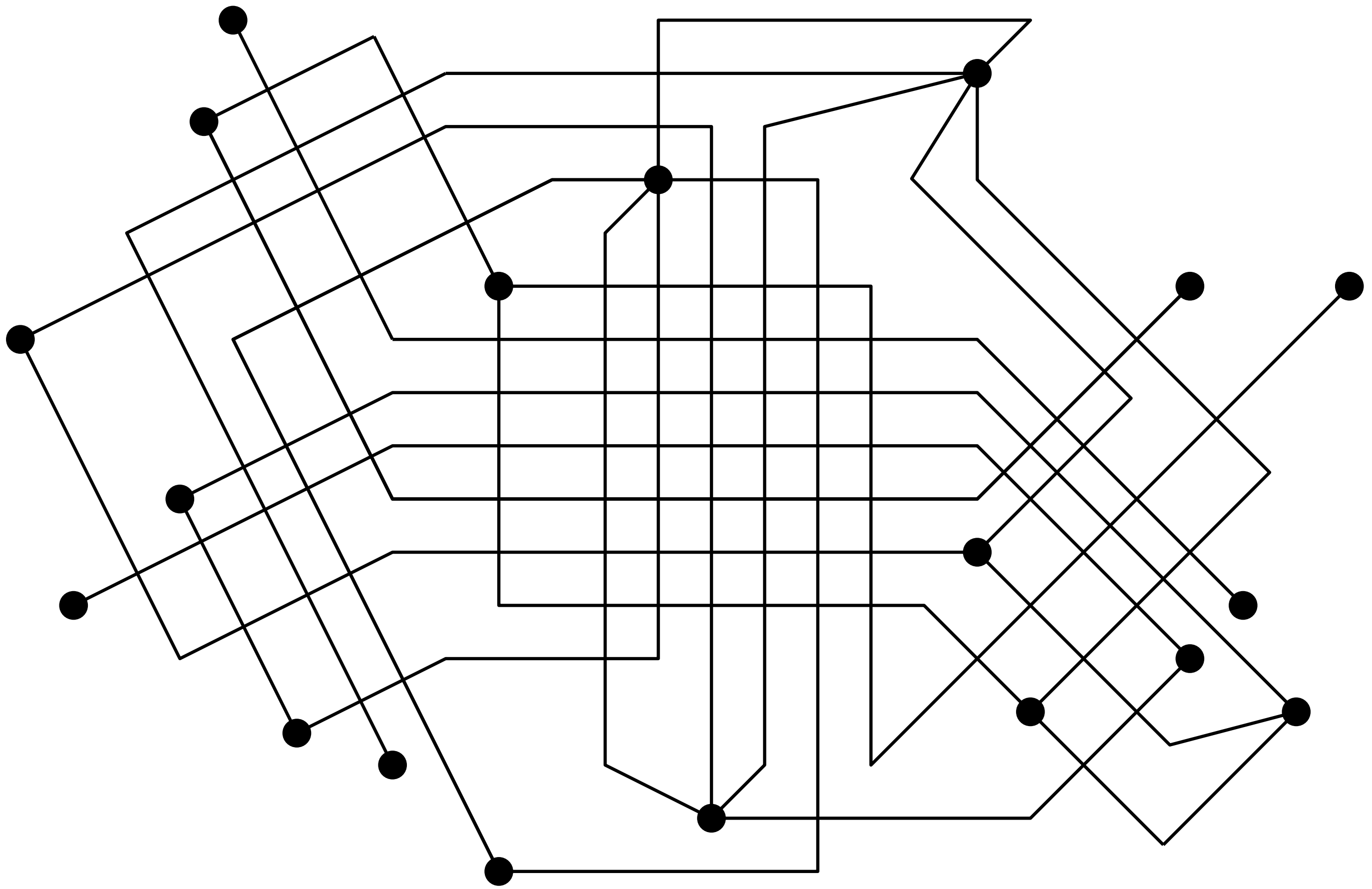


# The density of $RAC_2$ graphs

Let $G = (V, E)$ be a $RAC_2$ drawing.

Every edge has three segments:



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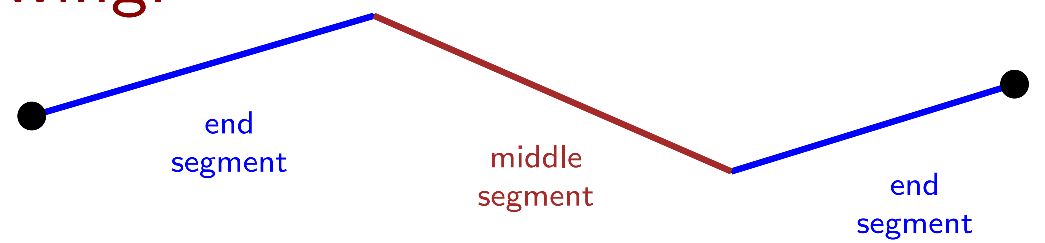
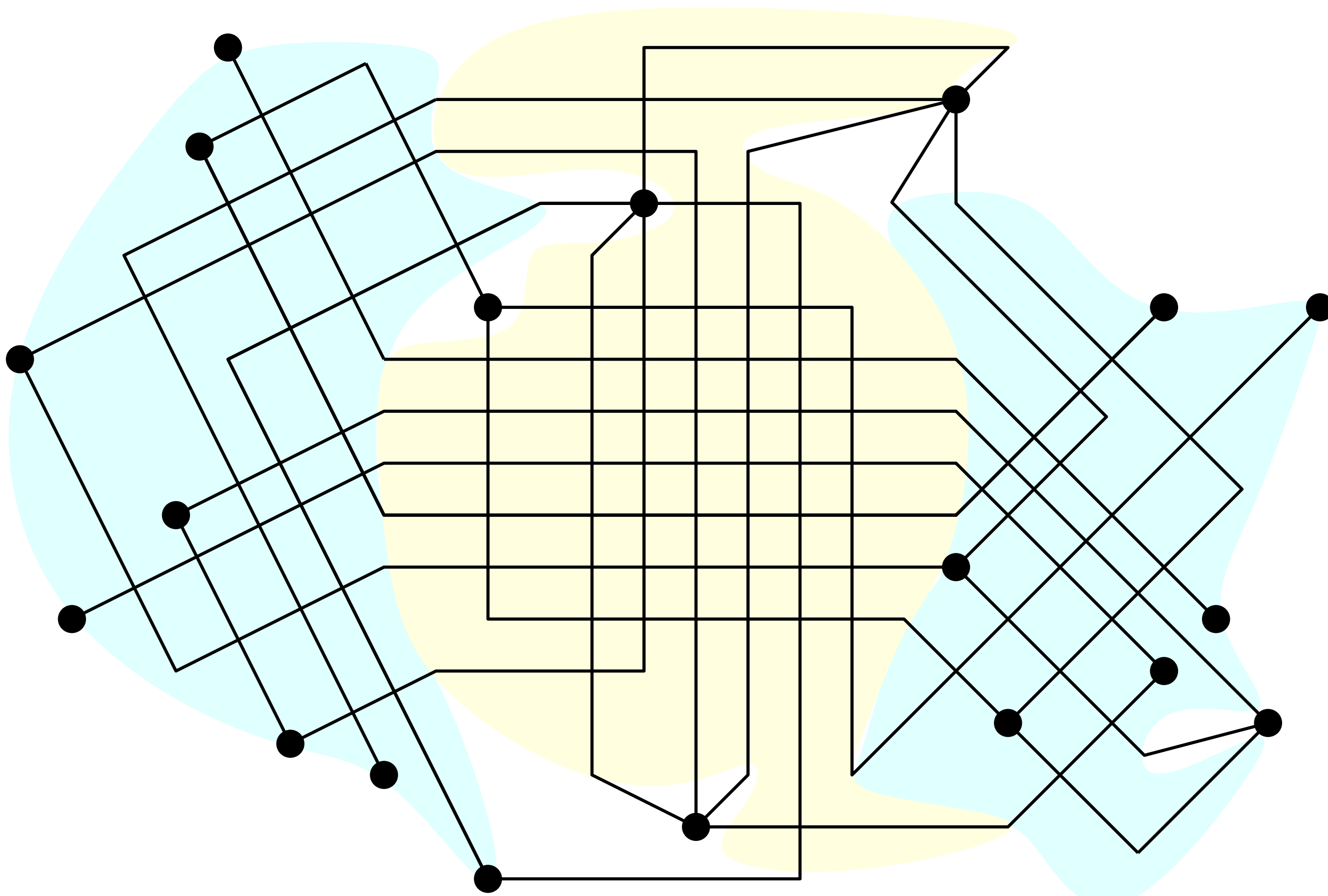


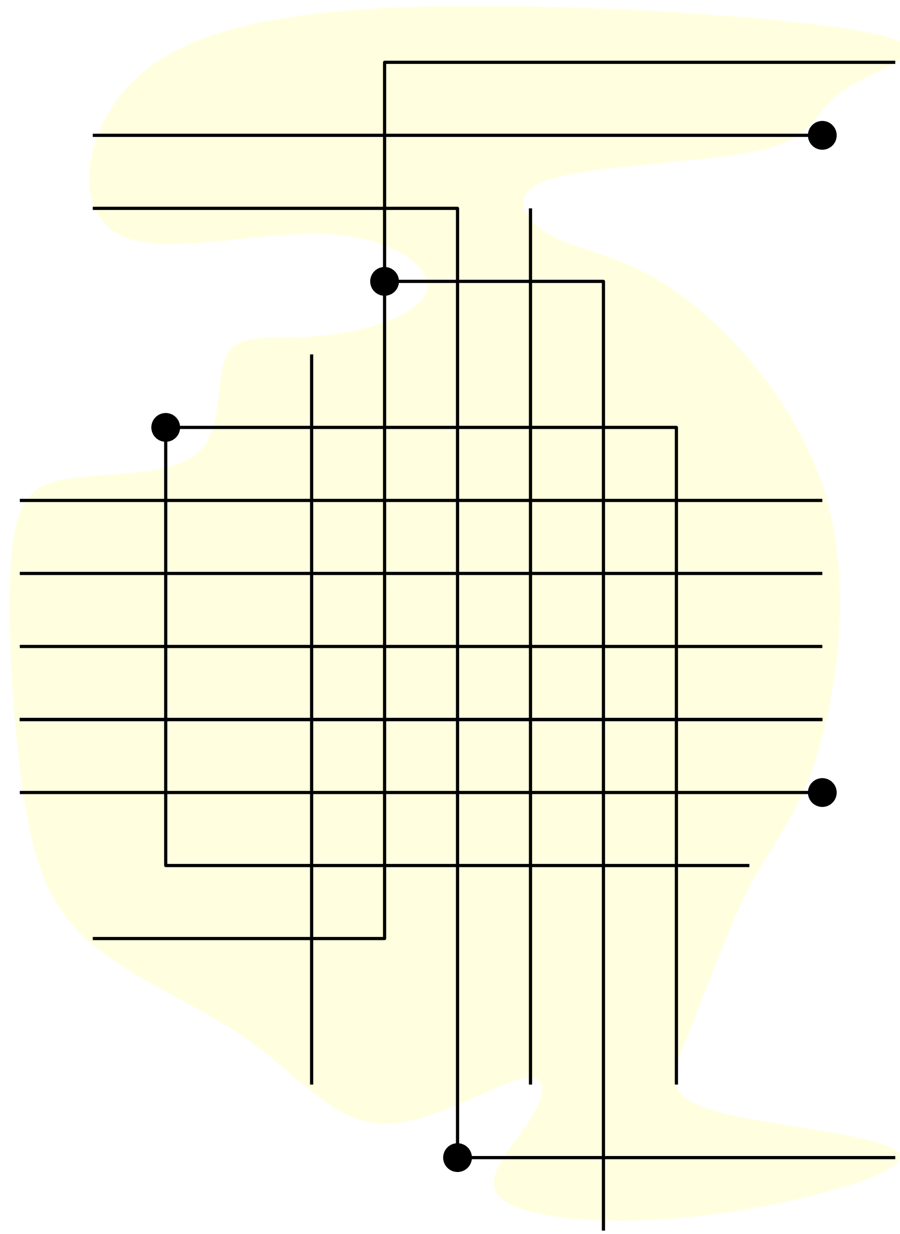



- *crossings* form a symmetric relation on the segments.
- Its transitive closure is an equivalence relation.
- A **block** is the set of segments in an equivalence class.

# The density of $RAC_2$ graphs

Consider one block of the $RAC_2$ drawing $G = (V, E)$.

- Every end segment is incident to a vertex,
- A vertex is incident to at most four end segments.



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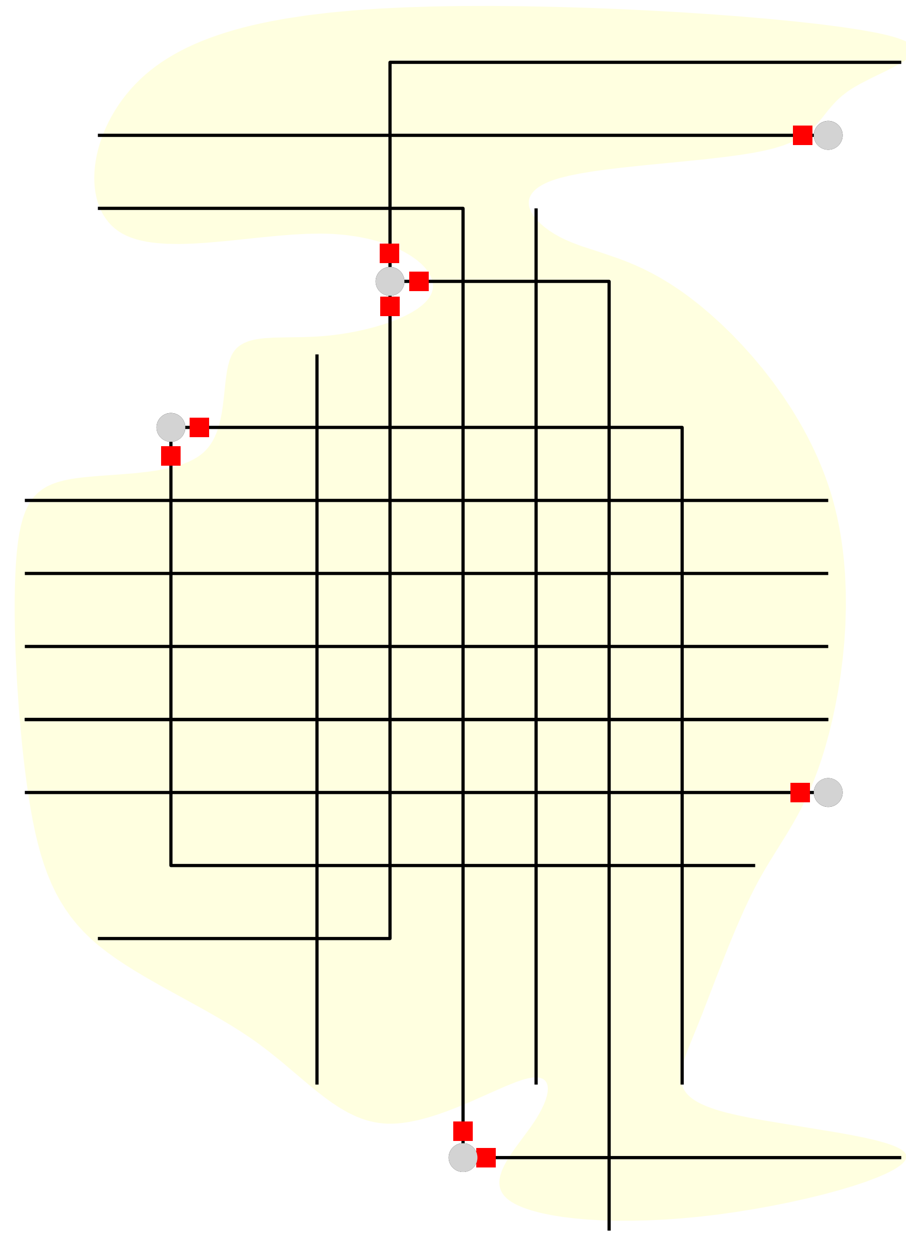
Truncate the neighborhood of vertices to create a unique **terminal** on each end segment.



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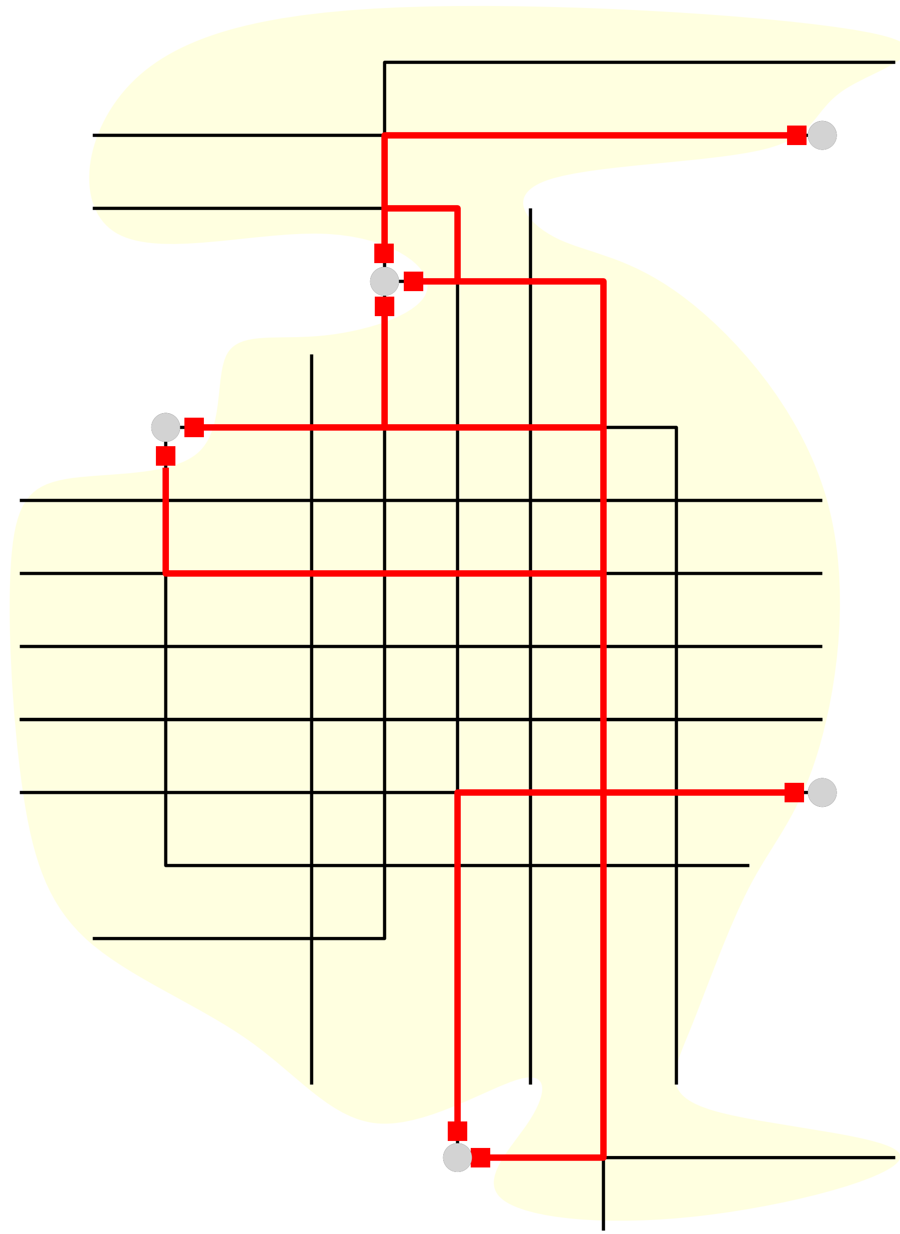

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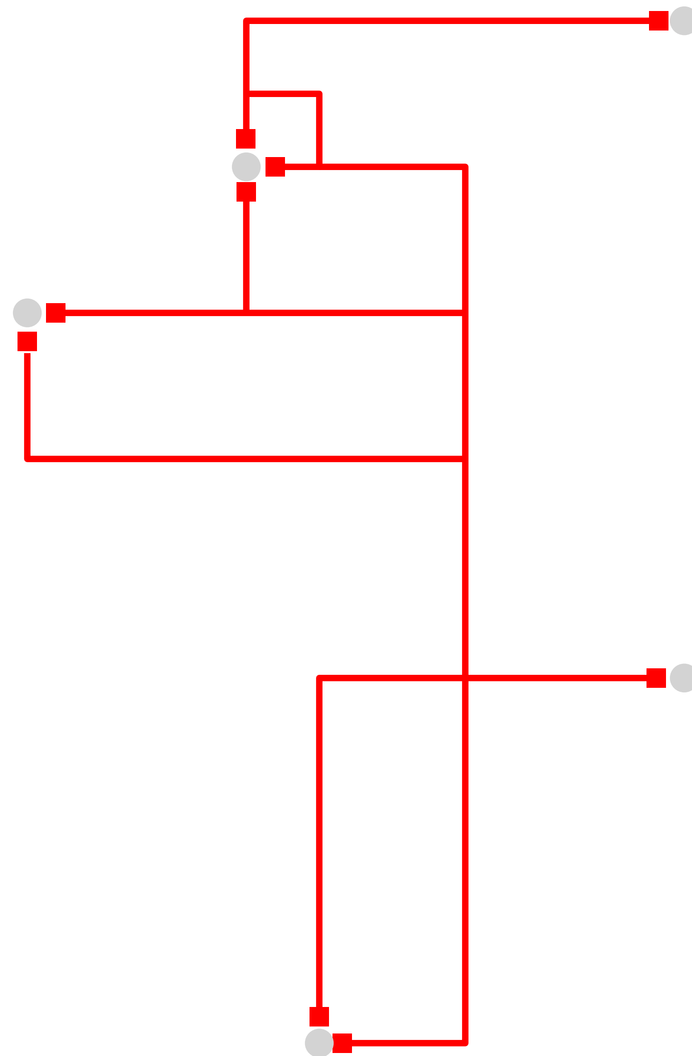

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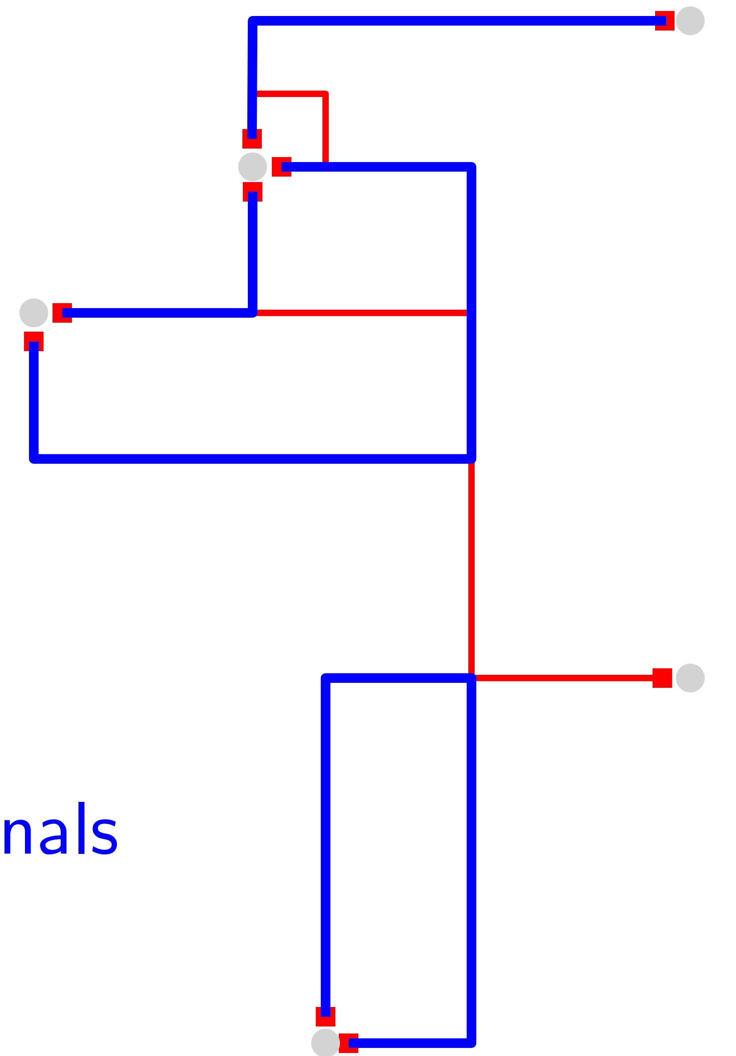

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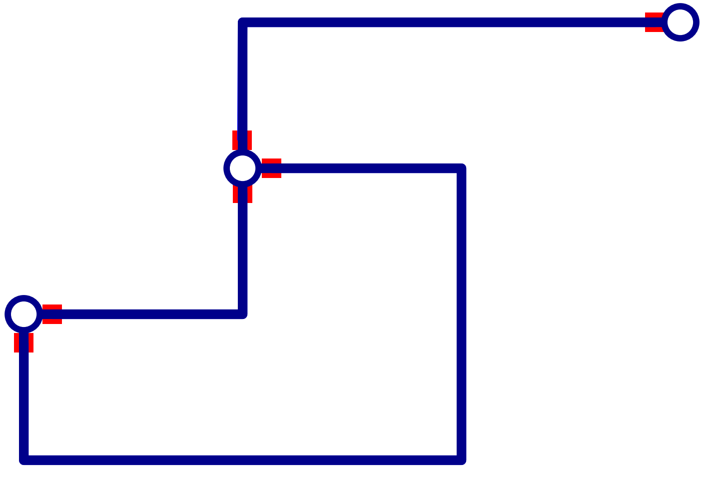

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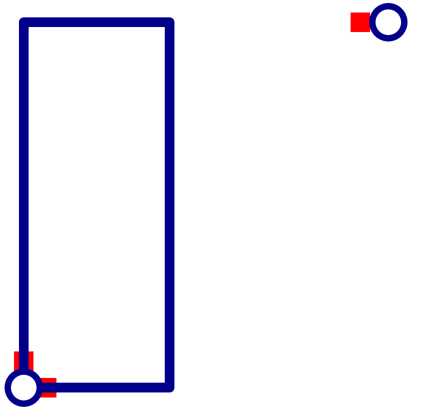

Extend the edges to the original vertices.

This is a **plane ortho-fin multigraph**!

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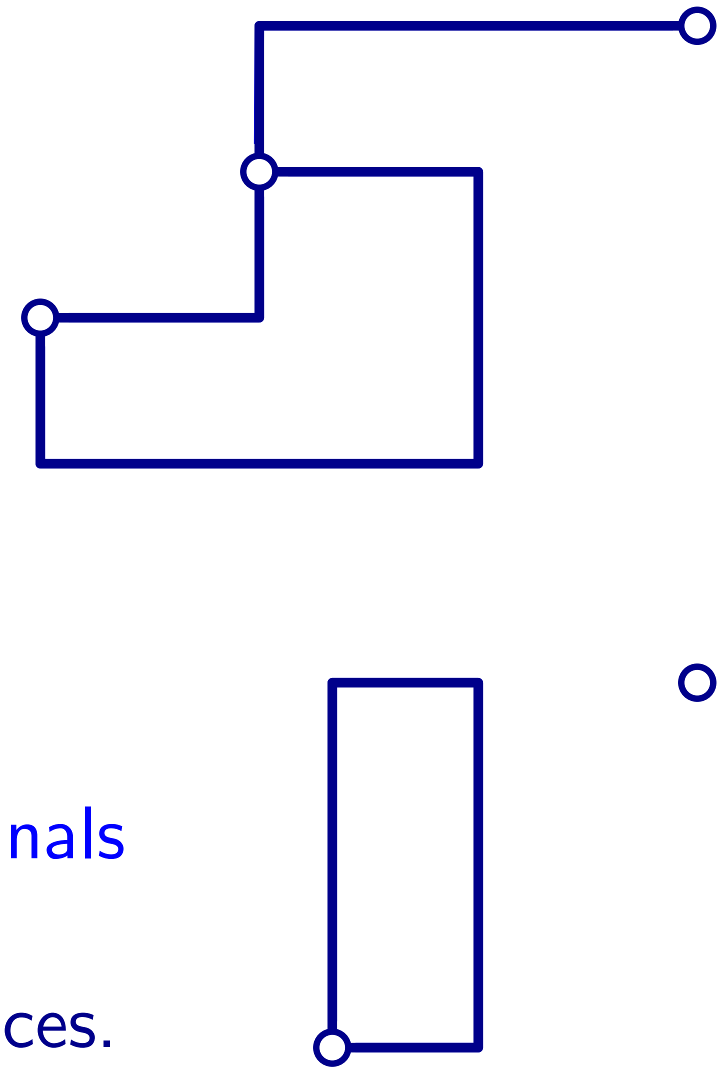
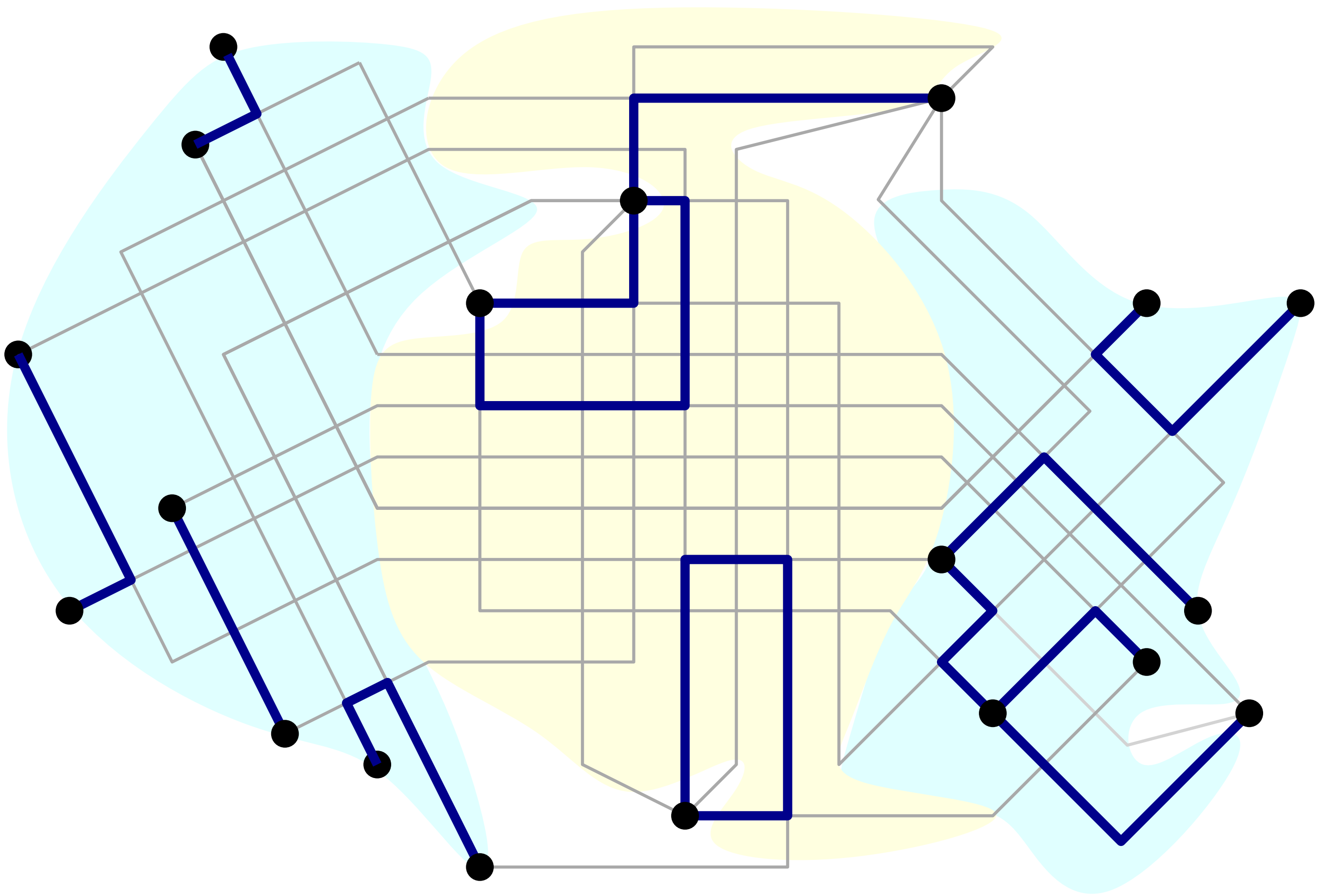


# The density of $RAC_2$ graphs

Let $G = (V, E)$ be a $RAC_2$ drawing.

The union of these "matchings" over all blocks is a plane ortho-fin multigraph $H$, with at most $5n - 2$ edges.



An edge $e$ of $H$ **represents** an edge $f$ of the $RAC_2$ drawing if an end segment of $e$ overlaps with an end segment of $f$.

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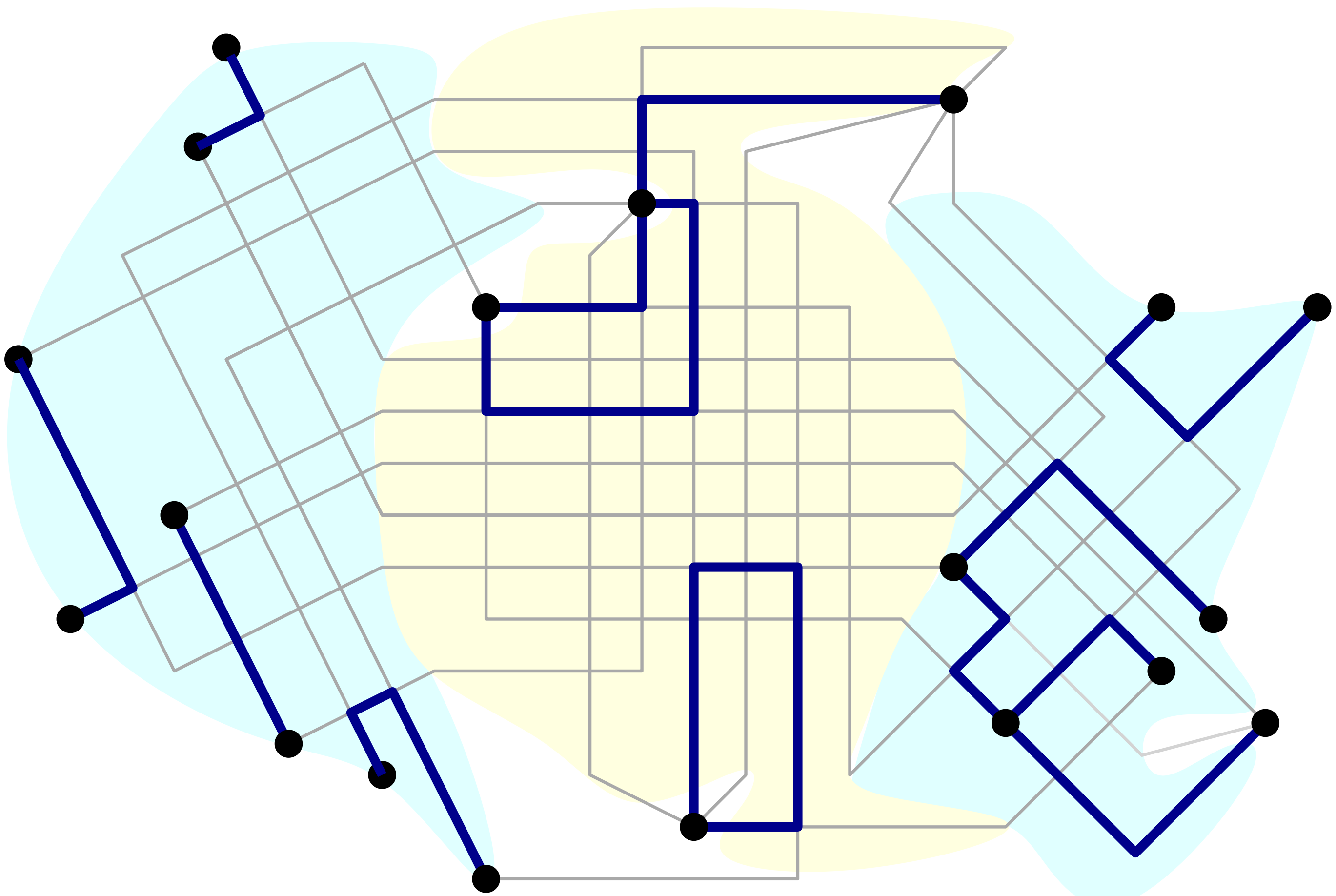

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The $H$ represents $\leq 2(5n-2) = 10n-4$ edges of $G$.

Consider the remaining edges of $G$ (that are not represented).

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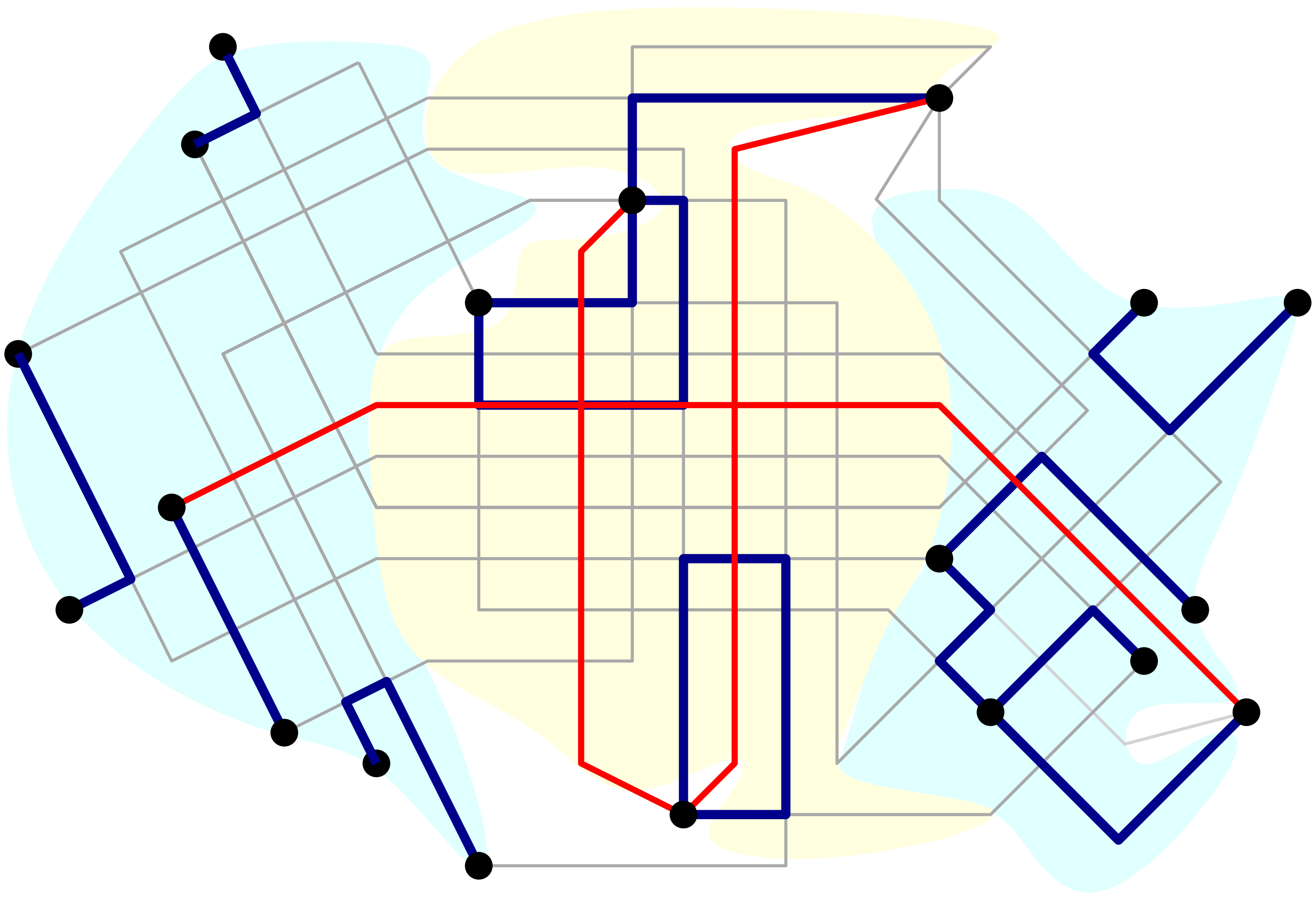

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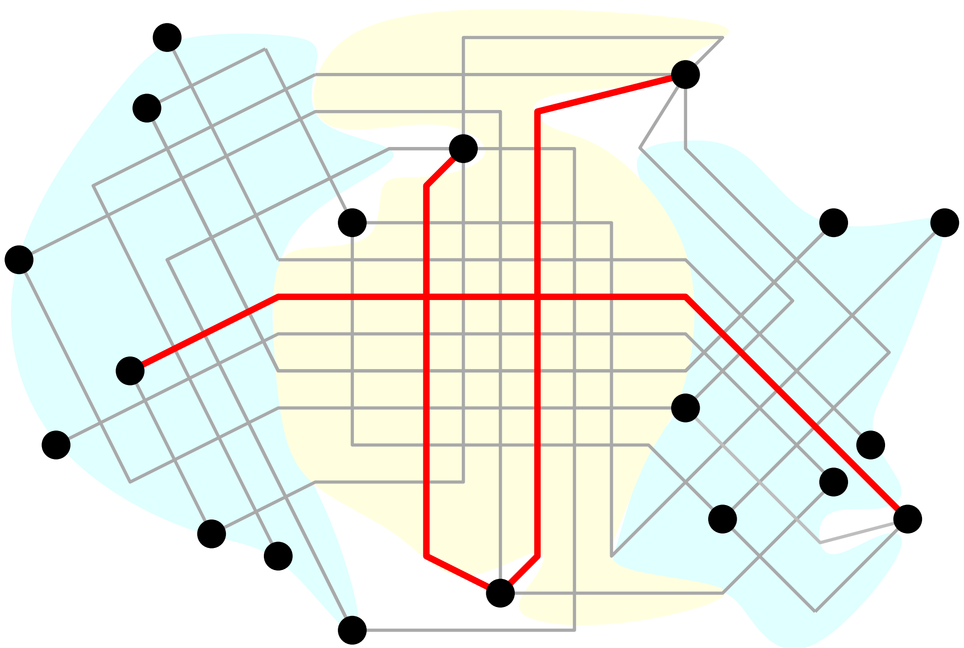
# The density of $RAC_2$ graphs

Let $G_0 = (V, E_0)$ be the graph of all edges not represented by $H$.

**Lemma.** In $G_0$, there is no end-end crossing,
and each middle segment is crossed by at most one end segment.



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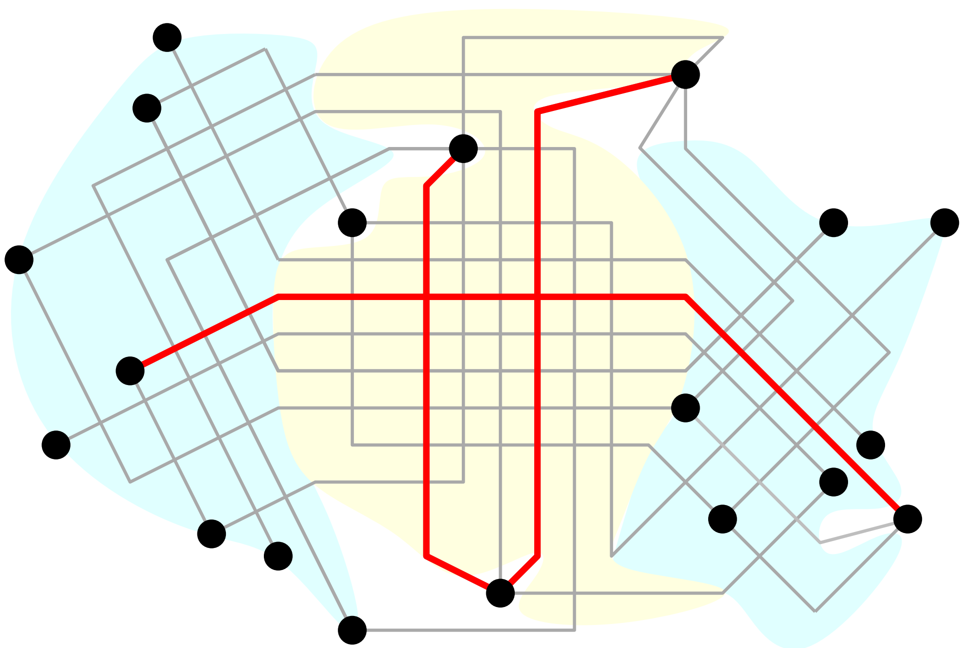

Partition $G_0$ into two subgraphs: $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, where $E_1 = \{e \in E$ : the middle segment of $e$ has negative slope$\}$, and $E_2 = E_0 \setminus E_1$.

**Lemma.** In each of $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, all crossings are end-middle crossings, and every middle segment has at most one crossing.

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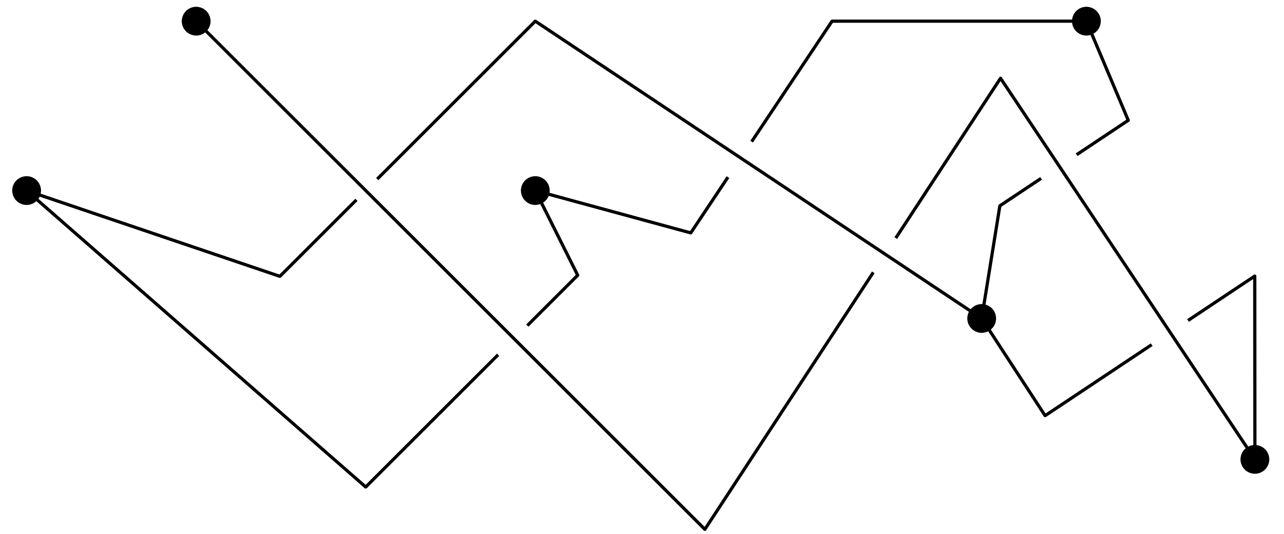
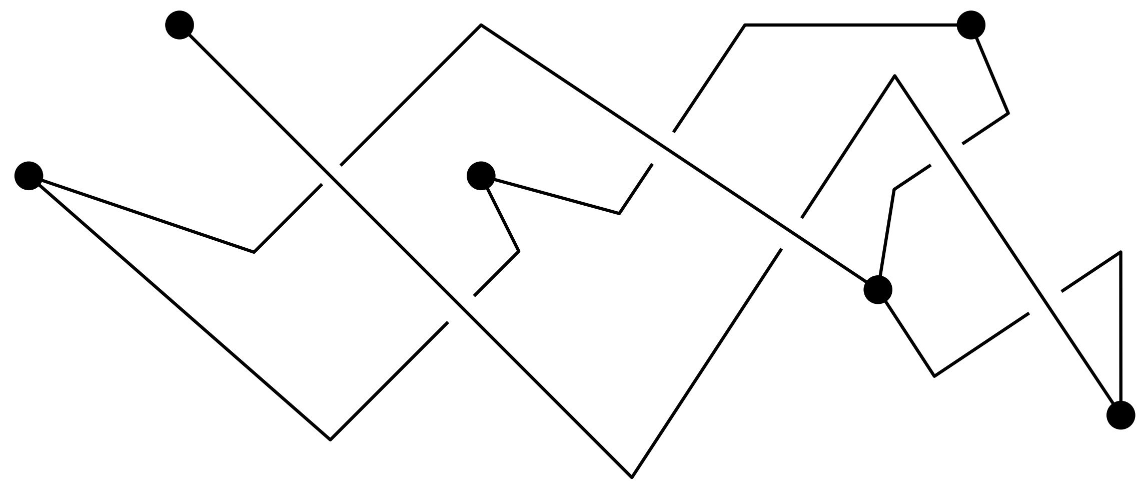


A graph is $k$-**gap planar** if it can be drawn such that
(1) exactly two edges of $G$ cross in any point,
(2) each crossing is *assigned* to one of its two crossing edges, and
(3) each edge is assigned with at most $k$ of its crossings.

**Lemma.** Both $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ are 1-gap planar.

# The density of $RAC_2$ graphs

**Theorem (Bae, Baffier, Chun, Eades, Eickmeyer, Grilli, Hong, Korman, Montecchiani, Rutter, and Tóth, GD 2017).** Every $1$-gap planar graph on $n \geq 3$ vertices has at most $5n - 10$ edges, and this bound is the best possible.

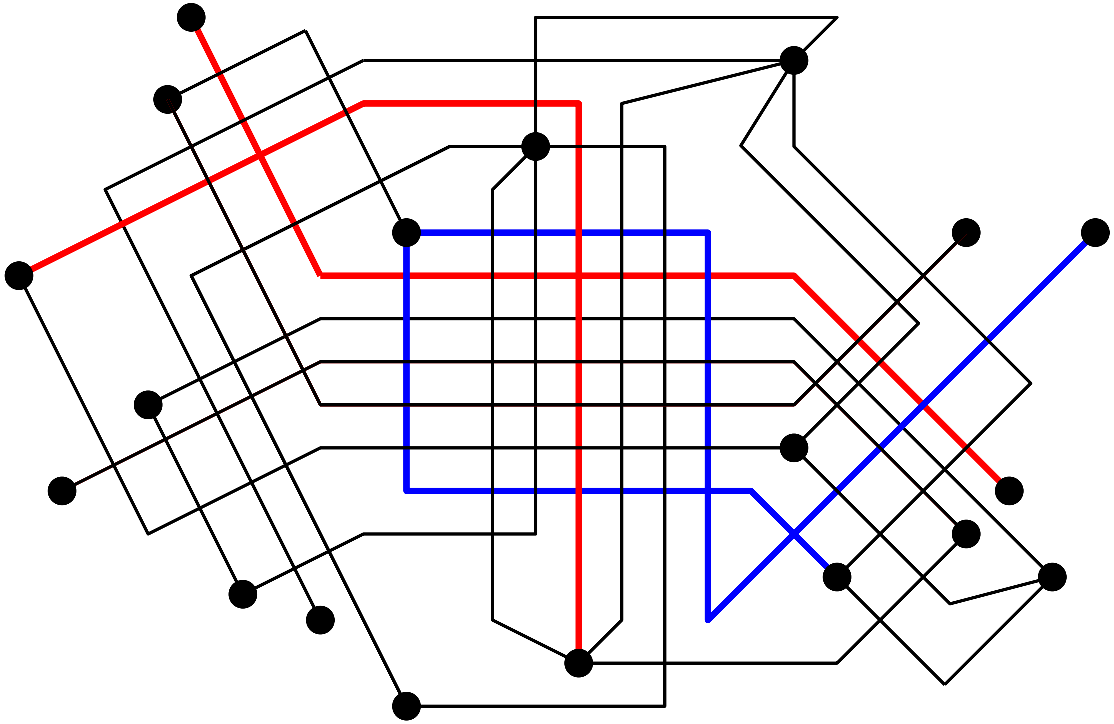


- $H$ represents at most $2(5n - 2)$ edges of $G$
- $G_1$ and $G_2$ each has at most $5n - 10$ edges.
- Overall, $G$ has at most $2(5n - 2) + 2(5n - 10) = 20n - 24$ edges. □

# Open Problems

- How many edges can a $RAC_2$ graph on $n$ vertices have? The maximum is between $10n - O(1)$ and $20n - 24$.

- Haw many edges can a $RAC_2$ drawing have if we require **simple** topological drawings, where any two edges meet at most once (at a common endpoint or a crossing)?



- Can $RAC_1$ or $RAC_2$ graphs be recognized efficiently? Can they be recognized if all crossing edge pairs are given? It is known that recgnizing $RAC_0$ graphs is NP-hard (Argyriou, Bekos, and Symvonis, SOFSEM 2011).