Decomposition of Geometric Graphs into Star-Forests

(Joint work with J. Pach and P. Schnider)

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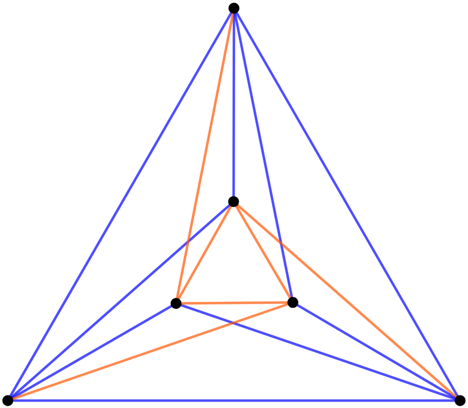
Background

Decompose the edges of graph G into minimum number of subgraphs

- Planar $\longrightarrow$ Thickness



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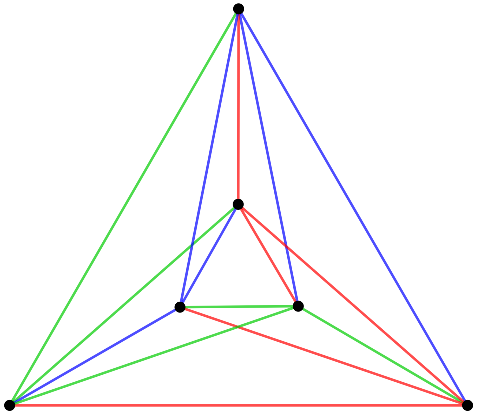
- Forest $\longrightarrow$ Arboricity



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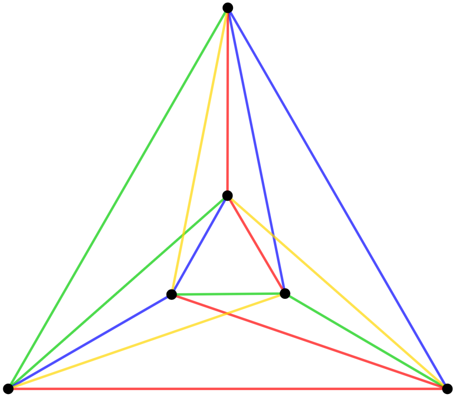
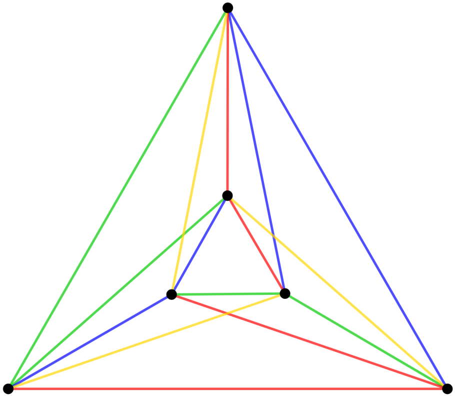
- Star-forest $\longrightarrow$ Star-Arboricity



Background

Decompose the edges of graph G into minimum number of subgraphs

- Planar $\longrightarrow$ Thickness
- Forest $\longrightarrow$ Arboricity
- Star-forest $\longrightarrow$ Star-Arboricity
- Abstract, Geometric, Convex



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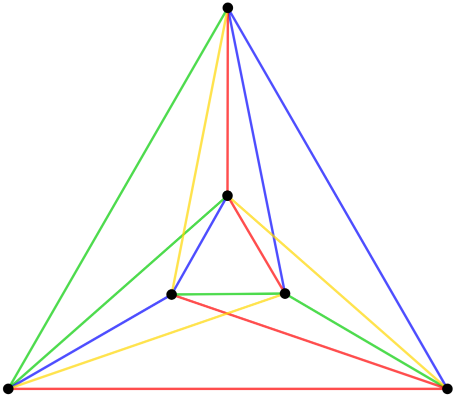

- Abstract, Geometric, Convex
- Covering $\Longleftrightarrow$ Decomposition

Background

Parameter	Abstract	Geometric	Convex (Book)
Thickness	$\theta(K_n) = \lfloor \frac{n+7}{6} \rfloor$ ([1])	$\lceil \frac{n}{5.646} \rceil \leq \bar{\theta}(K_n) \leq \lceil \frac{n}{4} \rceil$ ([2])	$bt(K_n) = \lceil \frac{n}{2} \rceil$ ([3])
Arboricity	$a(K_n) = \lceil \frac{n}{2} \rceil$ ([3])	$\bar{a}(K_n) = \lceil \frac{n}{2} \rceil$ ([3])	$ba(K_n) = \lceil \frac{n}{2} \rceil$ ([3])
Star-arboricity	$sa(K_n) = \lceil \frac{n}{2} \rceil + 1$ ([4])	$\bar{sa}(K_n) \leq n - 1$	$bsa(K_n) \leq n - 1$

- [1] Beineke, Harary (1965) - Alekseev, Gonchakov (1985)
- [2] Dillencourt, Eppstein, Hirschberg (2000)
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Star-arboricity	$sa(K_n) = \lceil \frac{n}{2} \rceil + 1$ ([4])	$\bar{sa}(K_n) \leq \lceil \frac{3n}{4} \rceil$	$bsa(K_n) = n - 1$

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- Pach, Saghafian, Schnider (2023)

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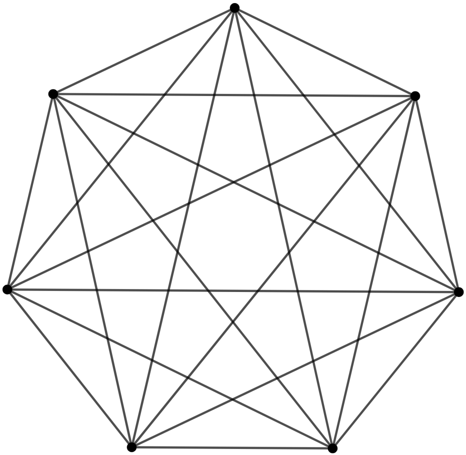
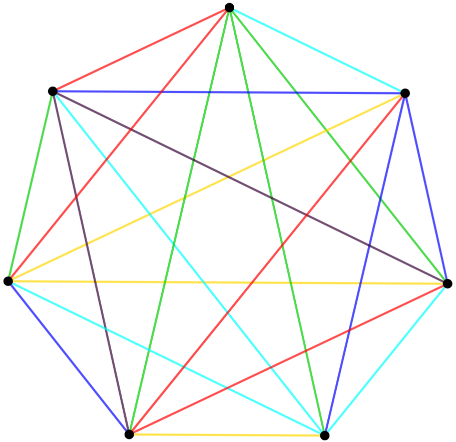
Complete Convex Geometric Graph



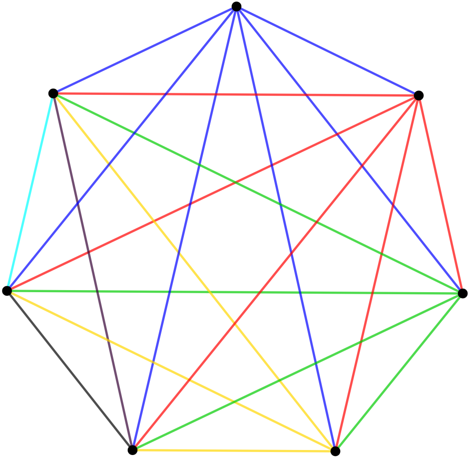
- n points in convex position
- Complete geometric graph

Decomposition into Star-Forests



- Decomposition into plane star-forests

Decomposition into Star-Forests



- Decomposition into plane star-forests
- Fewer than $n-1$? [Dujmović, Wood (2007)]

Book Star-Arborocity

Theorem (Book Star-Arboricity of K_n)

The complete convex geometric graph with n vertices cannot be decomposed into fewer than $n-1$ plane star-forests.

Approach:

- Recolor the edges wisely.
- Make sure the constraints remain satisfied.
  - Each color is a star-forest
  - Each color is crossing-free
  - All the edges are covered
- End up with a color being a single star.
- Remove and induction!

Supported Edges

- Vertices: $P_1, P_2, \cdots, P_n$ in clockwise order

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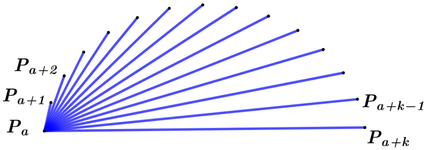
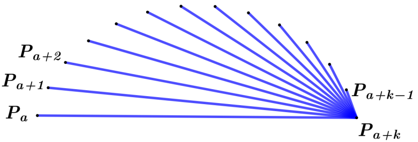
- Indices modulo n, so $P_{n+1} = P_1, P_{n+2} = P_2$, etc.
- **k-edge**: $P_a P_{a+k}$
- A k-edge $P_a P_{a+k}$ is **supported** if it belongs to one of the star-forests along with either all edges $P_a P_{a+1}, P_a P_{a+2}, \ldots, P_a P_{a+k-1}$, or all edges $P_{a+1} P_{a+k}, P_{a+2} P_{a+k}, \ldots, P_{a+k-1} P_{a+k}$.





Recoloring Process

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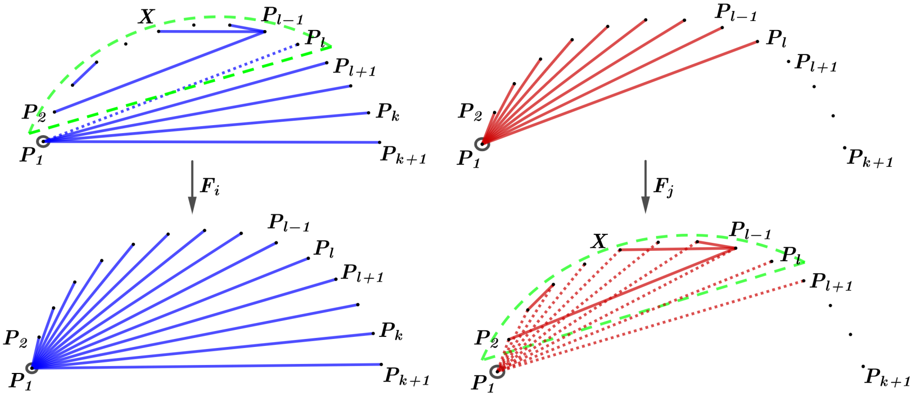
  - 2-edges, 3-edges, ..., k-edges
- Suppose that the complete convex geometric graph K_n can be covered by t crossing-free star-forests, for some positive integer t. Then, for every k, $1 < k < n$, there exists a covering of K_n by t crossing-free star-forests $F_1, F_2, \ldots, F_t$ such that every k'-edge with $1 < k' \leq k$ is supported.
- Induction on k

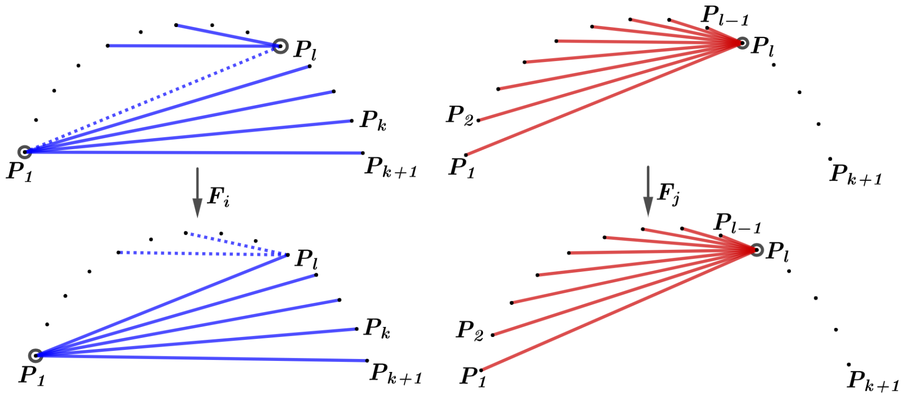
Induction on k

Case 1:



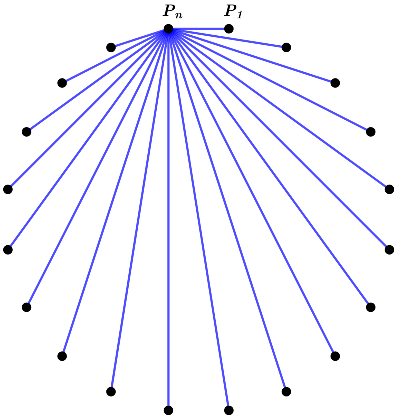
Induction on k

Case 2:



Induction on k

- In the end, the $(n-1)$-edge $P_1 P_n$ is supported.
- It means at least one star-forest is a single star.
- Remove it along with its center (either P_1 or P_n).
- Induction on n.



Fewer than $n-1$ is impossible!

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Non-convex Position

Theorem (Geometric Star-Arboricity of K_n)

There is a complete geometric graph with n vertices that can be decomposed into $\lceil \frac{3n}{4} \rceil$ plane star-forests.

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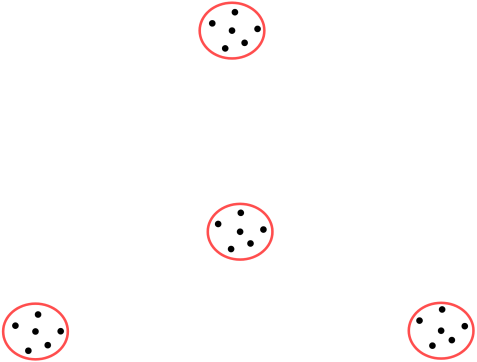

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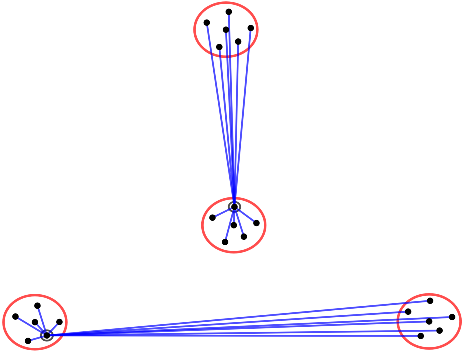

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Abstract Setting

- **k-Star-Forest**: at most k disjoint stars

Theorem (2-Star-Forests)

The complete graph with $n > 3$ vertices can be decomposed into $\lceil \frac{3n}{4} \rceil$ 2-star-forests. This bound cannot be improved.

Abstract Setting

- **k-Star-Forest**: at most k disjoint stars

Theorem (2-Star-Forests)

The complete graph with $n > 3$ vertices can be decomposed into $\lceil \frac{3n}{4} \rceil$ 2-star-forests. This bound cannot be improved.

Conjecture (k-Star-Forests)

For any $n \geq k \geq 2$, the number of k-star-forests needed to cover the complete graph K_n is at least $\lceil \frac{(k+1)n}{2k} \rceil$.

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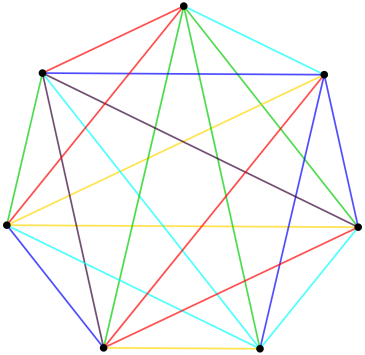
Open Problems

1. (Geometric Star-Arboricity) Is there any complete geometric graph on n vertices that can be decomposed into fewer than $\lceil \frac{3n}{4} \rceil$ plane star-forests?
2. (Abstract k-Star-Arboricity) Is it true that for any $n \geq k \geq 2$, we need at least $\lceil \frac{(k+1)n}{2k} \rceil$ k-star-forests to cover the complete graph K_n?
3. Recoloring Graph



Thank you!