Different Types of Isomorphisms of Drawings of Complete Multipartite Graphs

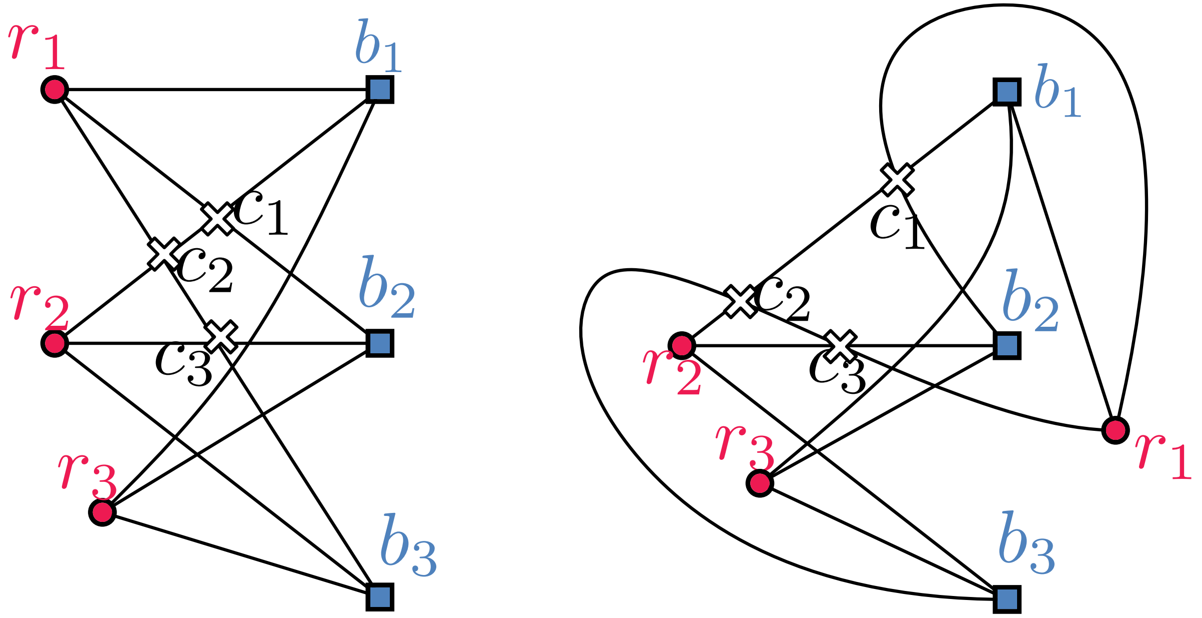
Oswin Aichholzer, Birgit Vogtenhuber, and Alexandra Weinberger

DOCTORAL PROGRAM
DISCRETE MATHEMATICS

TU & KFU GRAZ • MU LEOBEN
AUSTRIA

Isomorphisms of simple drawings of complete multipartite graphs

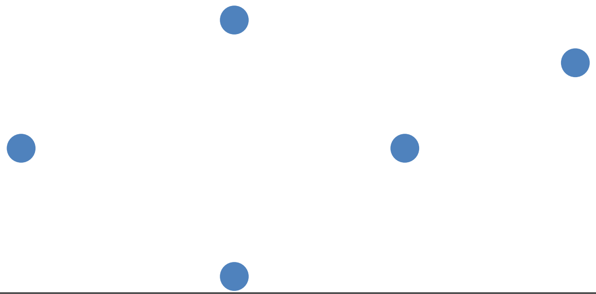
- Definitions
- What are we doing and why?
- Results
- Sketches of proofs



Simple drawings

Simple drawings (simple topological graphs, good drawings):

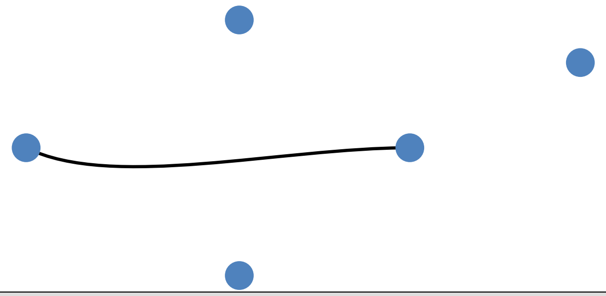
- Vertices are disjoint points, edges are Jordon arcs



Simple drawings

Simple drawings (simple topological graphs, good drawings):

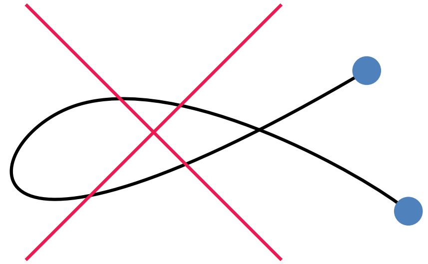
- Vertices are disjoint points, edges are Jordon arcs

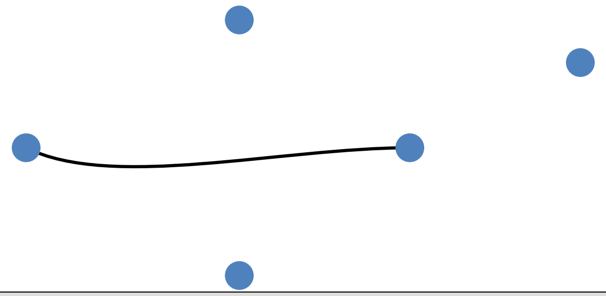


Simple drawings

Simple drawings (simple topological graphs, good drawings):

- Vertices are disjoint points, edges are Jordon arcs

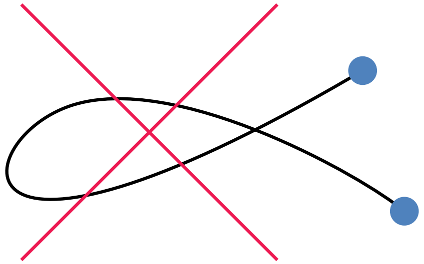




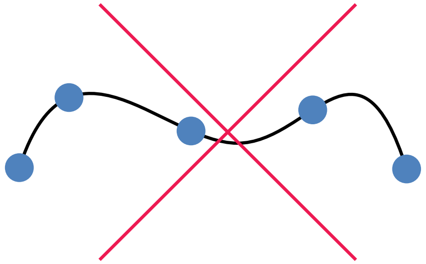
Simple drawings

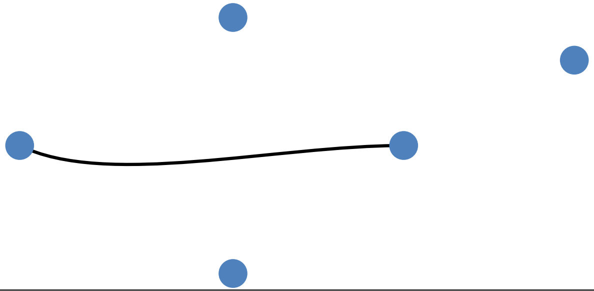
Simple drawings (simple topological graphs, good drawings):

- Vertices are disjoint points, edges are Jordon arcs



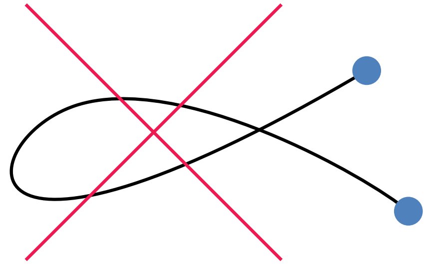
- Edges don't pass through other vertices



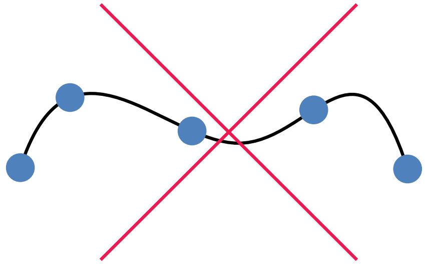


Simple drawings

Simple drawings (simple topological graphs, good drawings):



- Vertices are disjoint points, edges are Jordon arcs
- Edges don't pass through other vertices

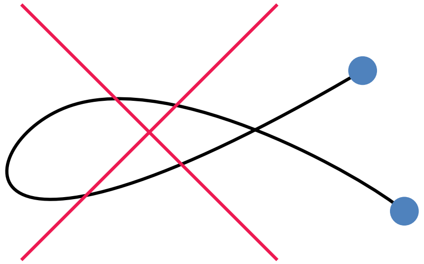


- Any pair of edges intersect at most once

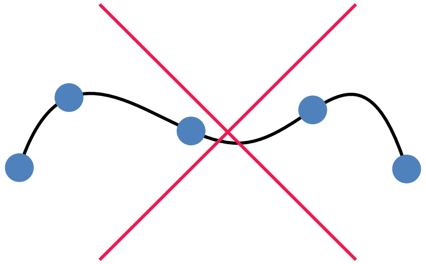
Simple drawings

Simple drawings (simple topological graphs, good drawings):

- Vertices are disjoint points, edges are Jordon arcs



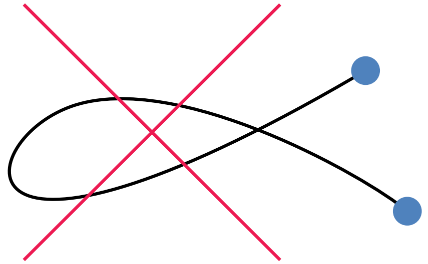
- Edges don't pass through other vertices



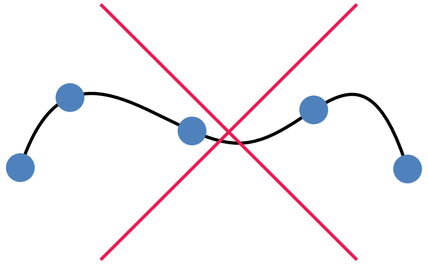
- Any pair of edges intersect at most once

Simple drawings

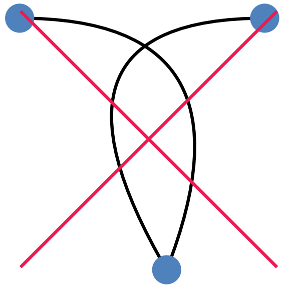
Simple drawings (simple topological graphs, good drawings):

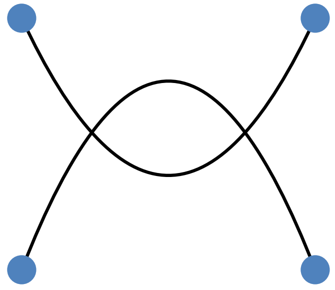


- Vertices are disjoint points, edges are Jordon arcs



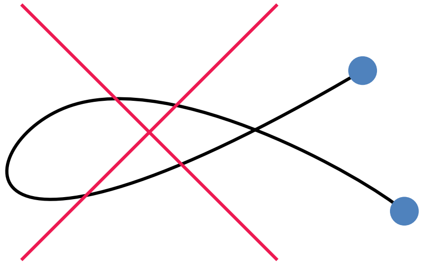
- Edges don't pass through other vertices
- Any pair of edges intersect at most once



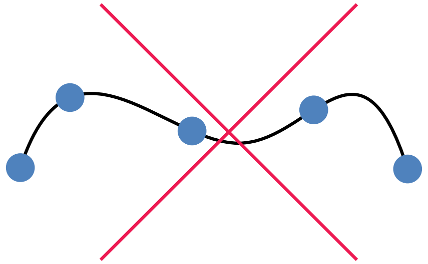


Simple drawings

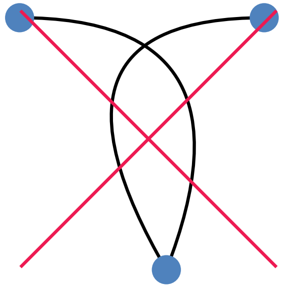
Simple drawings (simple topological graphs, good drawings):

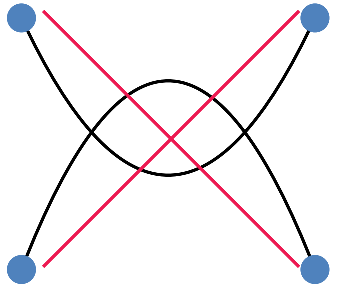


- Vertices are disjoint points, edges are Jordon arcs

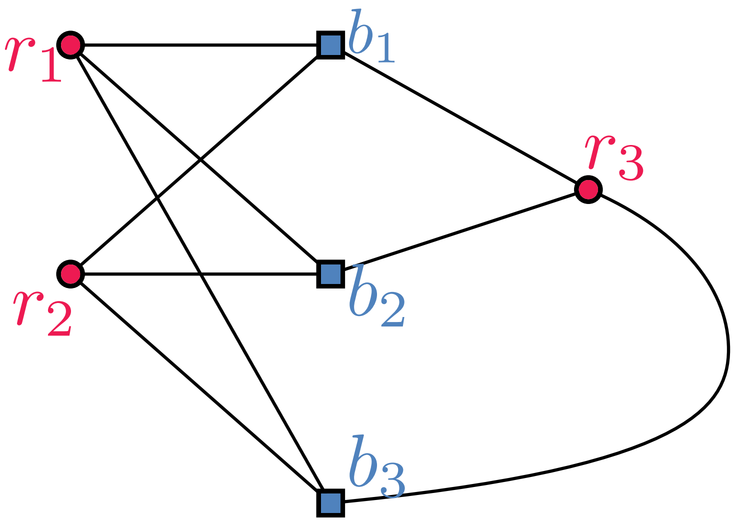


- Edges don't pass through other vertices
- Any pair of edges intersect at most once



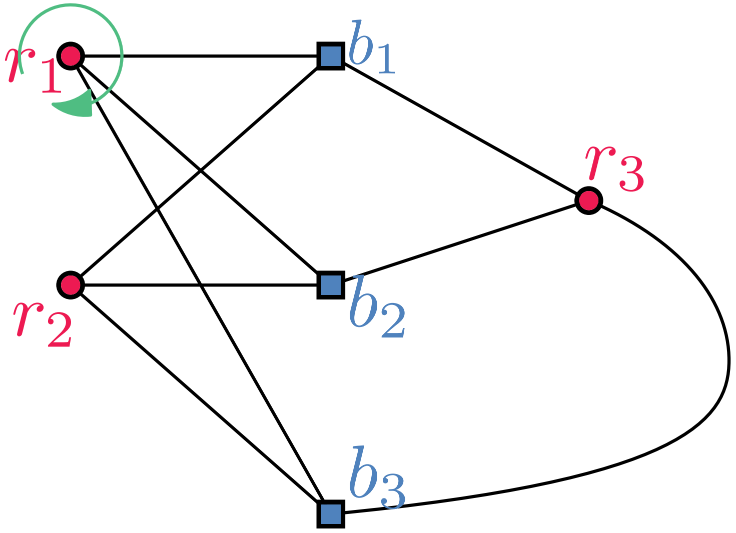


Describing simple drawings – types of isomorphisms



Rotation ... Cyclical order of incident edges

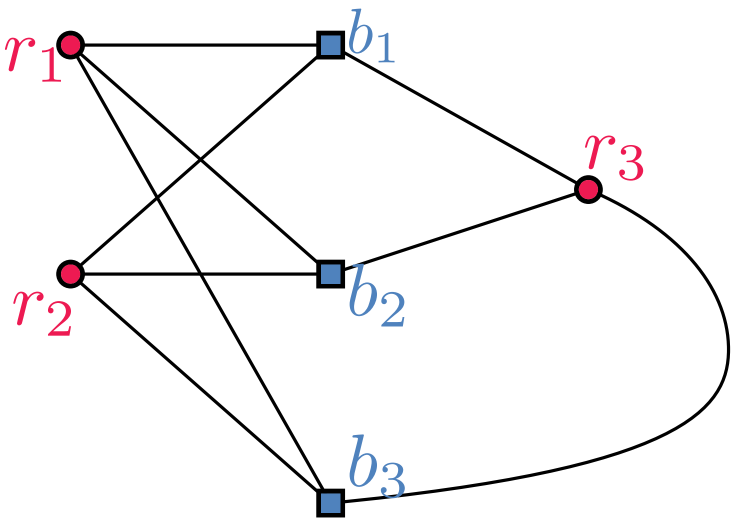
Describing simple drawings – types of isomorphisms



Rotation … Cyclical order of incident edges

Rotation around r_1: b_1 b_2 b_3

Describing simple drawings – types of isomorphisms



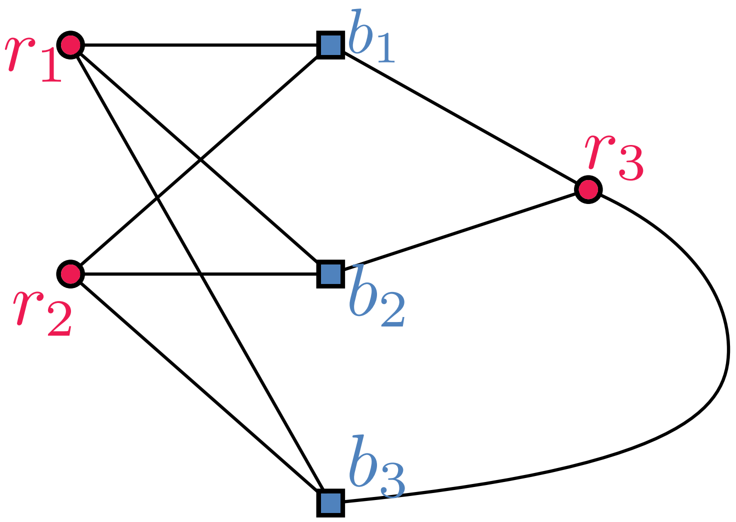
Rotation ... Cyclical order of incident edges

Rotation around r_1: $b_1\ b_2\ b_3$

Rotation System ... Collection of the rotations of all vertices.

r_1 :	b_1	b_2	b_3
r_2 :	b_1	b_2	b_3
r_3 :	b_1	b_3	b_2
b_1 :	r_1	r_3	r_2
b_2 :	r_1	r_3	r_2
b_3 :	r_1	r_3	r_2

Describing simple drawings – types of isomorphisms

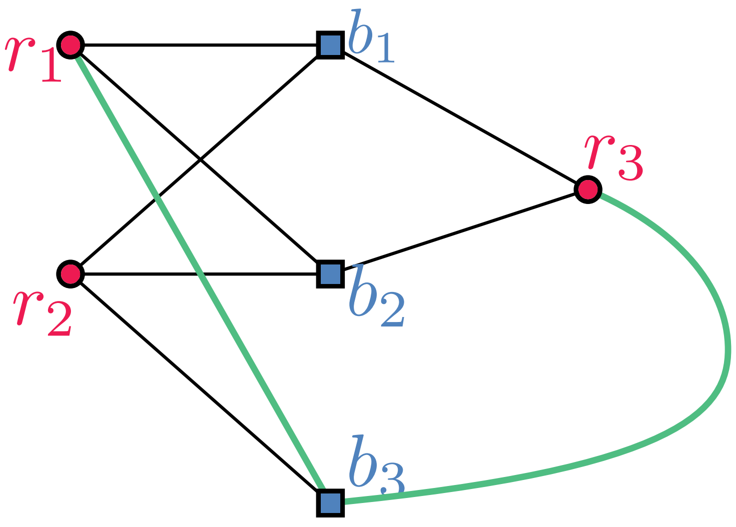


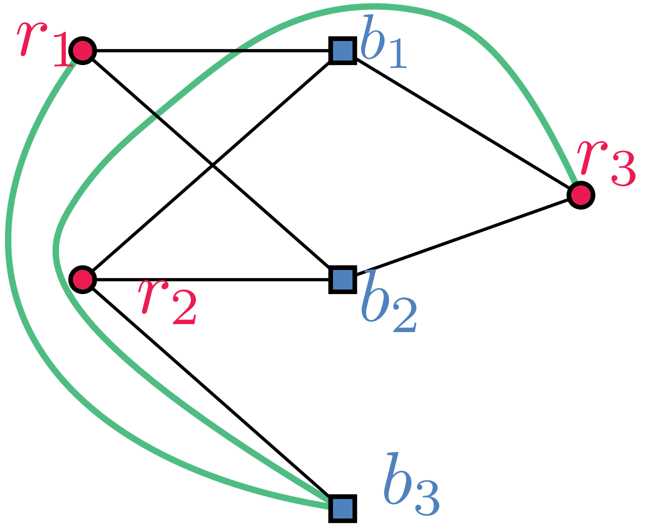
Rotation … Cyclical order of incident edges
Rotation around r_1: b_1 b_2 b_3

Rotation System … Collection of the rotations of all vertices.

Two labelled simple drawings are RS-isomorphic iff they have the same or inverse rotation systems.

Describing simple drawings – types of isomorphisms

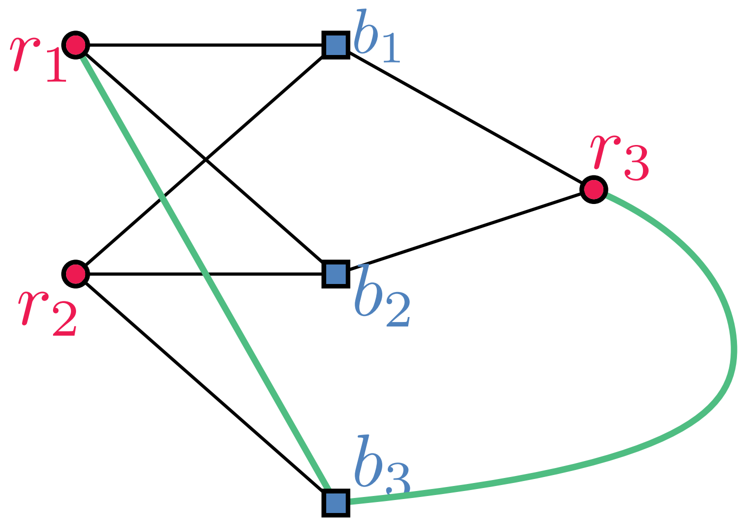


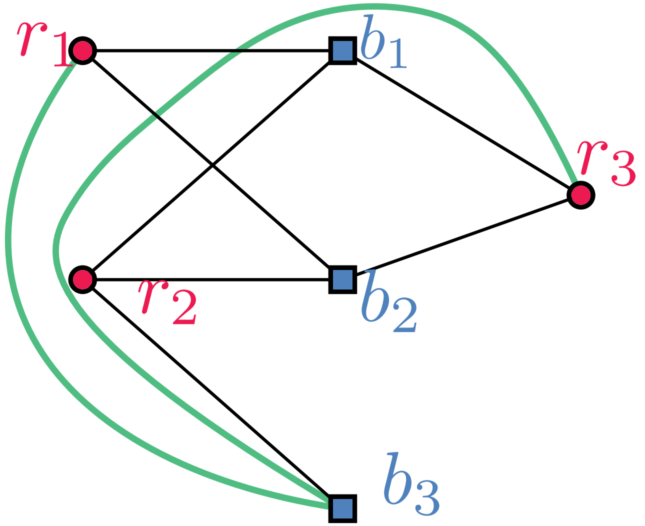


Two labelled simple drawings are RS-isomorphic iff they have the same or inverse rotation systems.

r_1 :	b_1	b_2	b_3
r_2 :	b_1	b_2	b_3
r_3 :	b_1	b_3	b_2
b_1 :	r_1	r_3	r_2
b_2 :	r_1	r_3	r_2
b_3 :	r_1	r_3	r_2

Describing simple drawings – types of isomorphisms



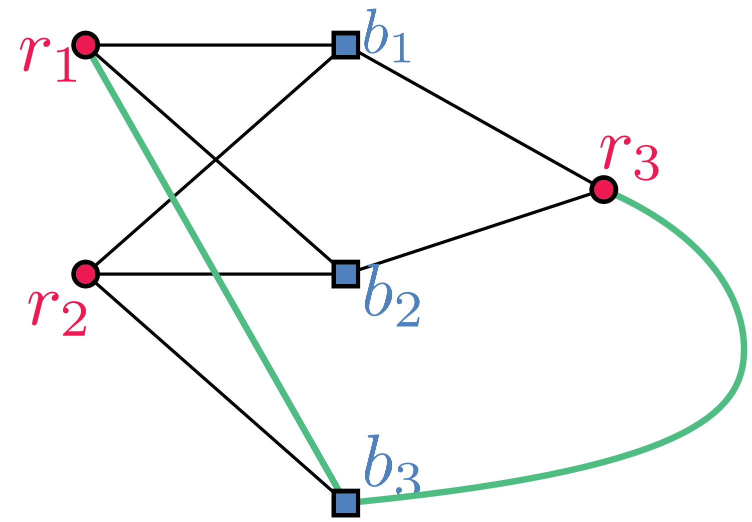


Two labelled simple drawings are RS-isomorphic iff they have the same or inverse rotation systems.

r_1 :	b_1	b_2	b_3
r_2 :	b_1	b_2	b_3
r_3 :	b_1	b_3	b_2
b_1 :	r_1	r_3	r_2
b_2 :	r_1	r_3	r_2
b_3 :	r_1	r_3	r_2

r_1 :	b_1	b_3	b_2
r_2 :	b_1	b_3	b_2
r_3 :	b_1	b_2	b_3
b_1 :	r_1	r_2	r_3
b_2 :	r_1	r_2	r_3
b_3 :	r_1	r_2	r_3

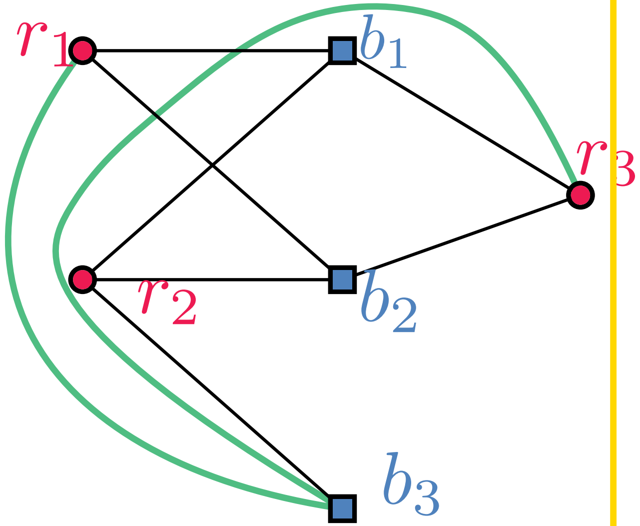
Describing simple drawings – types of isomorphisms

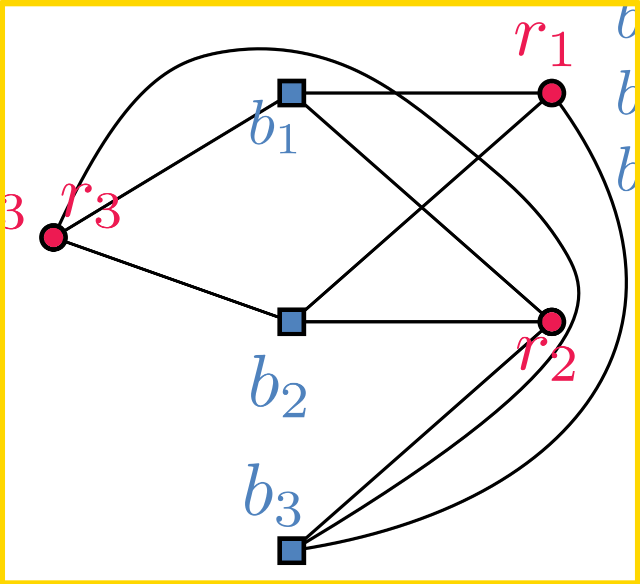


Two labelled simple drawings are RS-isomorphic iff they have the same or inverse rotation systems.

r_1 :	b_1	b_2	b_3
r_2 :	b_1	b_2	b_3
r_3 :	b_1	b_3	b_2
b_1 :	r_1	r_3	r_2
b_2 :	r_1	r_3	r_2
b_3 :	r_1	r_3	r_2

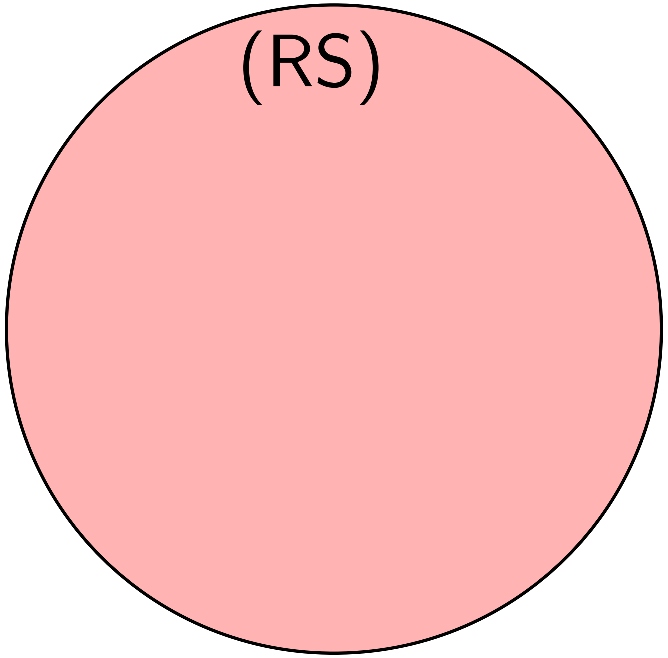
r_1 :	b_1	b_3	b_2
r_2 :	b_1	b_3	b_2
r_3 :	b_1	b_2	b_3
b_1 :	r_1	r_2	r_3
b_2 :	r_1	r_2	r_3
b_3 :	r_1	r_2	r_3





Implications between isomorphisms

RS.... Rotation System

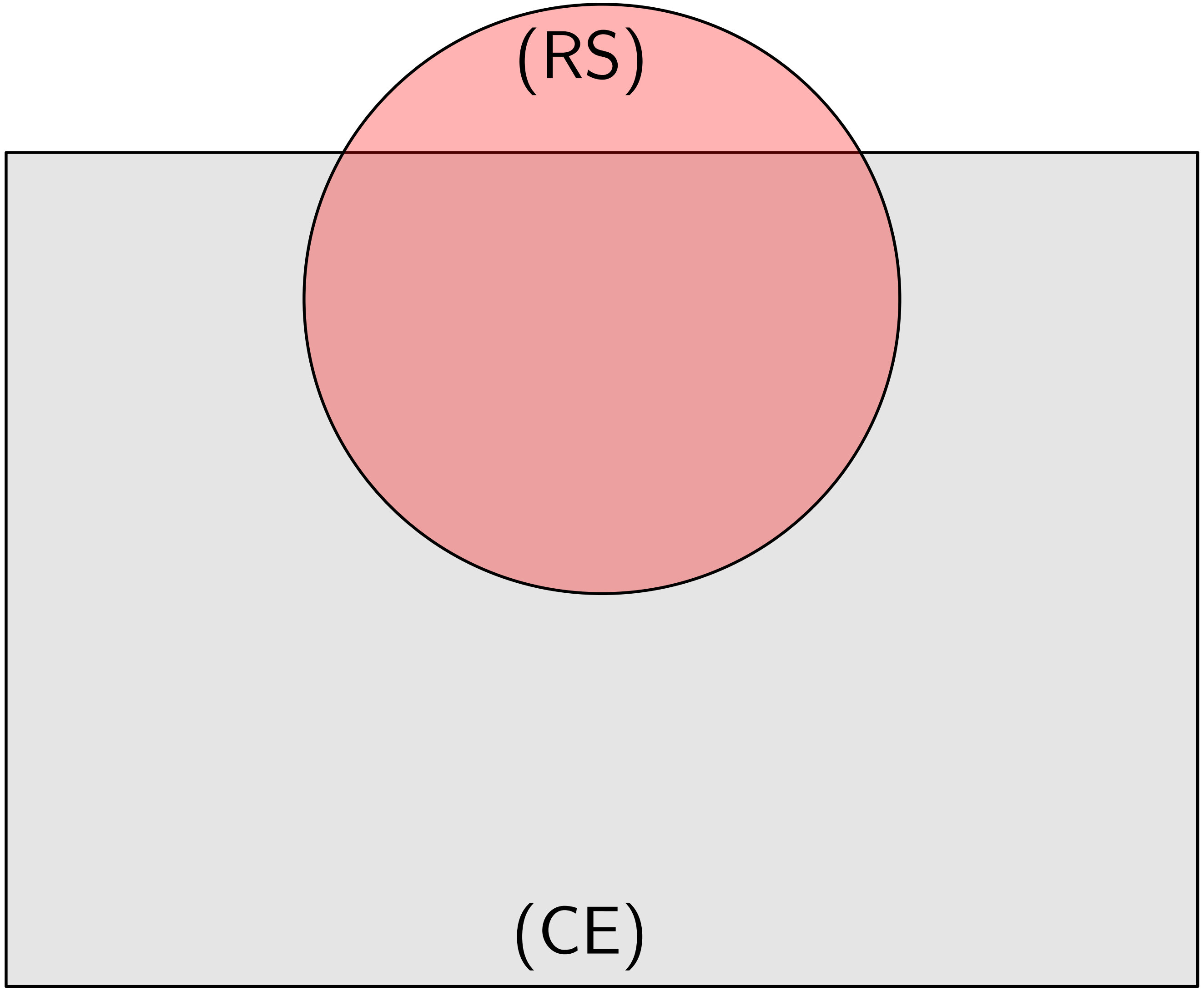


Describing simple drawings – types of isomorphisms

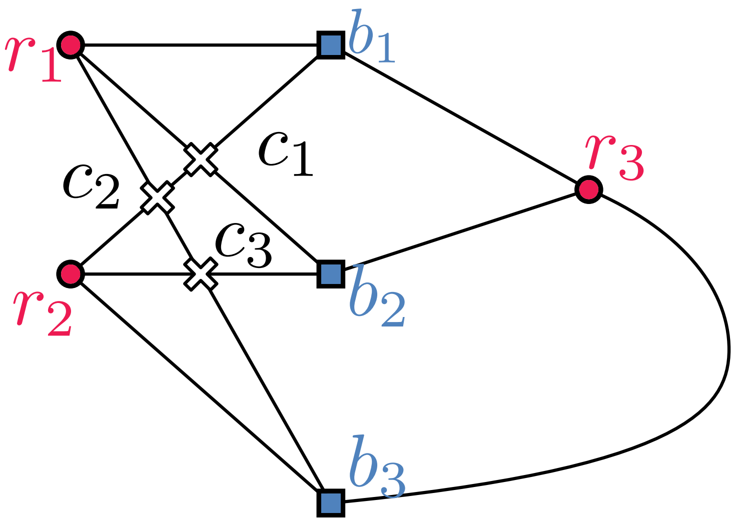
Two labelled simple drawings are CE-isomorphic (a.k.a. weakly isomorphic) iff they have the same crossing edge pairs.

Implications between isomorphisms

CE.... Crossing Edge pairs



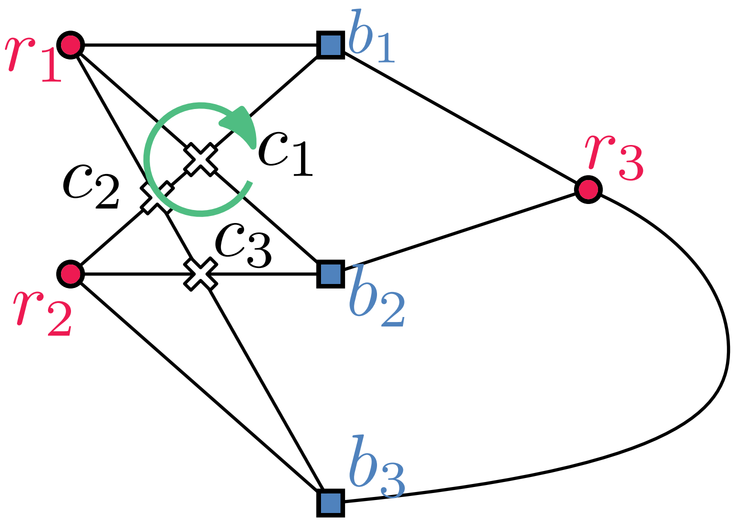
Describing simple drawings – types of isomorphisms



Rotation ... Cyclical order of incident edges

Rotation around r_1: b_1 b_2 b_3

Describing simple drawings – types of isomorphisms

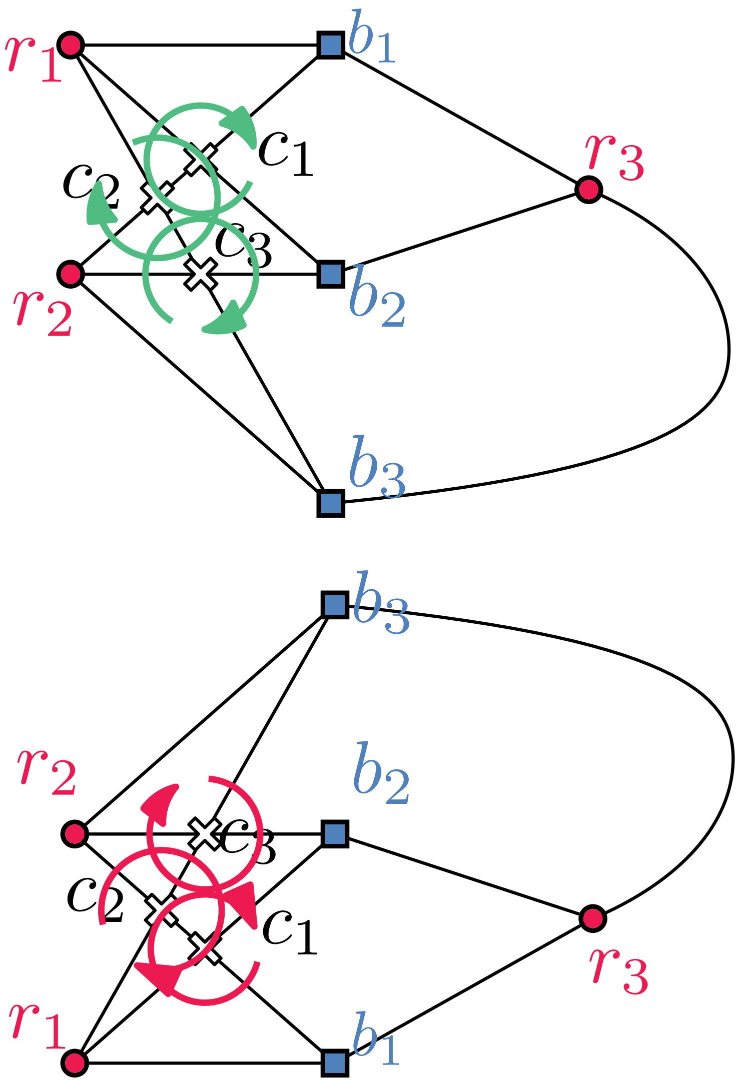


Rotation ... Cyclical order of incident edges

Rotation around r_1: b_1 b_2 b_3

Rotation around c_1: r_1 b_1 b_2 r_2

Describing simple drawings – types of isomorphisms



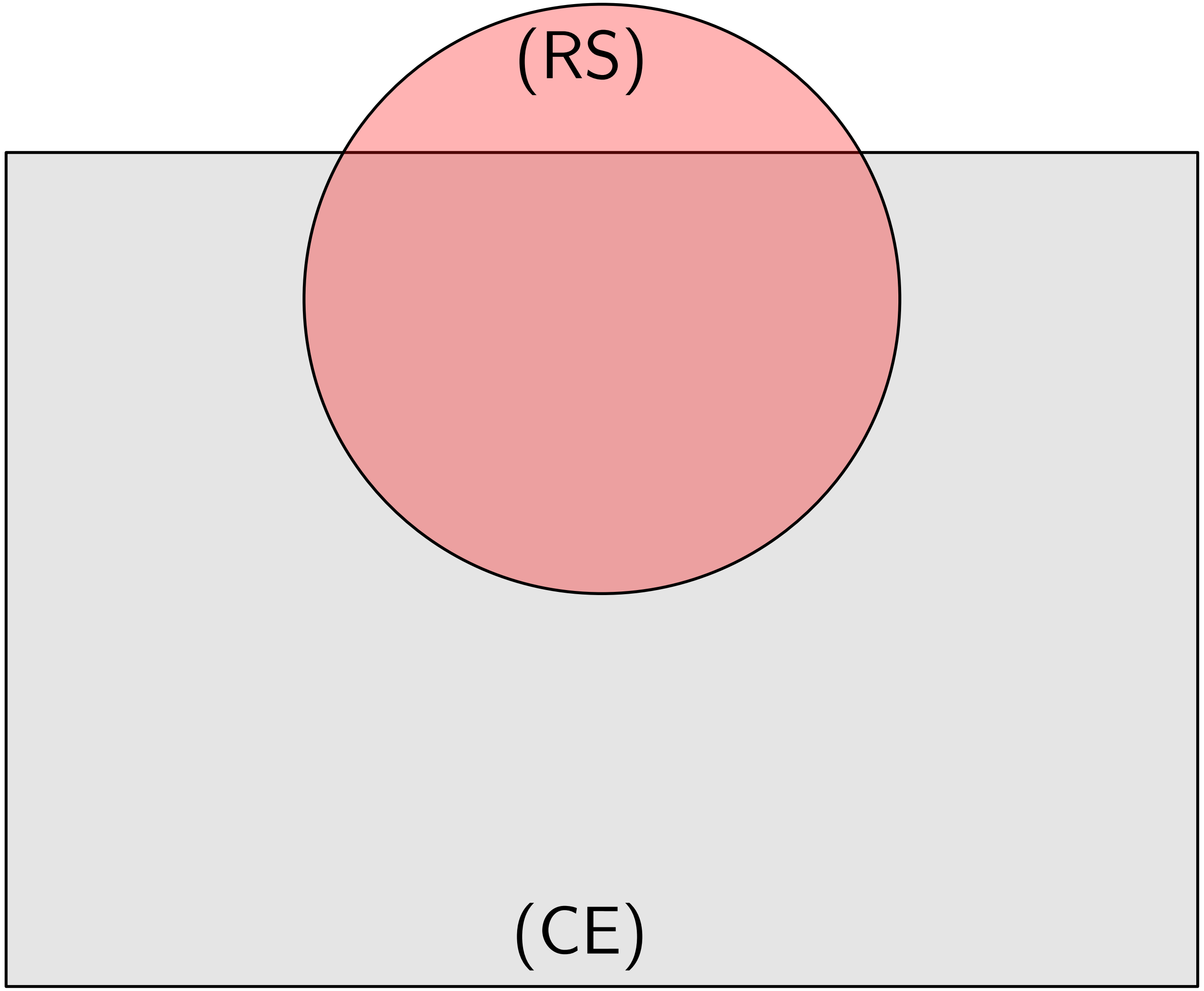
Rotation … Cyclical order of incident edges

Rotation around r_1: b_1 b_2 b_3

Rotation around c_1: r_1 b_1 b_2 r_2

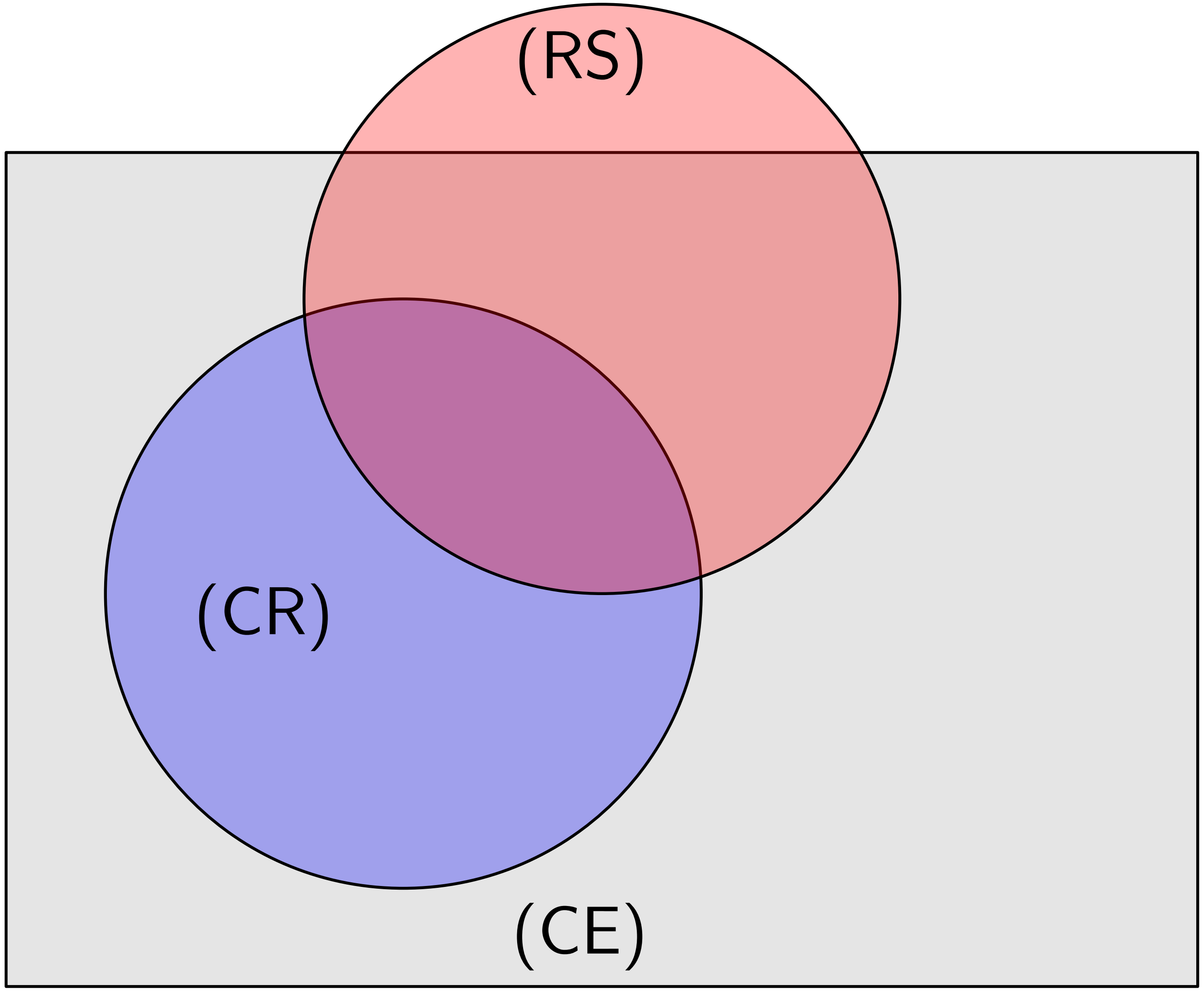
Two labelled simple drawings are CR-isomorphic iff either all crossings have the same rotations or all crossings have inverse rotations.

Implications between isomorphisms

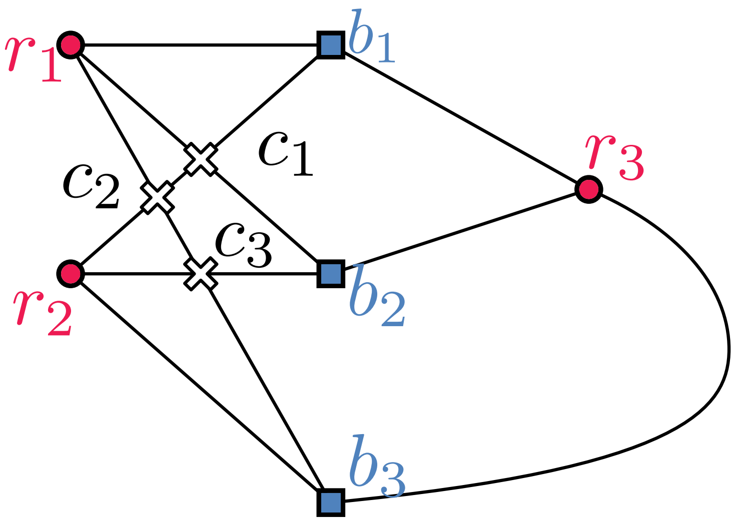


Implications between isomorphisms

CR.... Crossing Rotations



Describing simple drawings – types of isomorphisms



r_1 :	b_1	b_2	b_3	
r_2 :	b_1	b_2	b_3	
r_3 :	b_1	b_3	b_2	
b_1 :	r_1	r_3	r_2	
b_2 :	r_1	r_3	r_2	
b_3 :	r_1	r_3	r_2	
c_1 :	r_1	b_1	b_2	r_2
c_2 :	r_1	b_1	b_3	r_2
c_3 :	r_1	b_2	b_3	r_2

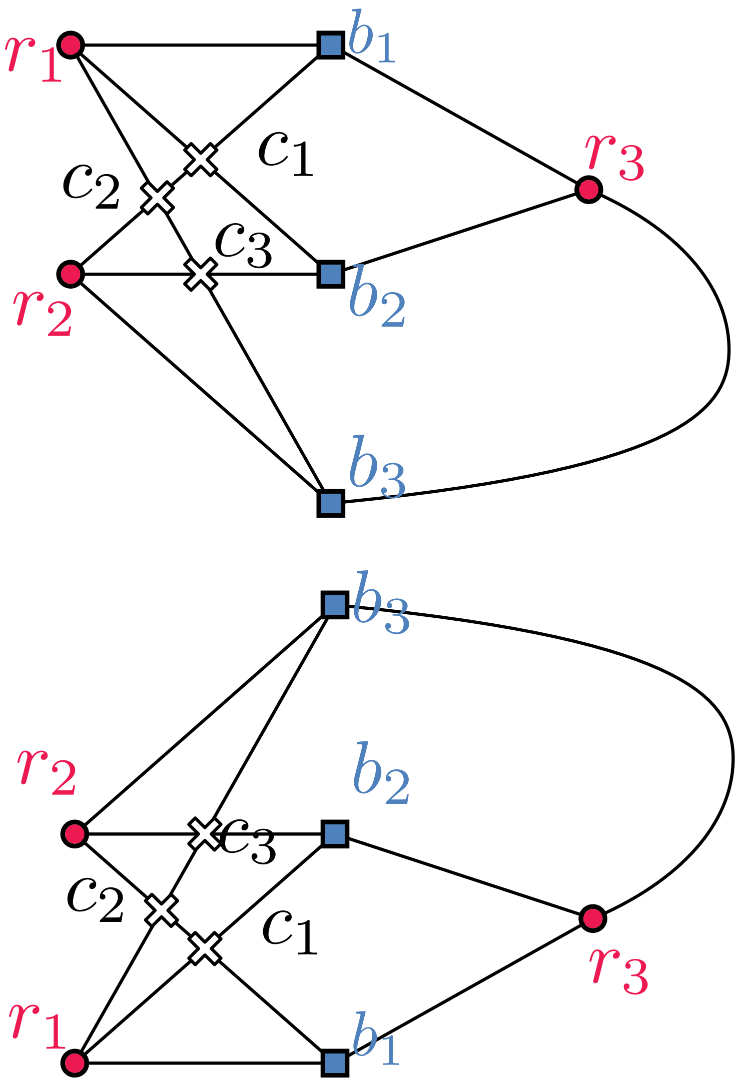
Rotation ... Cyclical order of incident edges

Rotation around r_1: b_1 b_2 b_3

Rotation around c_1: r_1 b_1 b_2 r_2

Extended rotation system ... Collection of the rotations of all vertices and crossings.

Describing simple drawings – types of isomorphisms

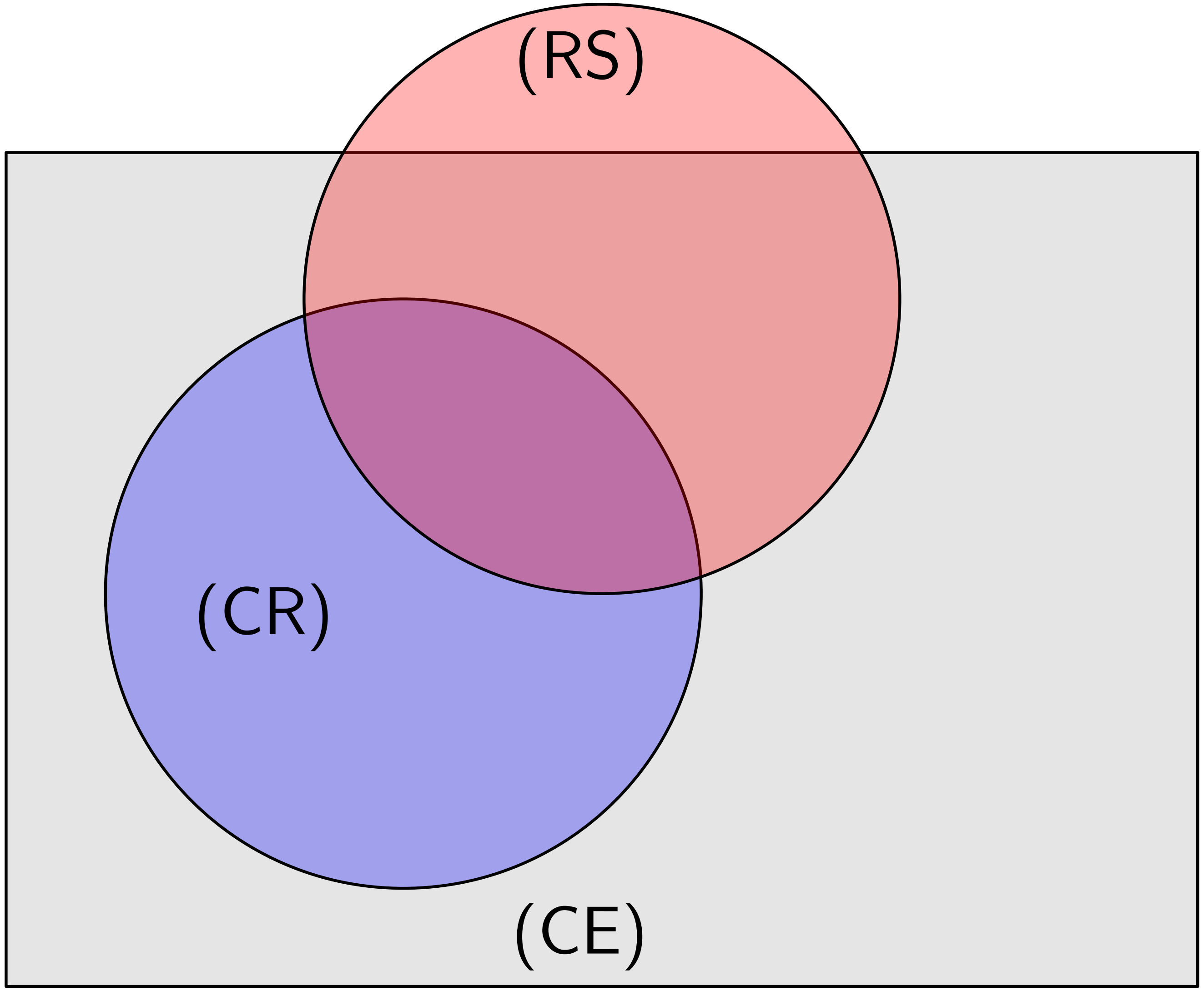


Rotation ... Cyclical order of incident edges
Rotation around r_1: $b_1\ b_2\ b_3$
Rotation around c_1: $r_1\ b_1\ b_2\ r_2$

Extended rotation system ... Collection of the rotations of all vertices and crossings.

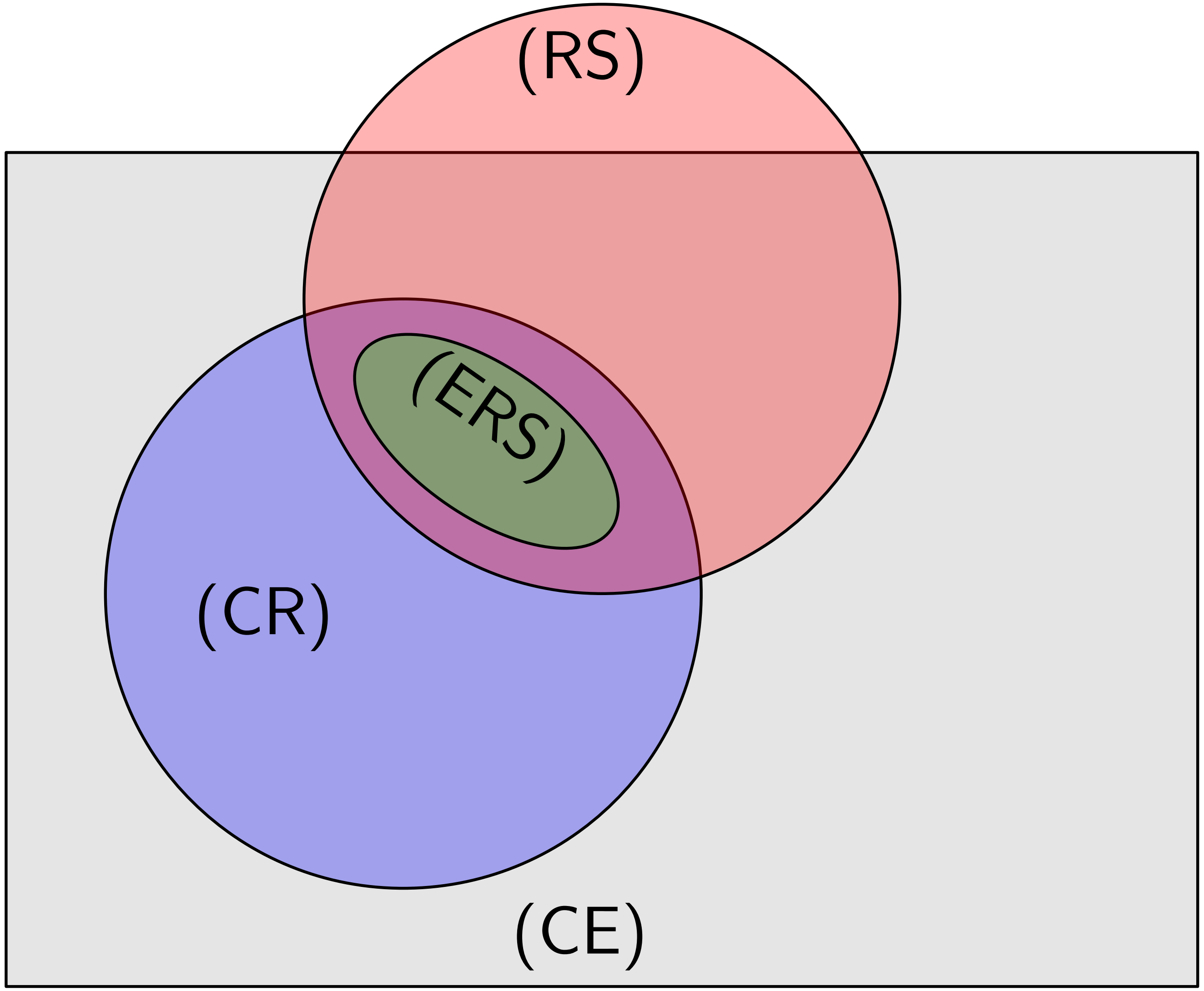
Two labelled simple drawings are ERS-isomorphic iff they have the same or inverse extended rotation systems.

Implications between isomorphisms



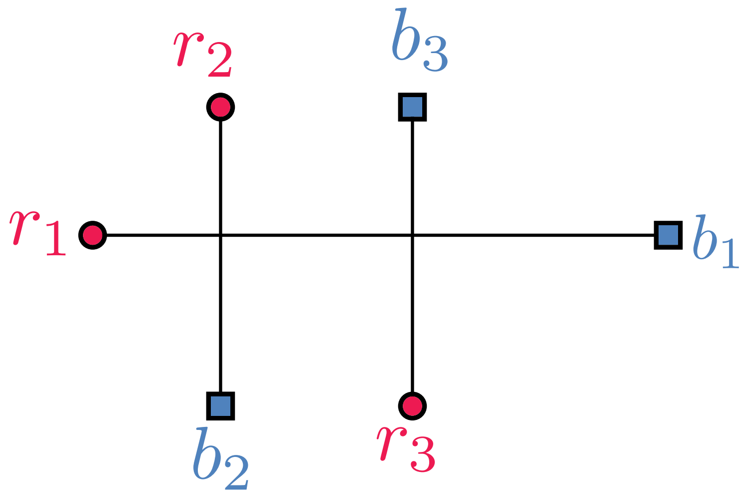
Implications between isomorphisms

ERS.... extended rotation system



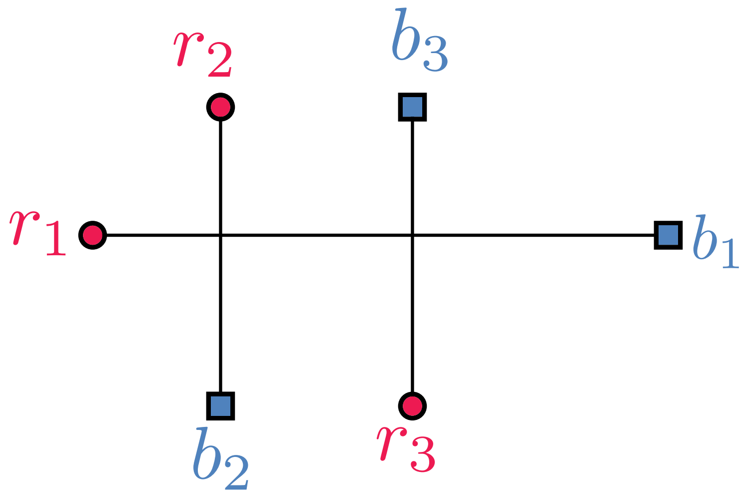
Describing simple drawings – types of isomorphisms

Two labelled simple drawings are CO-isomorphic iff for each edge the order in which it crosses other edges is the same.

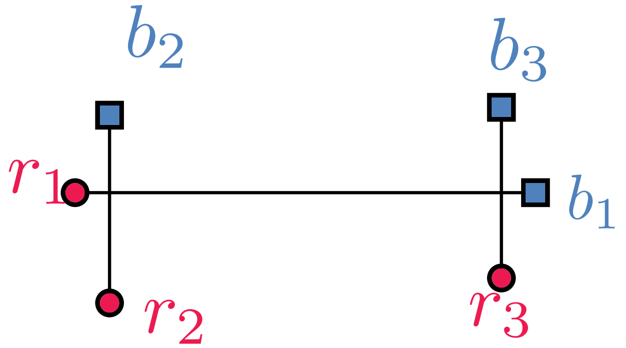


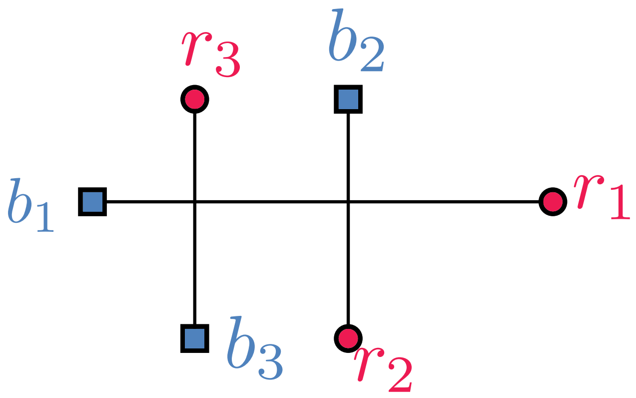
Describing simple drawings – types of isomorphisms

Two labelled simple drawings are CO-isomorphic iff for each edge the order in which it crosses other edges is the same.



$\cong$

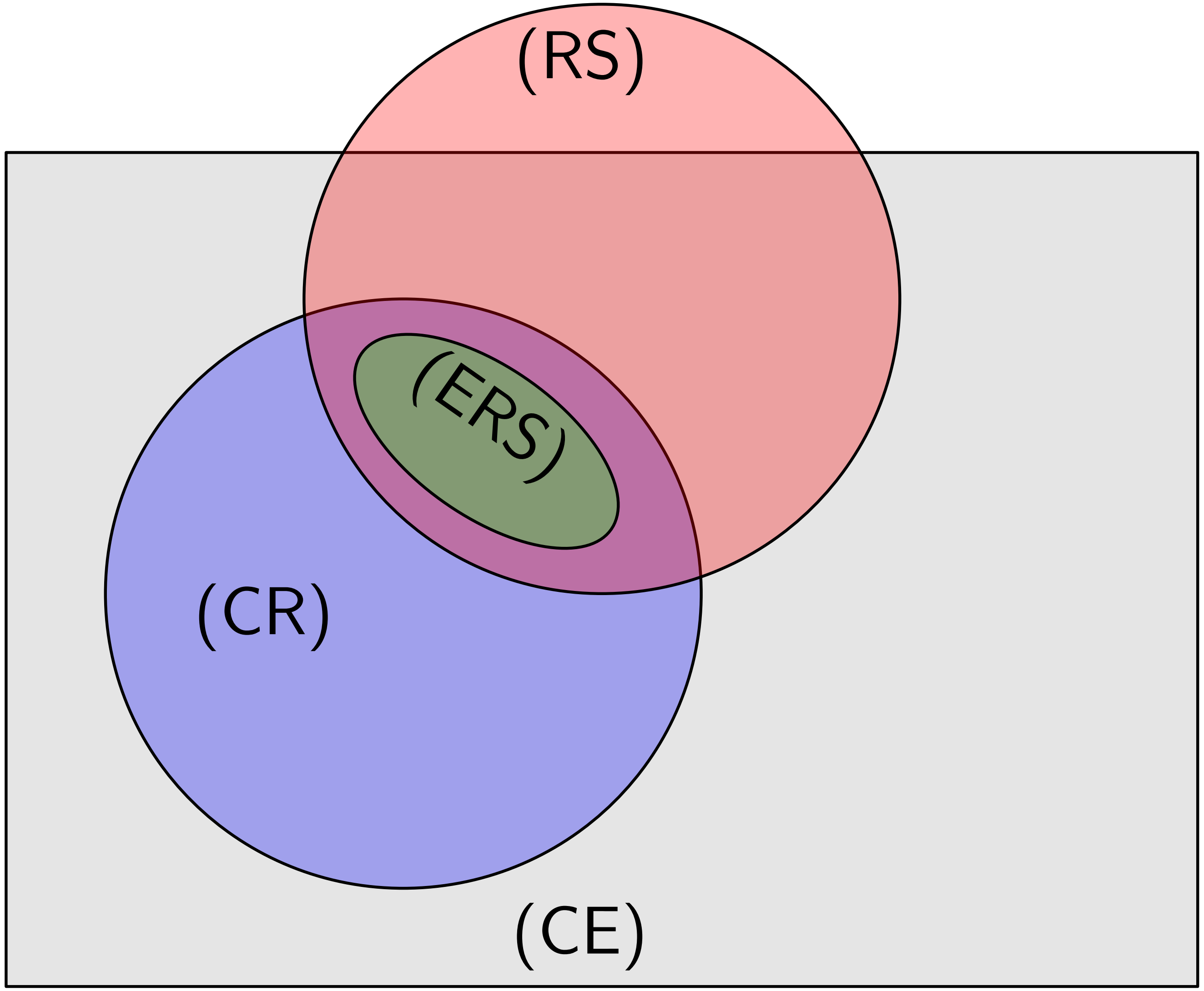




Describing simple drawings – types of isomorphisms

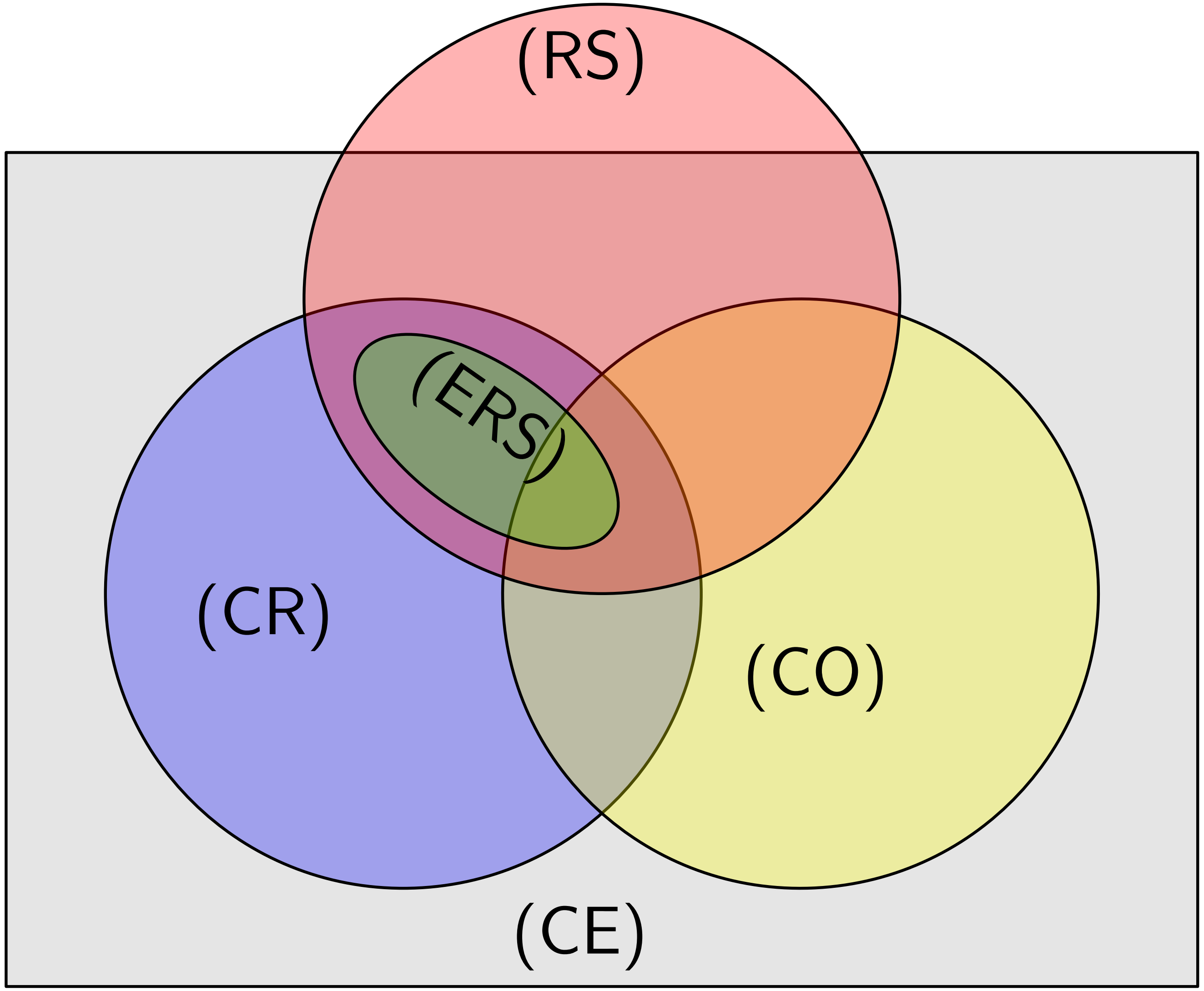
Two labelled simple drawings are CO-isomorphic iff for each edge the order in which it crosses other edges is the same.

Implications between isomorphisms



Implications between isomorphisms

CO.... Crossing Order



Describing simple drawings – Types of isomorphism

Two labelled simple drawings are strongly isomorphic iff
there exists a homeomorphism of the sphere such that one drawing is mapped to the other.

Describing simple drawings – Types of isomorphism

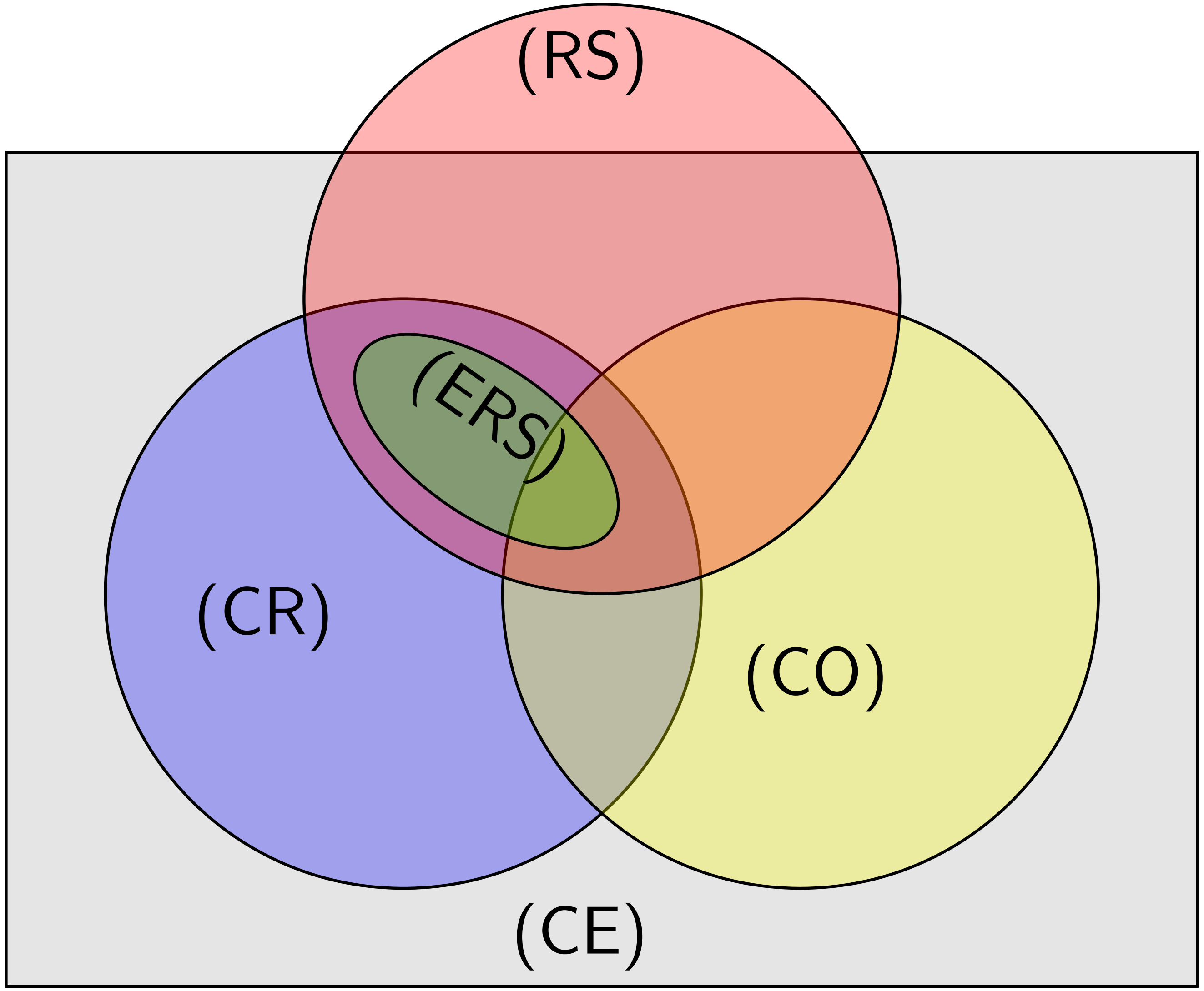
Two labelled simple drawings are strongly isomorphic iff
there exists a homeomorphism of the sphere such that one drawing is mapped to the other.

Two labelled simple drawings of connected graphs on the sphere are strongly isomorphic iff

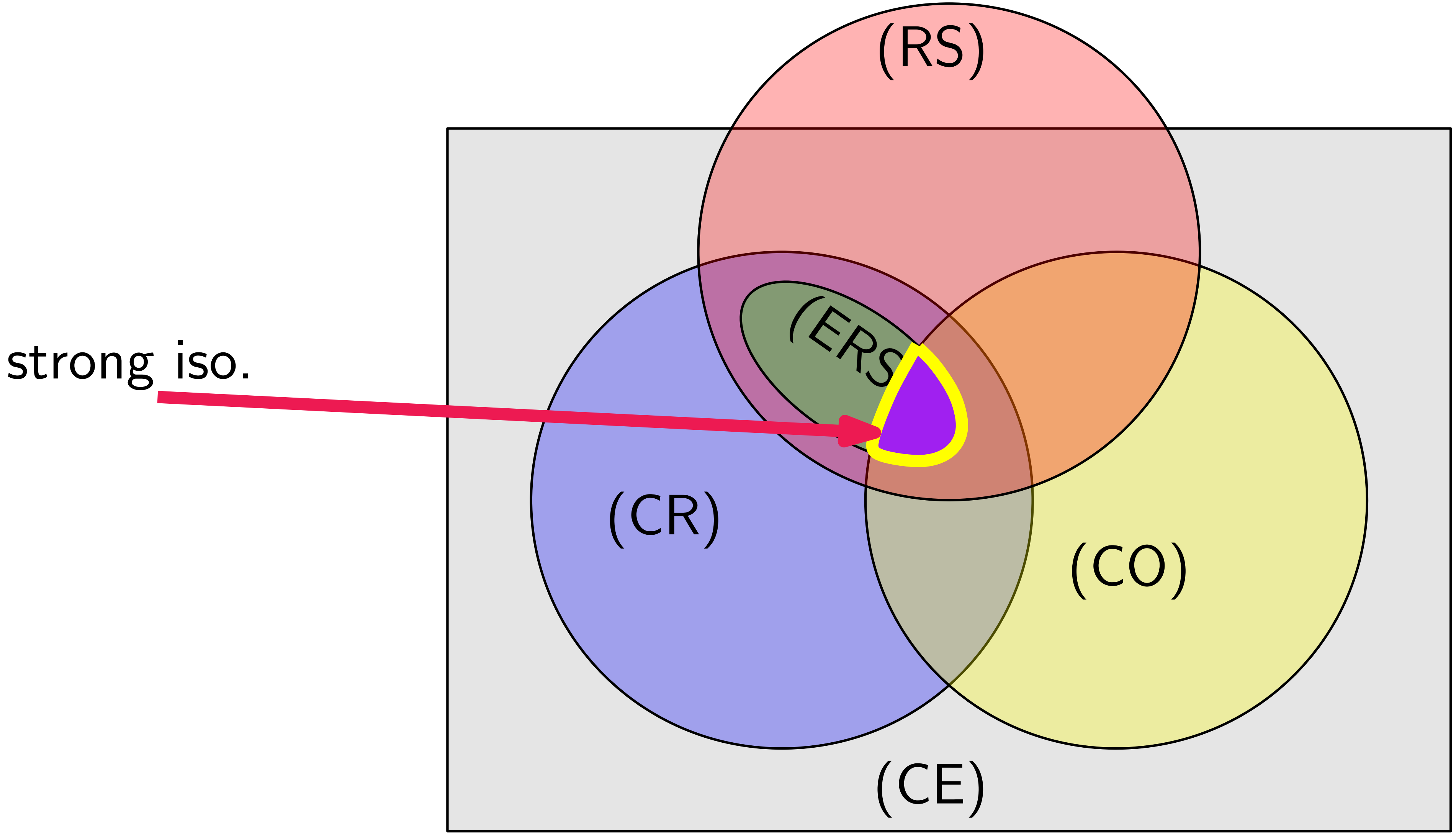
1. They are ERS-isomorphic and
2. CO-isomorphic.[1)]

1) [J. Kynčl 2011]

Implications between isomorphisms



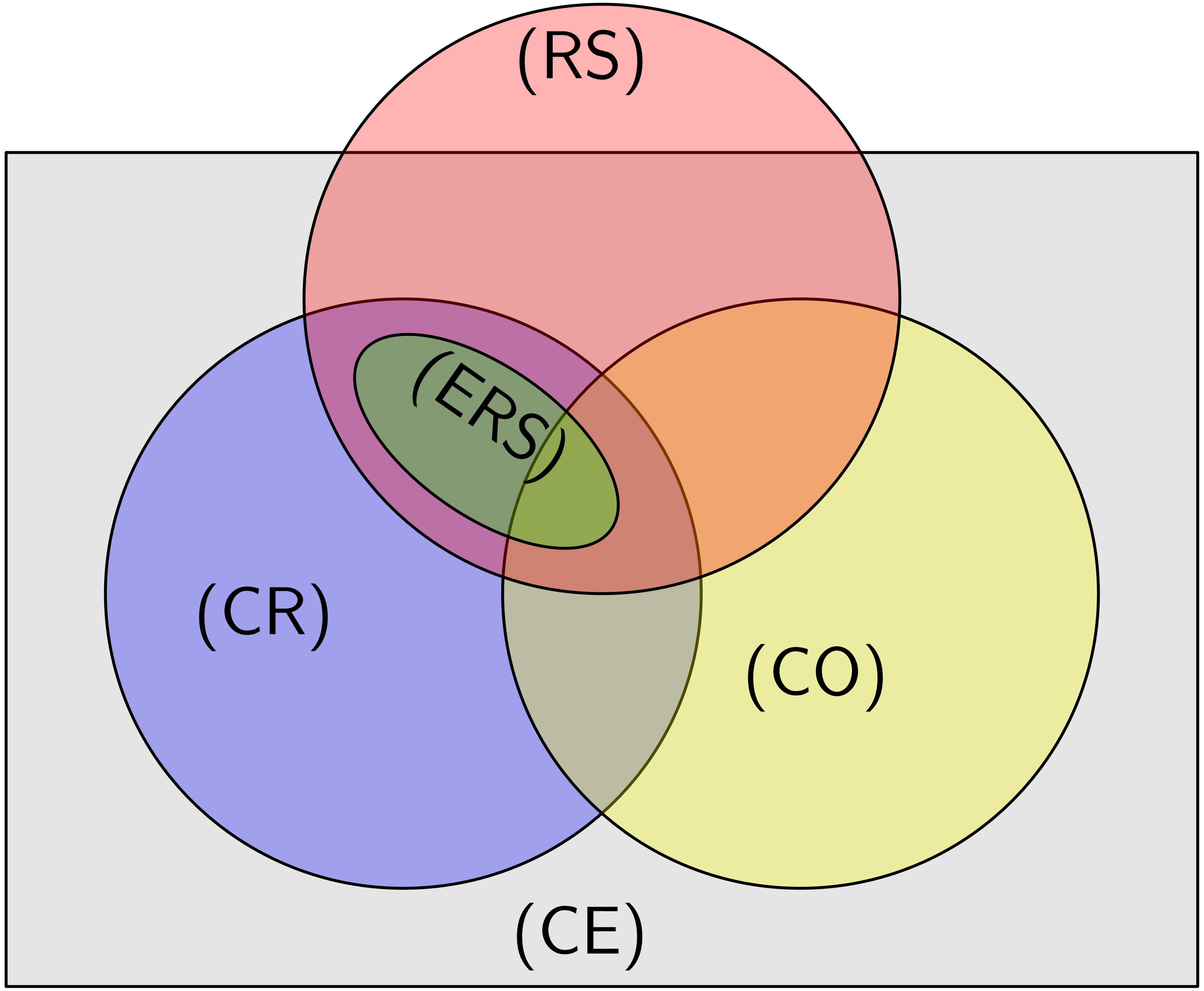
Implications between isomorphisms



Describing simple drawings – Types of isomorphism

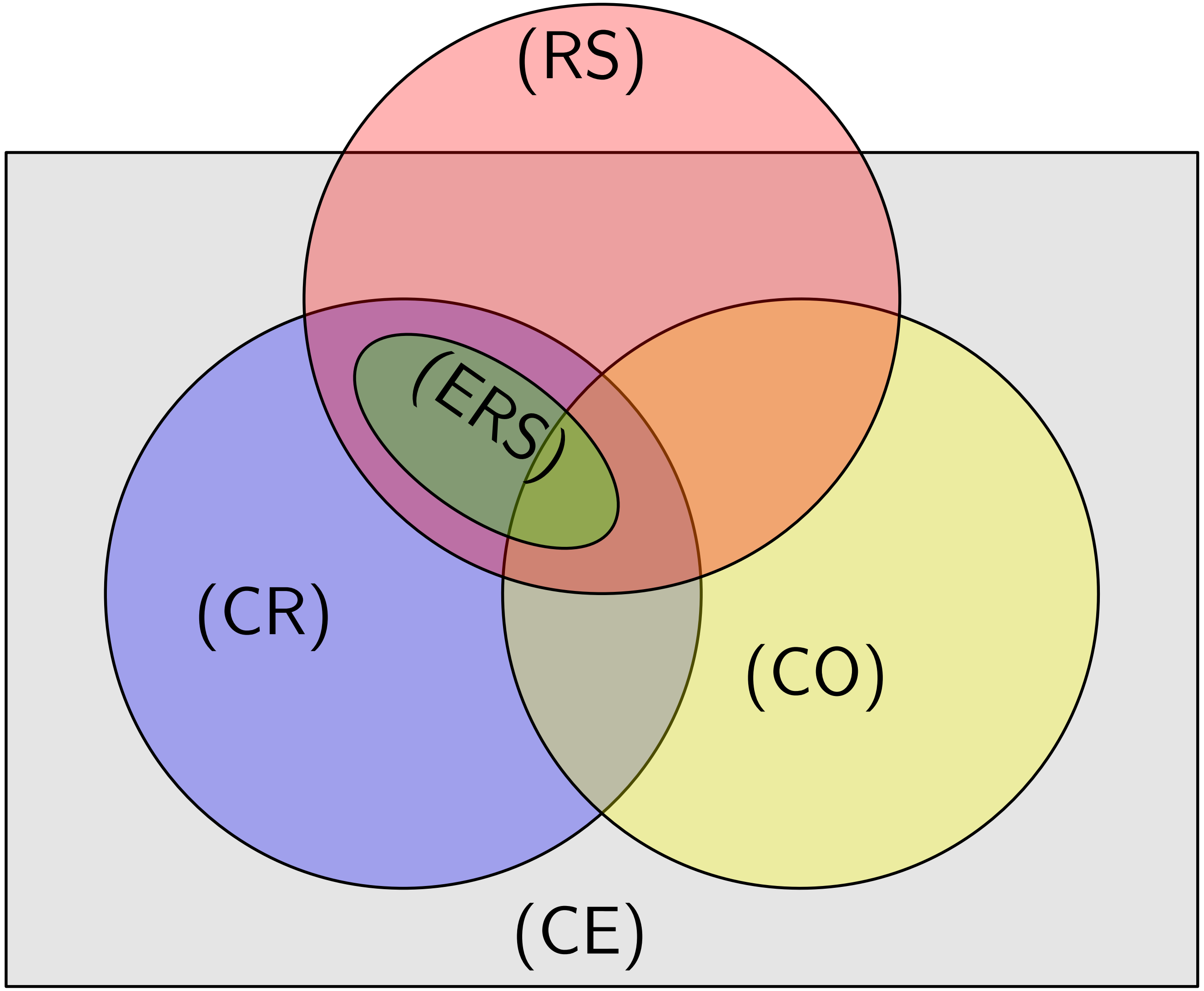
Unlabelled simple drawings are isomorphic w.r.t. some type of isomorphism iff $\exists$ labeling s.t. labelled drawings are isomorphic w.r.t. that type.

Implications between isomorphisms



Implications between isomorphisms

For the complete graph:

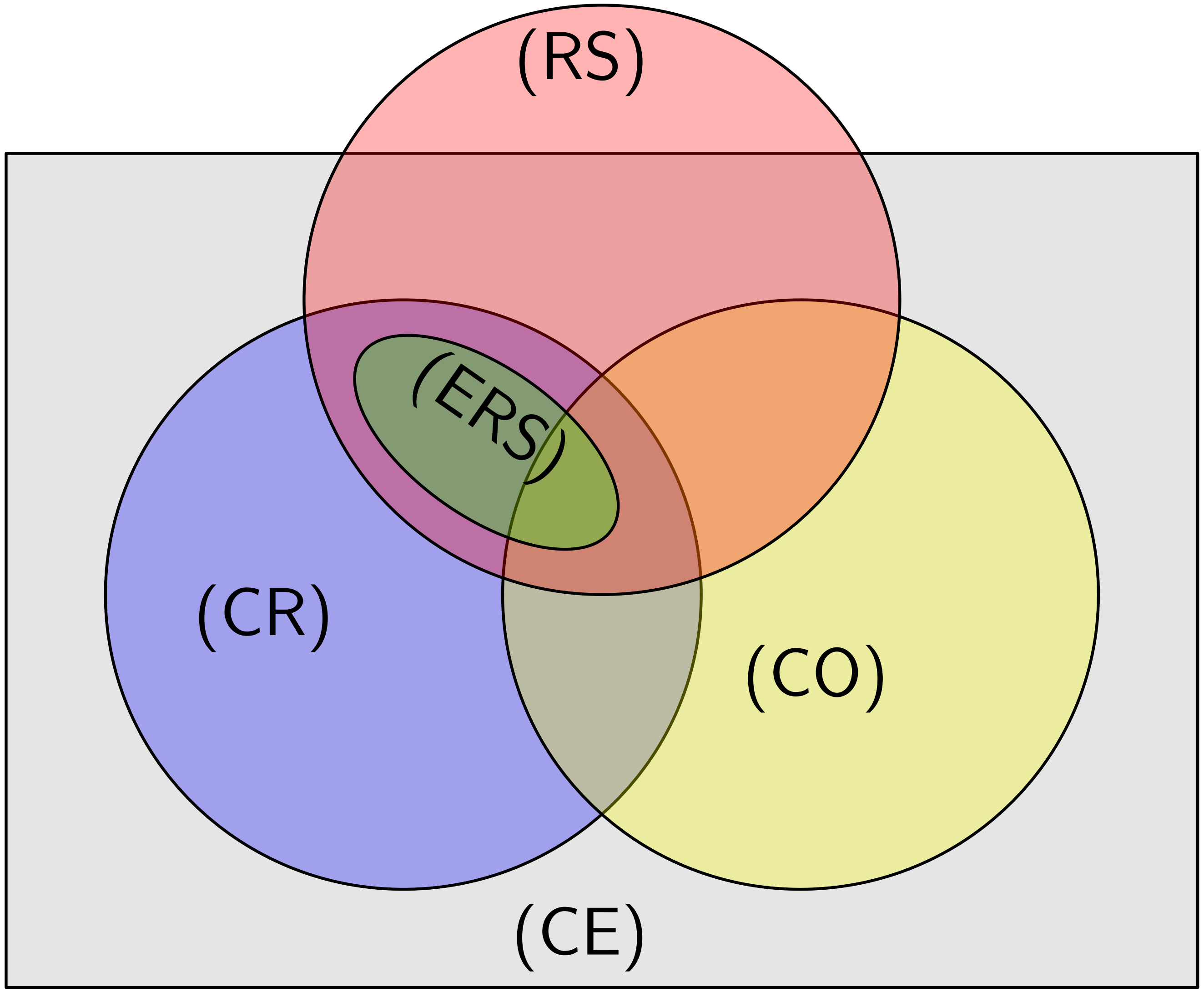


Implications between isomorphisms

For the complete graph:

ERS <=> CE <=> RS <=> CR

[E. Gioan 2005, 2022], [J. Kynčl 2013]



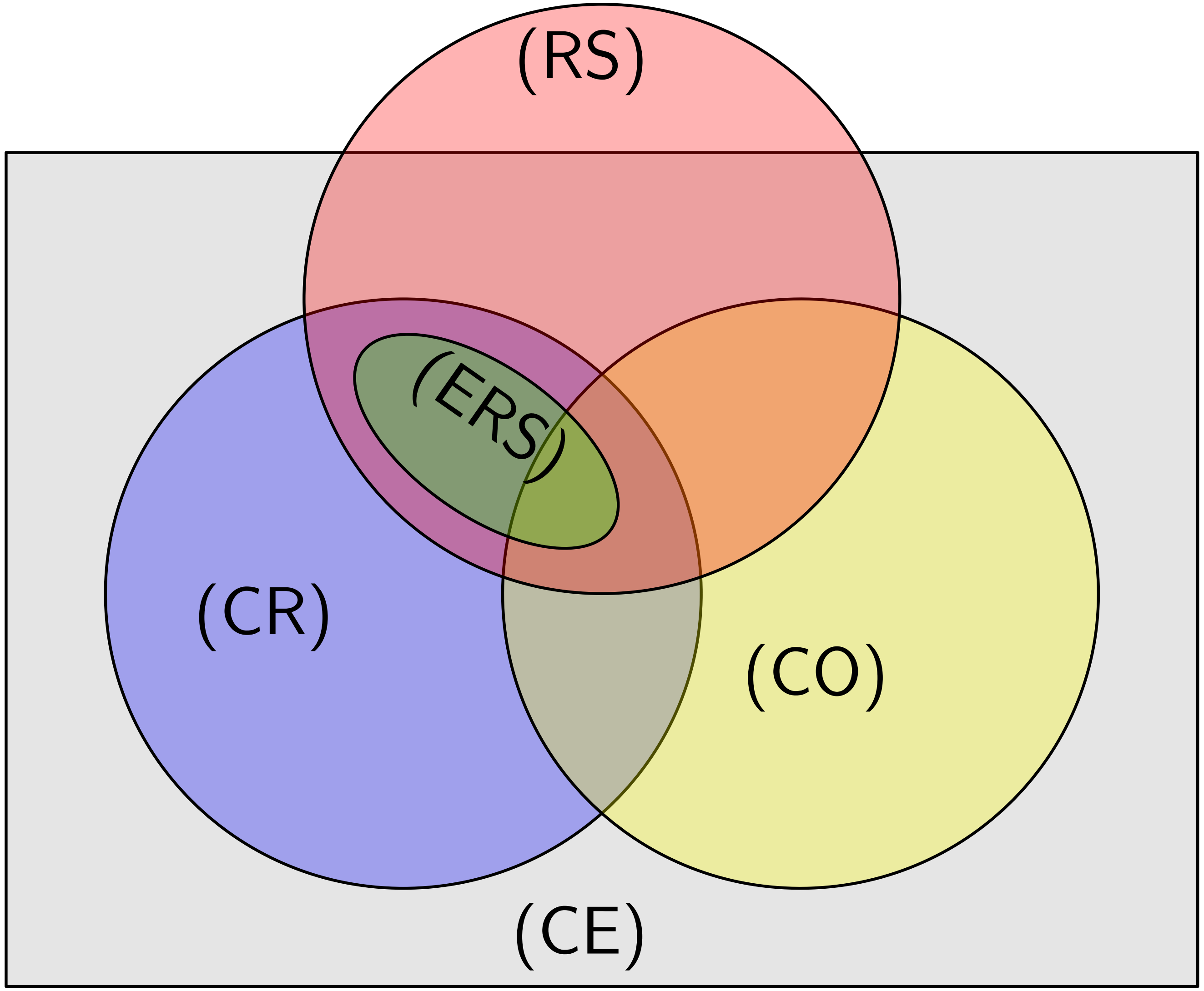
Implications between isomorphisms

For the complete graph:

ERS <=> CE <=> RS <=> CR

[E. Gioan 2005, 2022], [J. Kynčl 2013]

CO <=> strong iso.



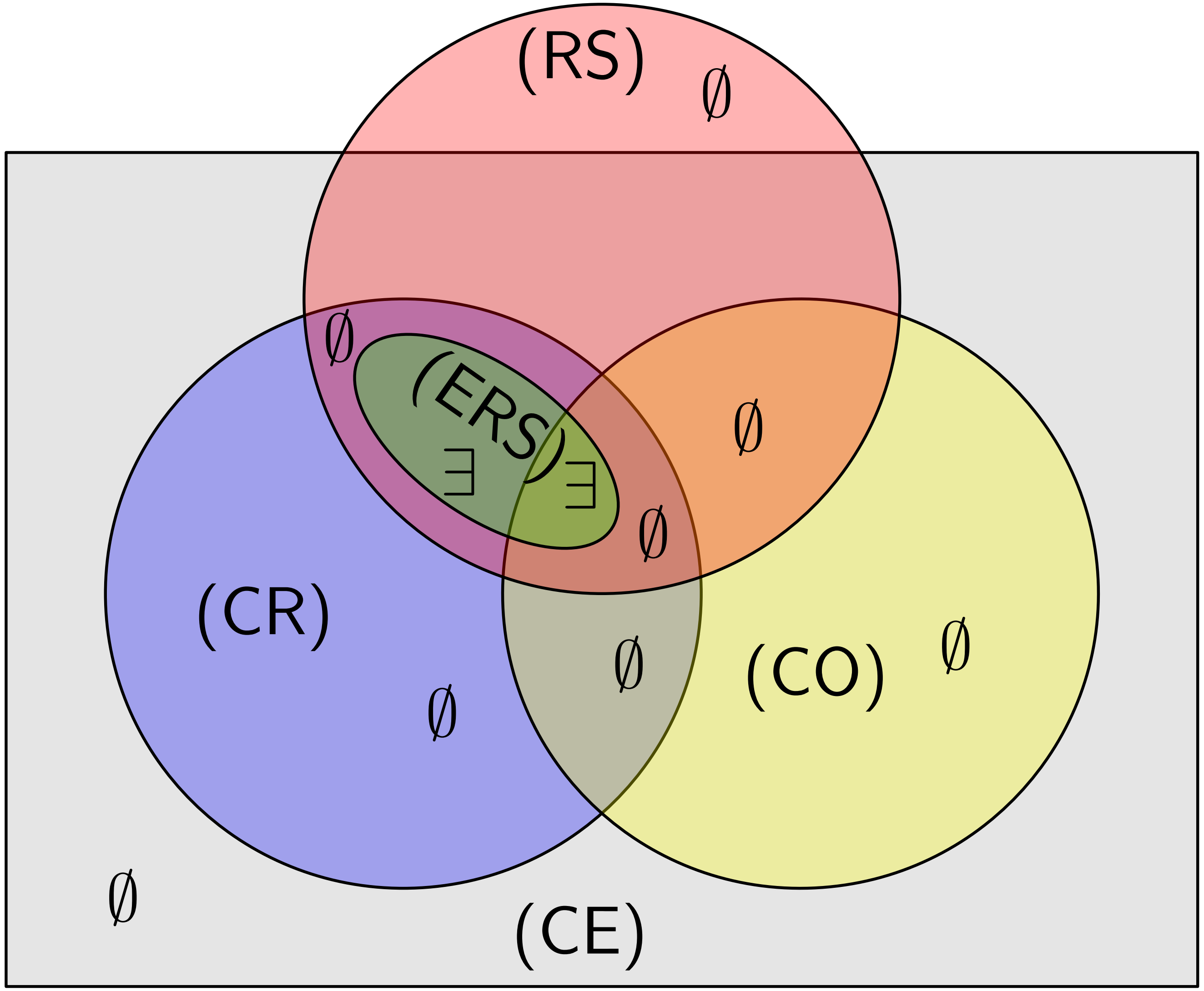
Implications between isomorphisms

For the complete graph:

ERS <=> CE <=> RS <=> CR

[E. Gioan 2005, 2022], [J. Kynčl 2013]

CO <=> strong iso.



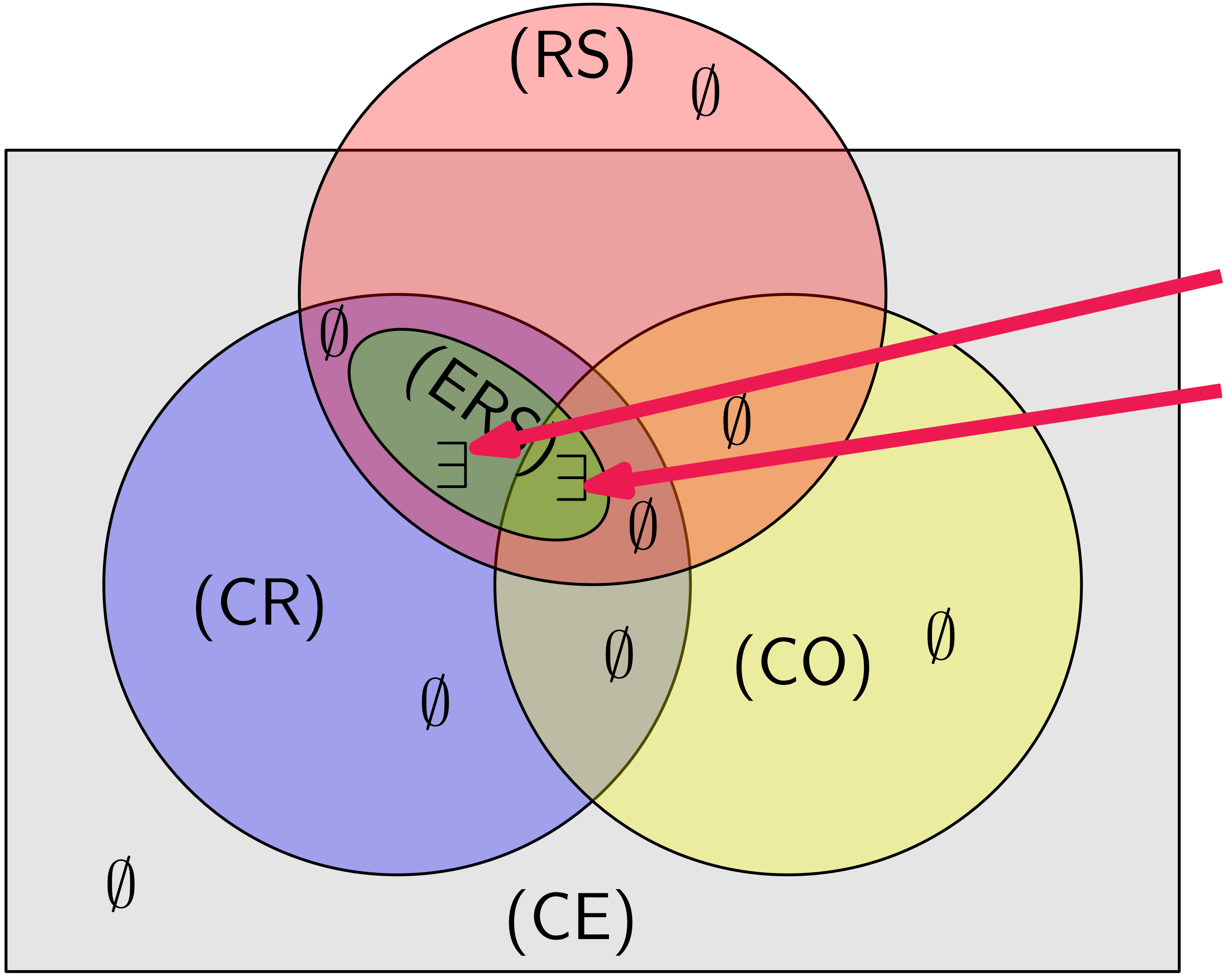
Implications between isomorphisms

For the complete graph:

ERS <=> CE <=> RS <=> CR

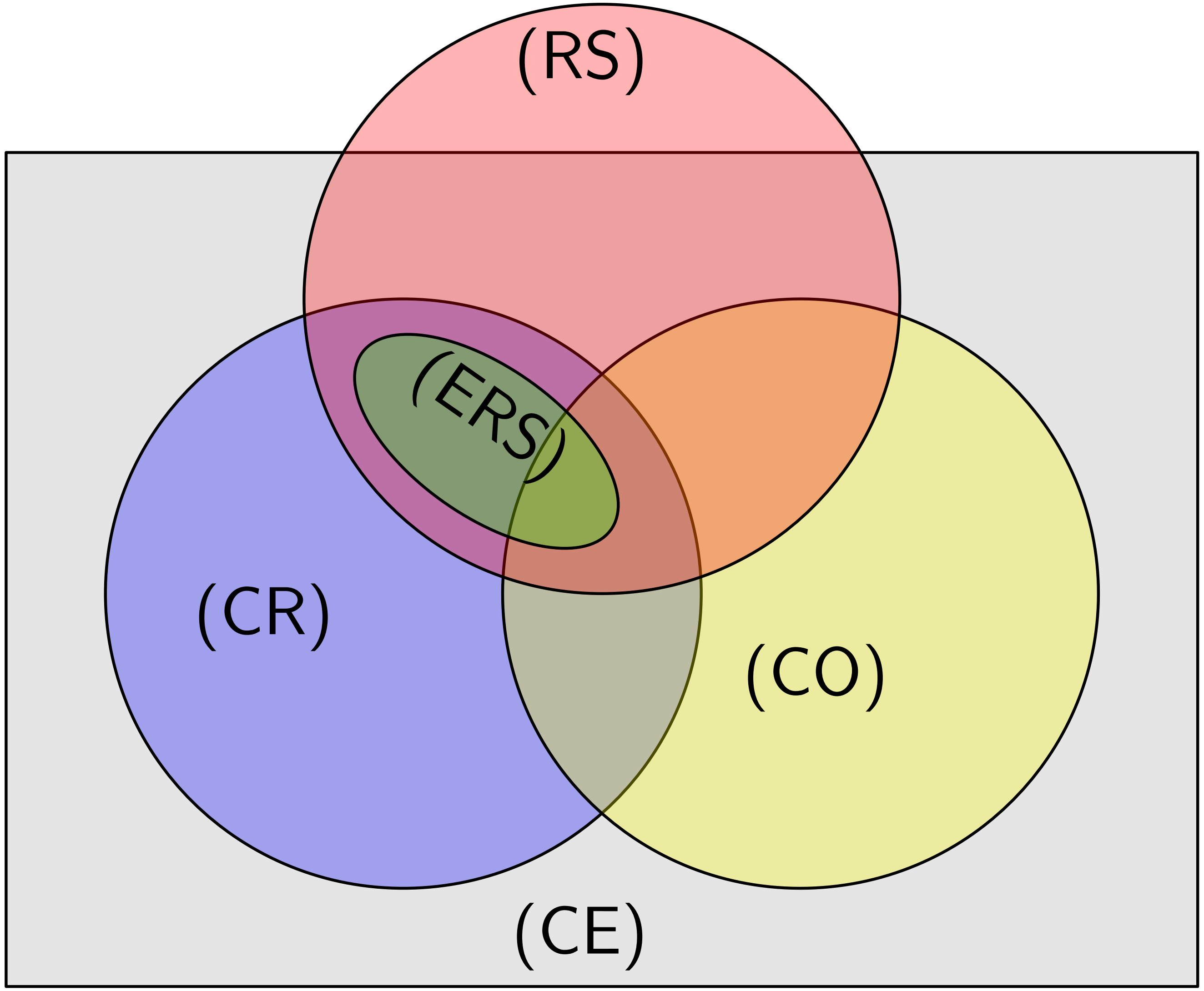
[E. Gioan 2005, 2022], [J. Kynčl 2013]

CO <=> strong iso.



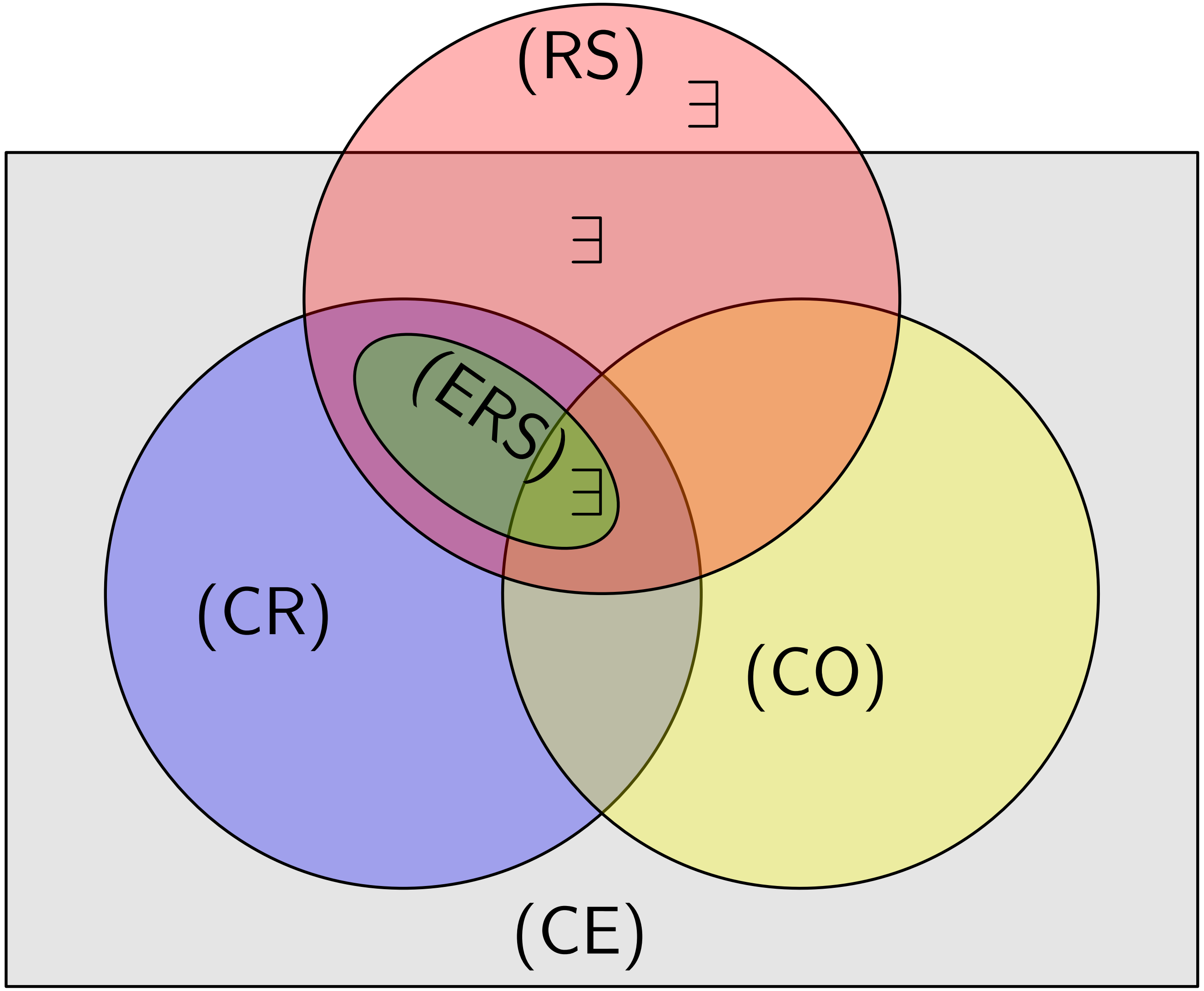
Implications between isomorphisms

For complete
multipartite graphs:



Implications between isomorphisms

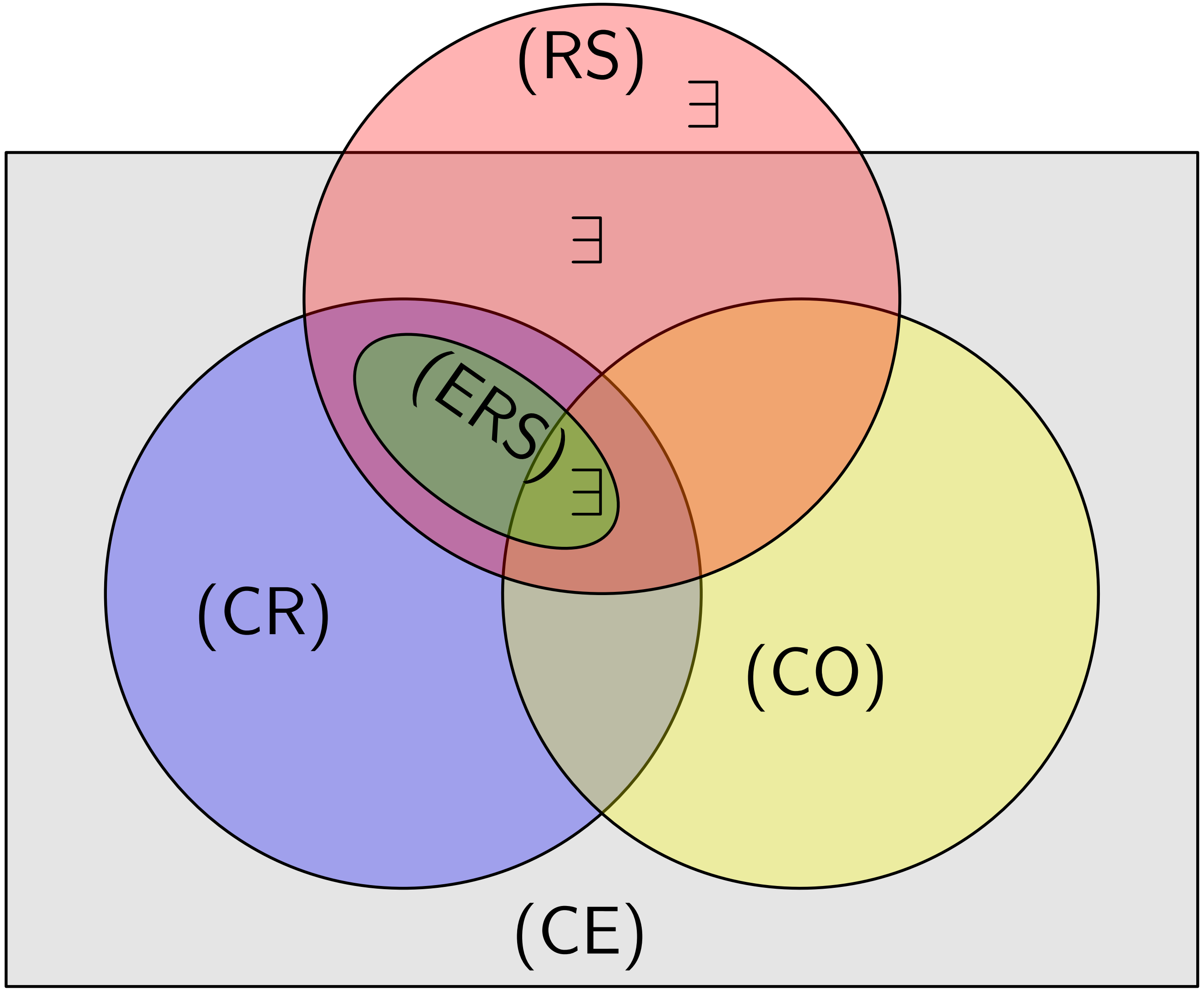
For complete multipartite graphs:



Implications between isomorphisms

For complete multipartite graphs:

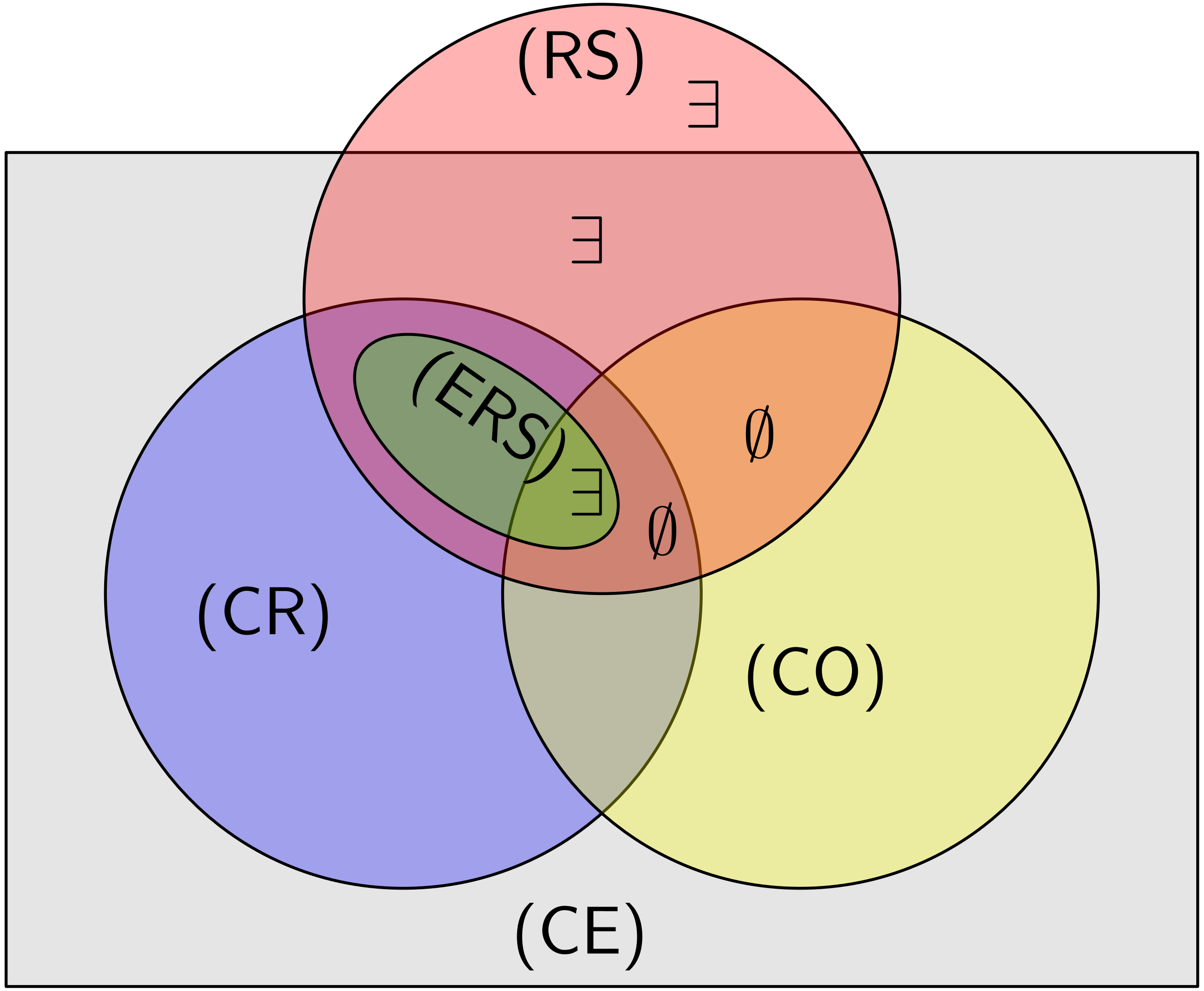
RS + CO => strong iso.



Implications between isomorphisms

For complete multipartite graphs:

RS + CO => strong iso.

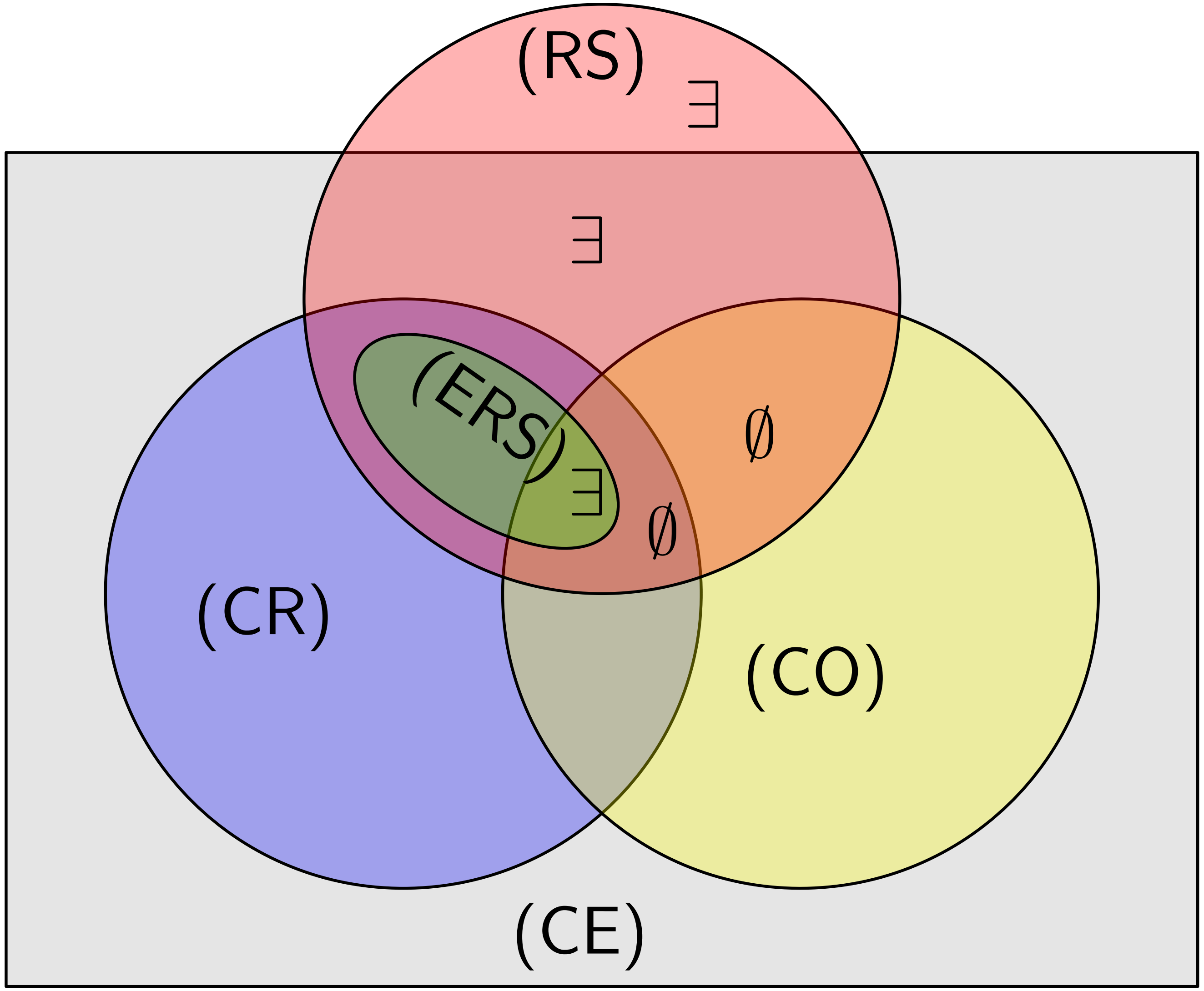


Implications between isomorphisms

For complete multipartite graphs:

RS + CO => strong iso.

If each partition class has ≥ 3 vertices:
CE => RS
CR => ERS
CO => strong iso.

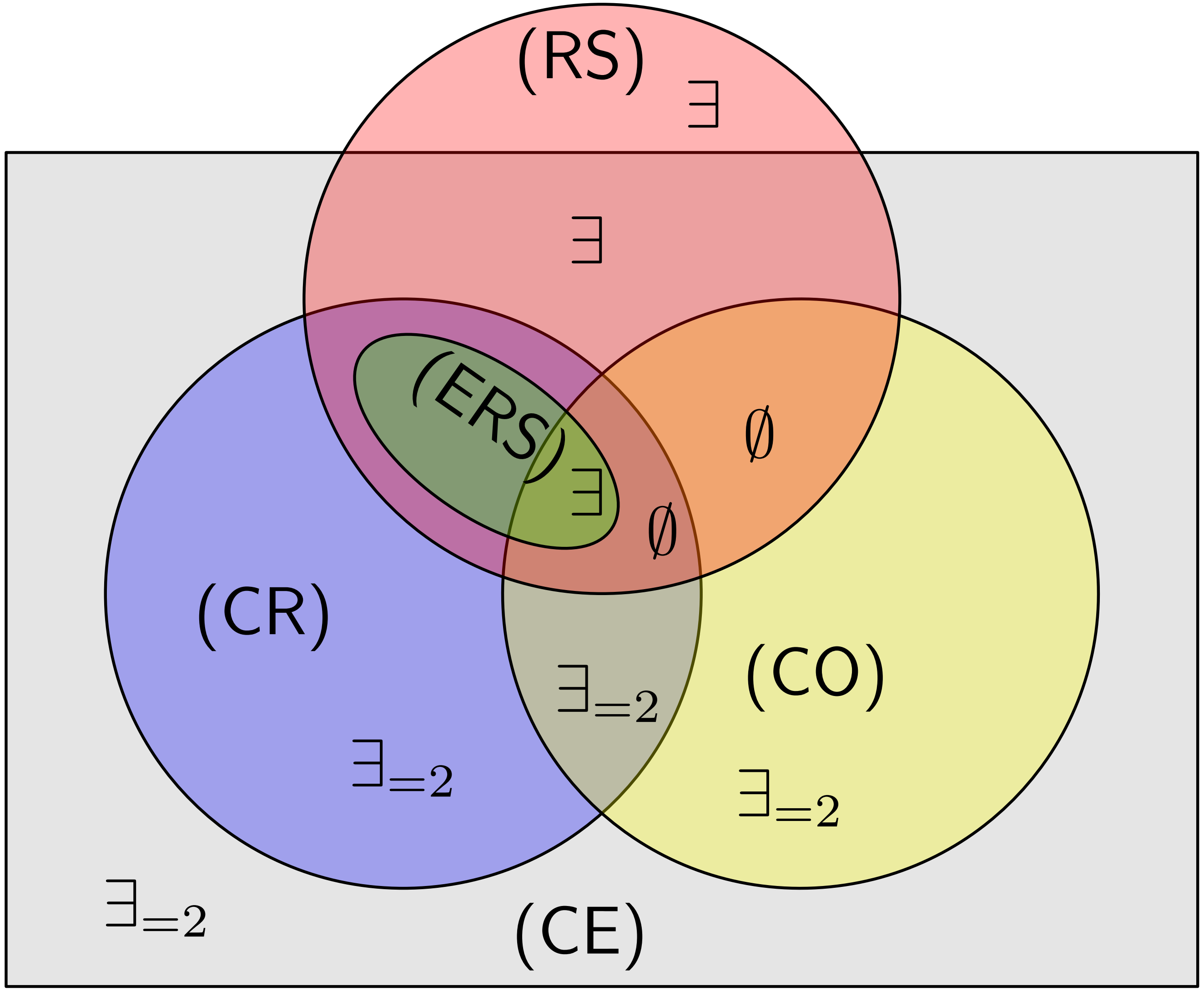


Implications between isomorphisms

For complete multipartite graphs:

RS + CO => strong iso.

If each partition class has ≥ 3 vertices:
CE => RS
CR => ERS
CO => strong iso.

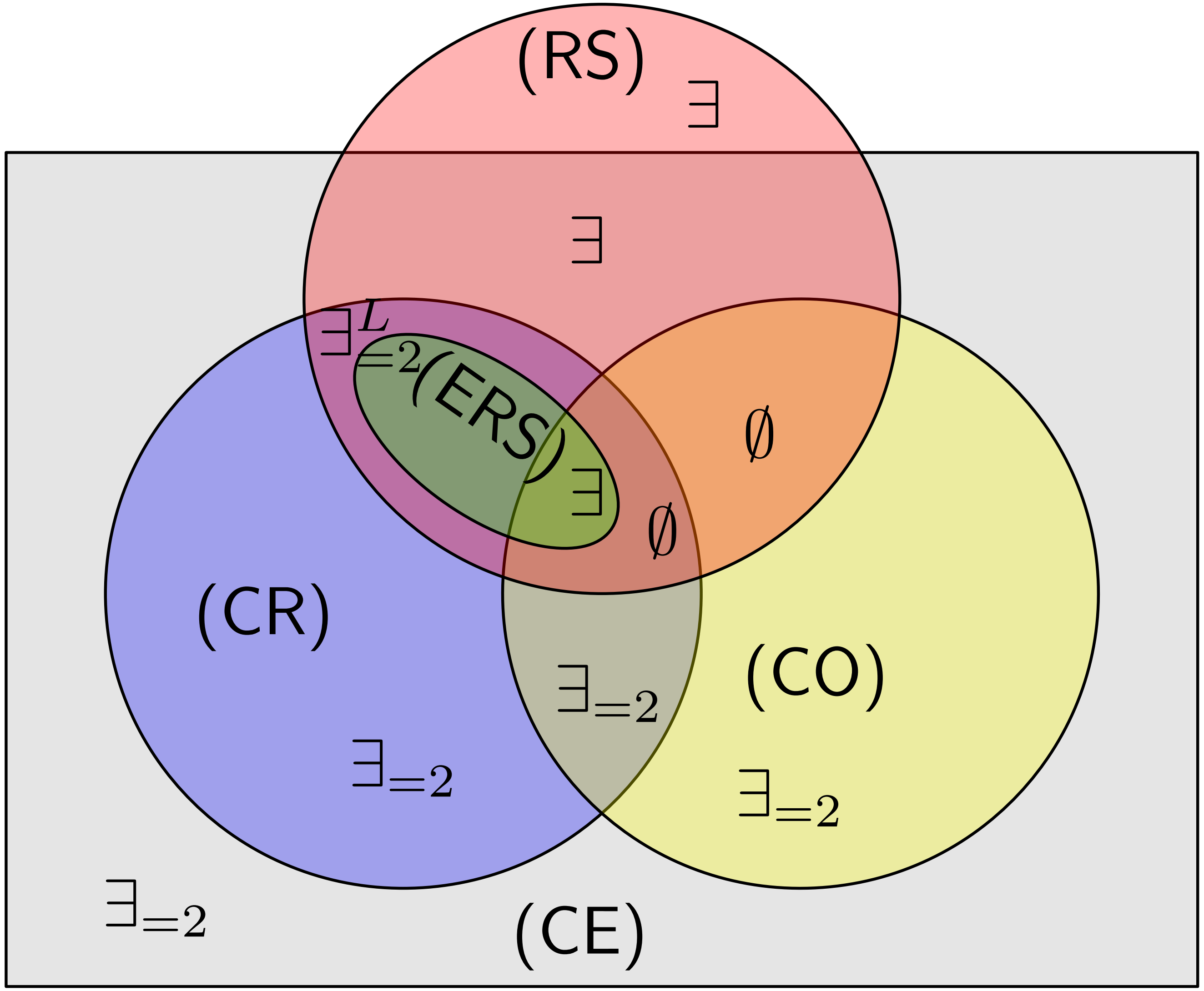


Implications between isomorphisms

For complete multipartite graphs:

RS + CO => strong iso.

If each partition class has ≥ 3 vertices:
CE => RS
CR => ERS
CO => strong iso.



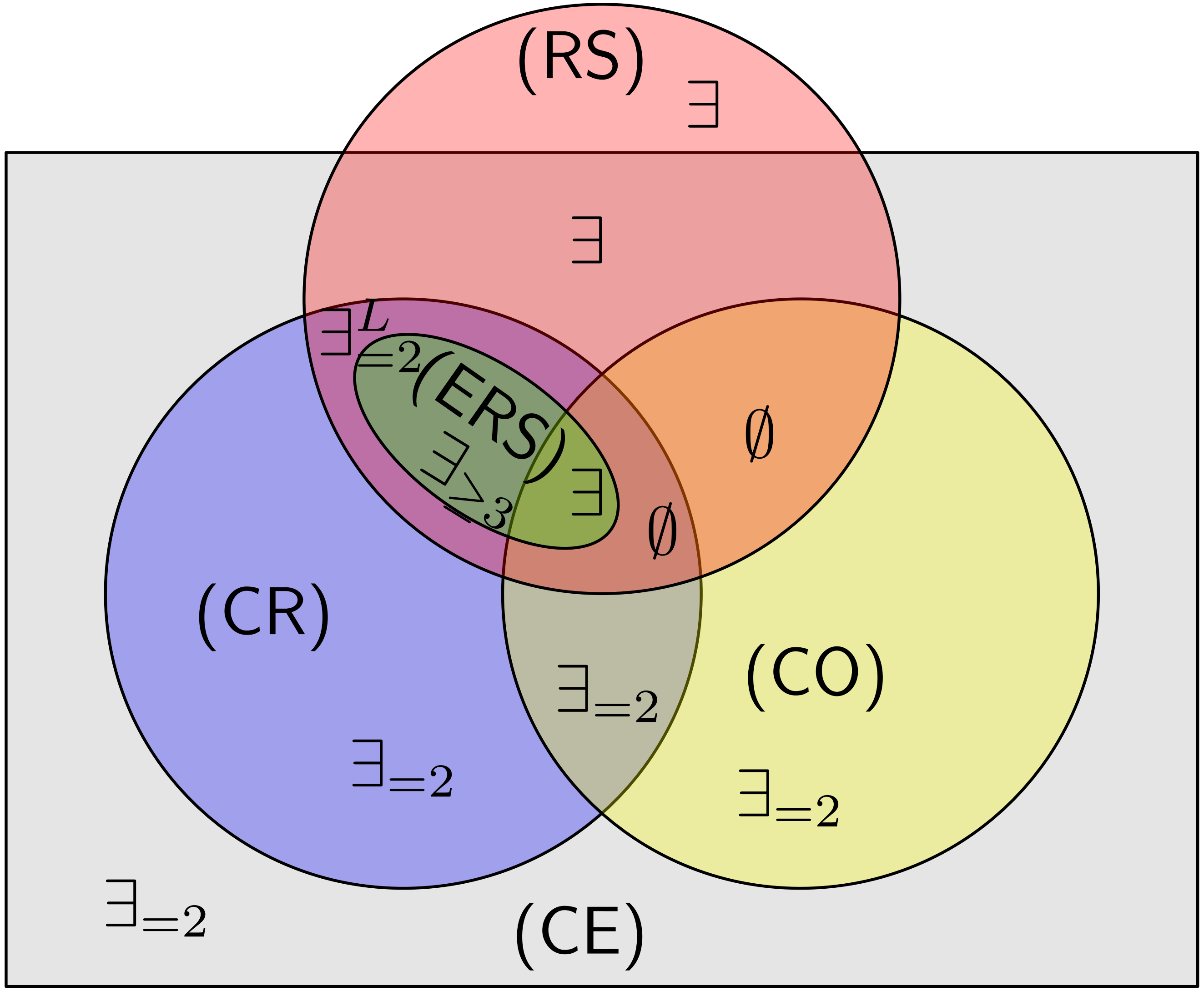
Implications between isomorphisms

For complete multipartite graphs:

RS + CO => strong iso.

If each partition class has ≥ 3 vertices:
CE => RS
CR => ERS
CO => strong iso.

For $K_{2,n}$: ERS => strong iso.[1]



1) [O. Aichholzer, M.K. Chiu, H. Hoang, M. Hoffmann, J. Kynčl, Y. Maus, B. Vogtenhuber, A.W. 2023]

Implications between isomorphisms

For complete multipartite graphs:

Proof sketch time :-)

RS + CO => strong iso.

If each partition class has ≥ 3 vertices:
CE => RS
CR => ERS
CO => strong iso.

For $K_{2,n}$: ERS => strong iso.[1]

1) [O. Aichholzer, M.K. Chiu, H. Hoang, M. Hoffmann, J. Kynčl, Y. Maus, B. Vogtenhuber, A.W. 2023]

Implications between isomorphisms

For complete multipartite graphs:

Proof sketch time :-)

RS + CO => strong iso.

If each partition class has ≥ 3 vertices:

CE => RS ← Let's prove this one!

CR => ERS

CO => strong iso.

For $K_{2,n}$: ERS => strong iso.[1]

1) [O. Aichholzer, M.K. Chiu, H. Hoang, M. Hoffmann, J. Kynčl, Y. Maus, B. Vogtenhuber, A.W. 2023]

In each partition class at least 3 vertices: CE $\implies$ RS

In each partition class at least 3 vertices: CE $\implies$ RS

True for (labelled) drawings of $K_{3,3}$.

In each partition class at least 3 vertices: CE $\implies$ RS

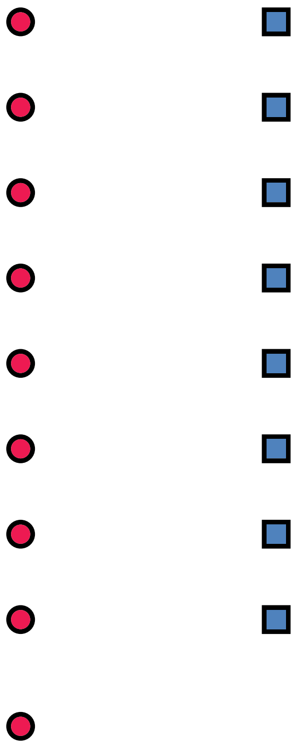
True for (labelled) drawings of $K_{3,3}$.

Computationally check all[1)] 102 unlabelled drawings with all labellings (72 each).

1) [Harborth 1976]

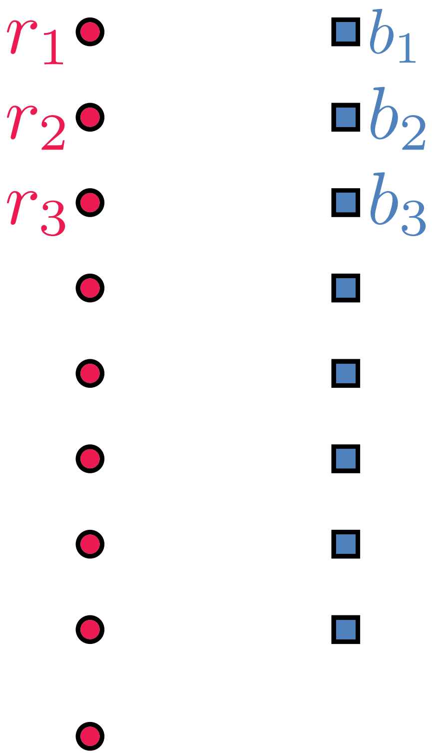
In each partition class at least 3 vertices: CE $\implies$ RS

True for complete bipartite graphs.



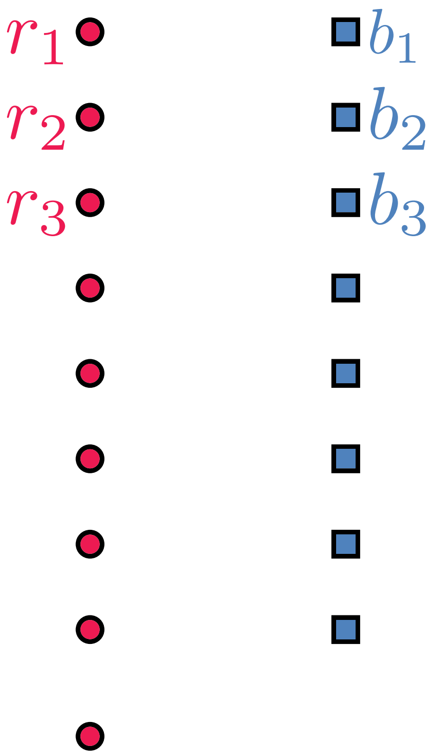
In each partition class at least 3 vertices: CE $\implies$ RS

True for complete bipartite graphs.



In each partition class at least 3 vertices: CE $\implies$ RS

True for complete bipartite graphs.

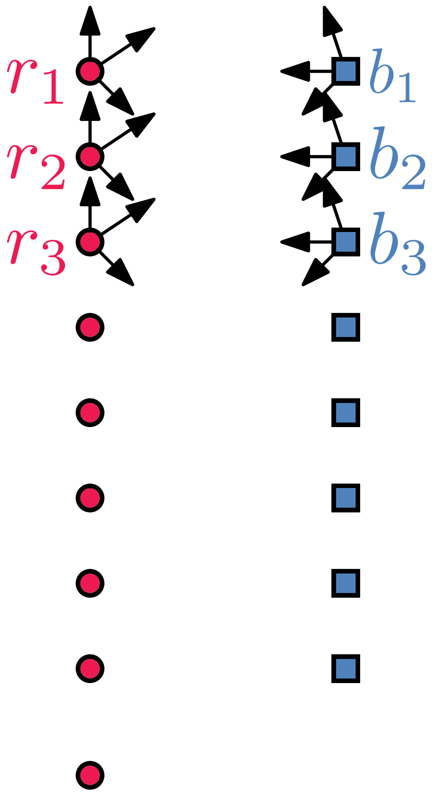


Use $K_{3,3}$:

$r_1, r_2, r_3, b_1, b_2, b_3$

In each partition class at least 3 vertices: CE $\implies$ RS

True for complete bipartite graphs.

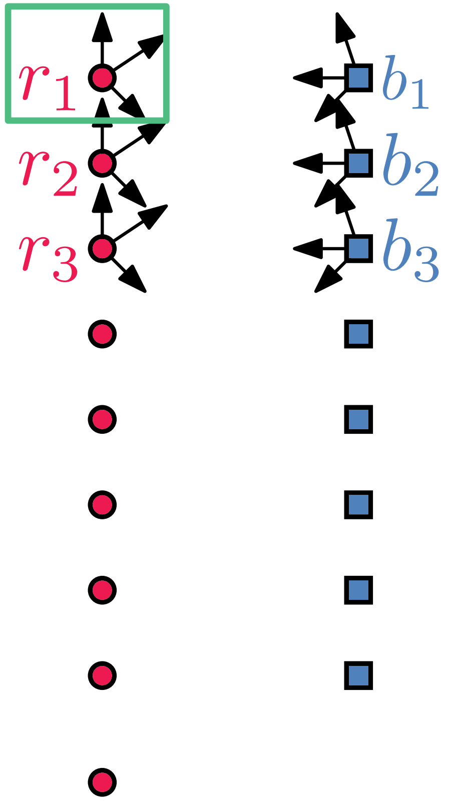


Use $K_{3,3}$:

$r_1, r_2, r_3, b_1, b_2, b_3$

In each partition class at least 3 vertices: CE $\implies$ RS

True for complete bipartite graphs.



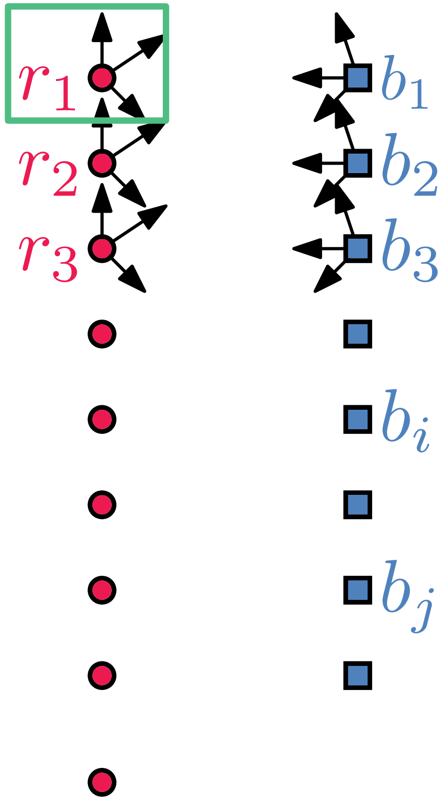
Use $K_{3,3}$:

$r_1, r_2, r_3, b_1, b_2, b_3$

$r_1, r_2, r_3, b_1, b_i, b_j$

In each partition class at least 3 vertices: CE $\implies$ RS

True for complete bipartite graphs.



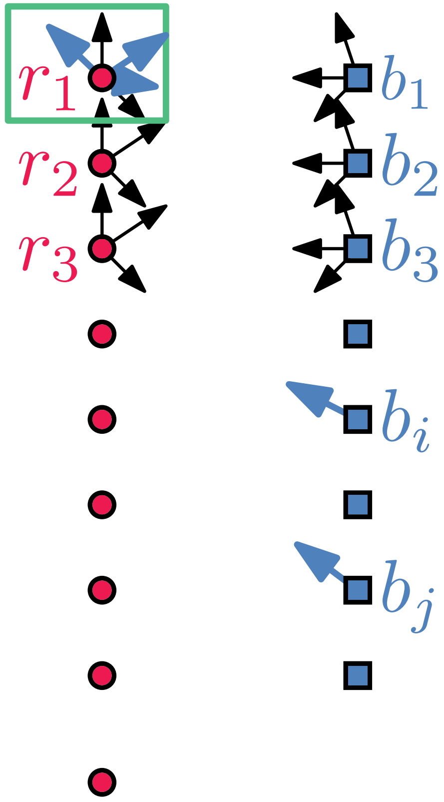
Use $K_{3,3}$:

$r_1, r_2, r_3, b_1, b_2, b_3$

$r_1, r_2, r_3, b_1, b_i, b_j$

In each partition class at least 3 vertices: CE $\Longrightarrow$ RS

True for complete bipartite graphs.



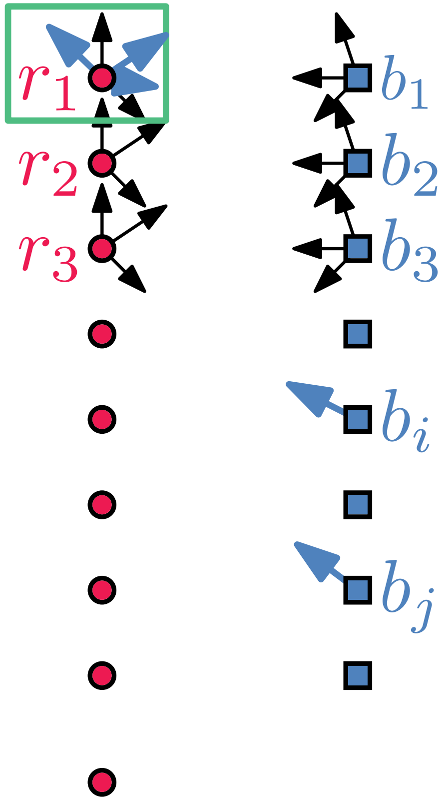
Use $K_{3,3}$:

$r_1, r_2, r_3, b_1, b_2, b_3$

$r_1, r_2, r_3, b_1, b_i, b_j$

In each partition class at least 3 vertices: CE $\implies$ RS

True for complete bipartite graphs.



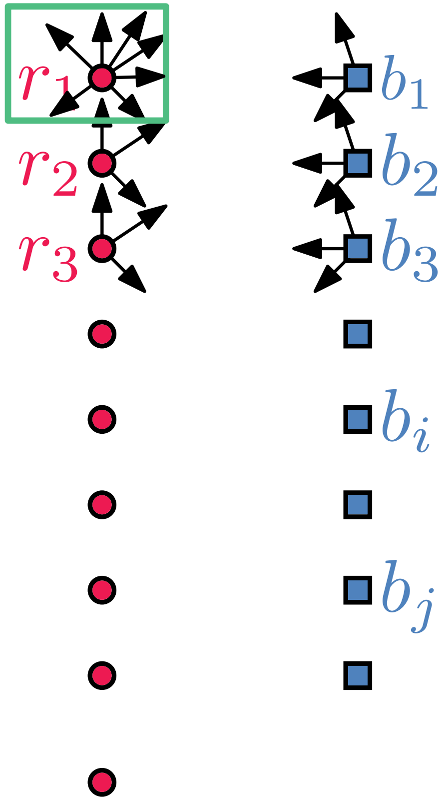
Use $K_{3,3}$:

$r_1, r_2, r_3, b_1, b_2, b_3$

$r_1, r_2, r_3, b_1, b_i, b_j$

In each partition class at least 3 vertices: CE $\implies$ RS

True for complete bipartite graphs.



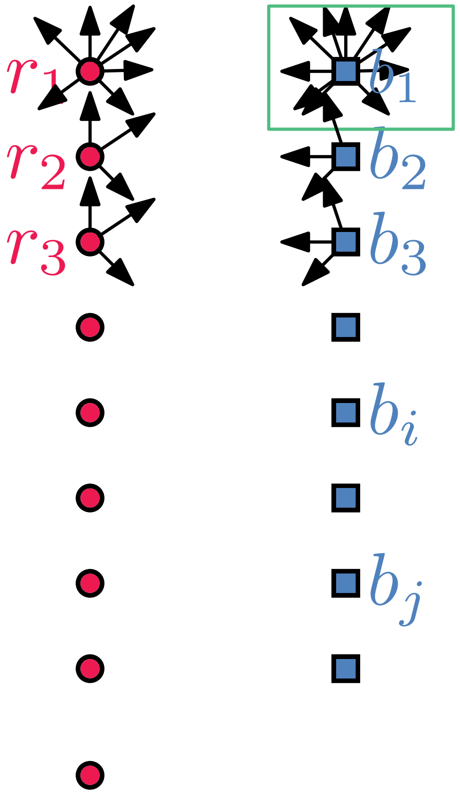
Use $K_{3,3}$:

$r_1, r_2, r_3, b_1, b_2, b_3$

$r_1, r_2, r_3, b_1, b_i, b_j$

In each partition class at least 3 vertices: CE $\implies$ RS

True for complete bipartite graphs.



Use $K_{3,3}$:

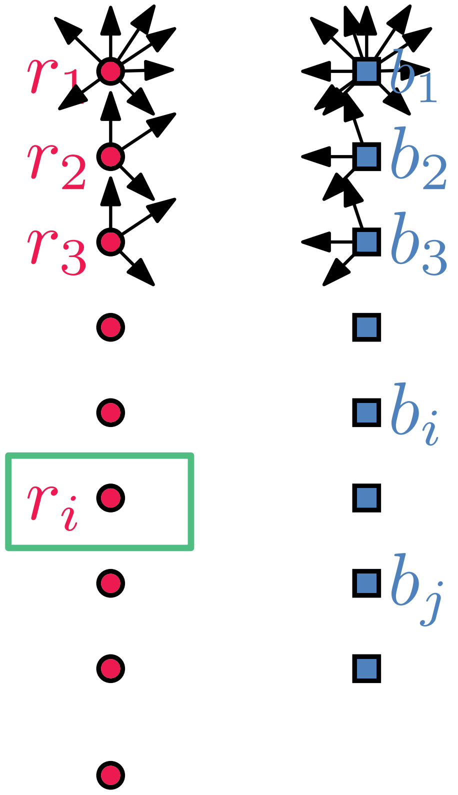
$r_1, r_2, r_3, b_1, b_2, b_3$

$r_1, r_2, r_3, b_1, b_i, b_j$

$r_1, r_i, r_j, b_1, b_2, b_3$

In each partition class at least 3 vertices: CE $\implies$ RS

True for complete bipartite graphs.



Use $K_{3,3}$:

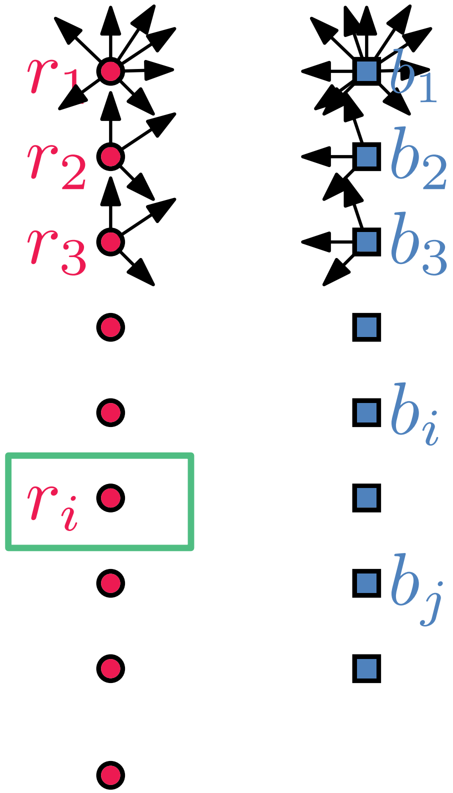
$r_1, r_2, r_3, b_1, b_2, b_3$

$r_1, r_2, r_3, b_1, b_i, b_j$

$r_1, r_i, r_j, b_1, b_2, b_3$

In each partition class at least 3 vertices: CE $\implies$ RS

True for complete bipartite graphs.



Use $K_{3,3}$:

$r_1, r_2, r_3, b_1, b_2, b_3$

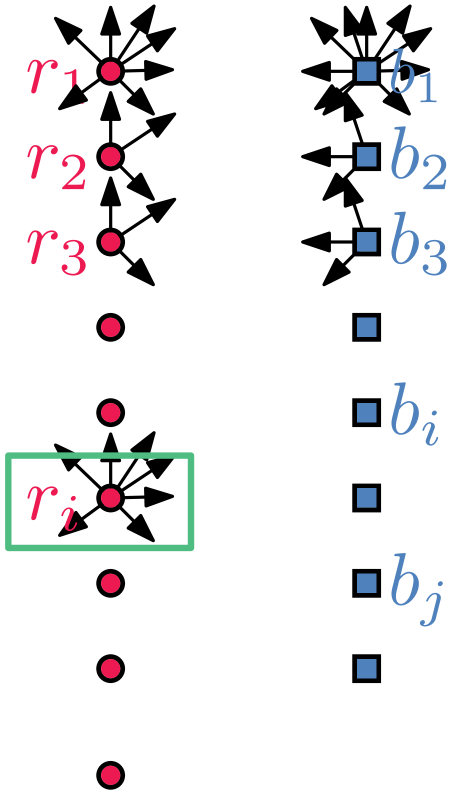
$r_1, r_2, r_3, b_1, b_i, b_j$

$r_1, r_i, r_j, b_1, b_2, b_3$

$r_1, r_2, r_i, b_1, b_k, b_l$

In each partition class at least 3 vertices: CE $\implies$ RS

True for complete bipartite graphs.



Use $K_{3,3}$:

$r_1, r_2, r_3, b_1, b_2, b_3$

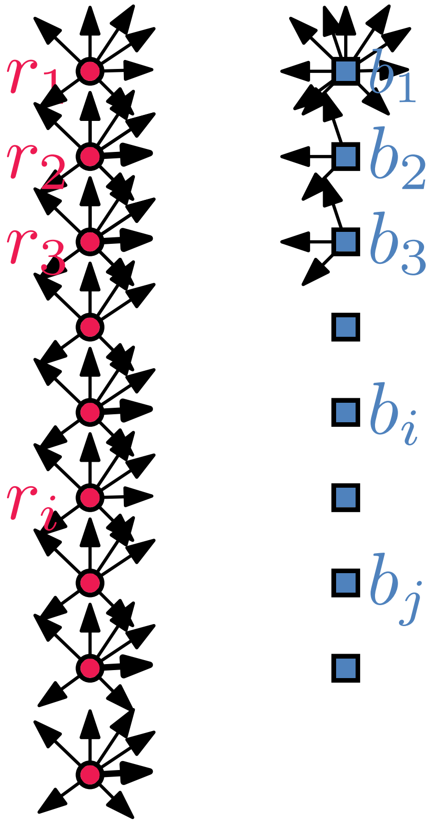
$r_1, r_2, r_3, b_1, b_i, b_j$

$r_1, r_i, r_j, b_1, b_2, b_3$

$r_1, r_2, r_i, b_1, b_k, b_l$

In each partition class at least 3 vertices: CE $\Longrightarrow$ RS

True for complete bipartite graphs.



Use $K_{3,3}$:

$r_1, r_2, r_3, b_1, b_2, b_3$

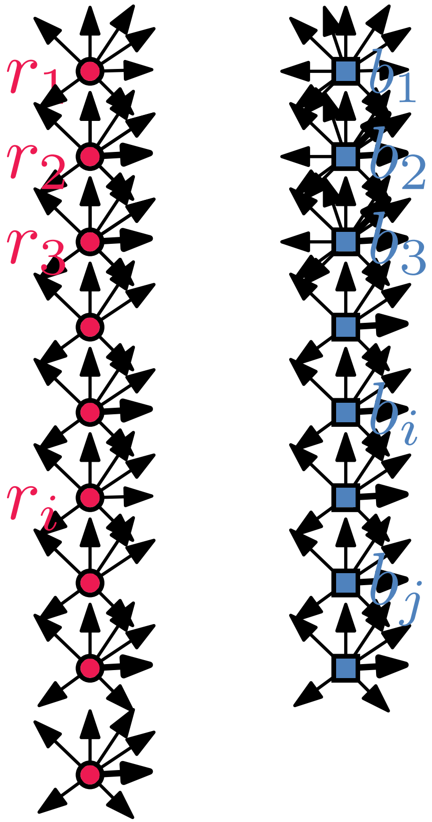
$r_1, r_2, r_3, b_1, b_i, b_j$

$r_1, r_i, r_j, b_1, b_2, b_3$

$r_1, r_2, r_i, b_1, b_k, b_l$

In each partition class at least 3 vertices: CE $\Longrightarrow$ RS

True for complete bipartite graphs.



Use $K_{3,3}$:

$r_1, r_2, r_3, b_1, b_2, b_3$

$r_1, r_2, r_3, b_1, b_i, b_j$

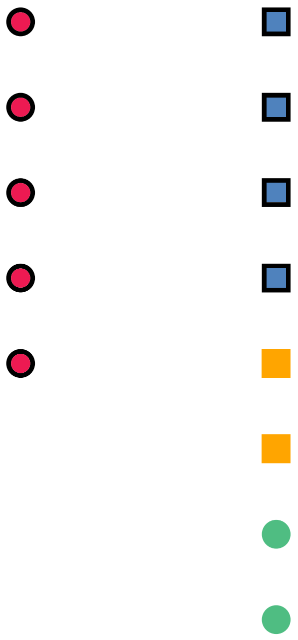
$r_1, r_i, r_j, b_1, b_2, b_3$

$r_1, r_2, r_i, b_1, b_k, b_l$

$r_1, r_i, r_j, b_1, b_2, b_l$

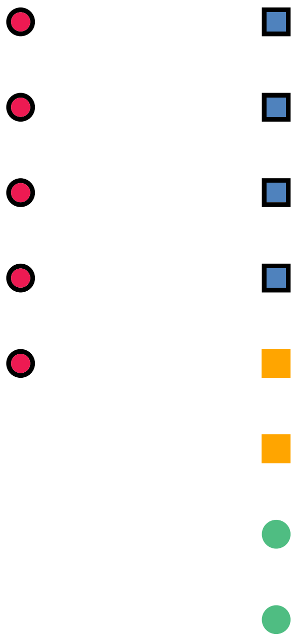
In each partition class at least 3 vertices: CE $\implies$ RS

True for complete multipartite graphs.



In each partition class at least 3 vertices: CE $\implies$ RS

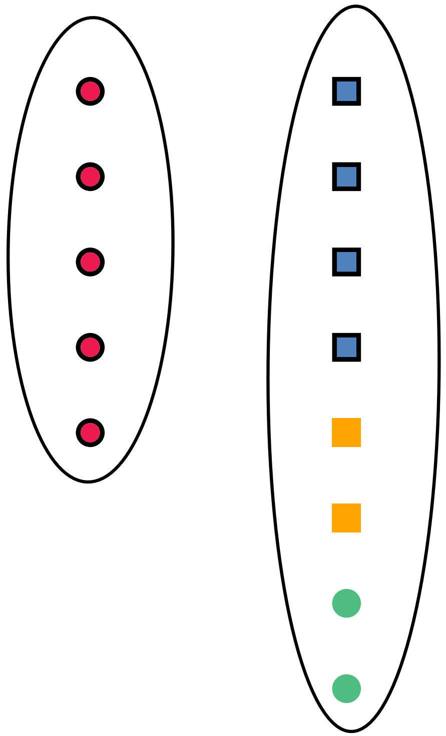
True for complete multipartite graphs.



Use bipartite graphs.

In each partition class at least 3 vertices: CE $\implies$ RS

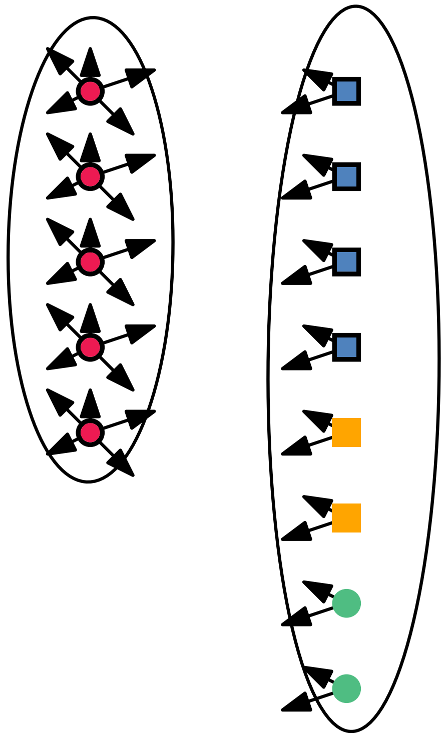
True for complete multipartite graphs.



Use bipartite graphs.

In each partition class at least 3 vertices: CE $\Longrightarrow$ RS

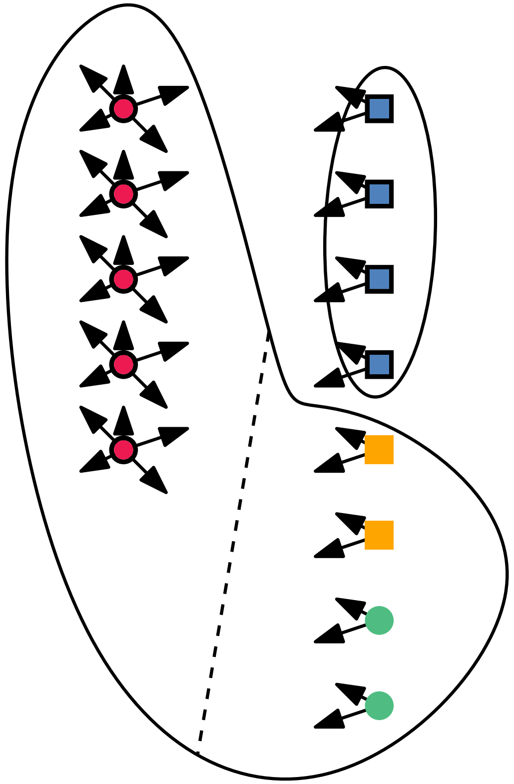
True for complete multipartite graphs.



Use bipartite graphs.

In each partition class at least 3 vertices: CE $\implies$ RS

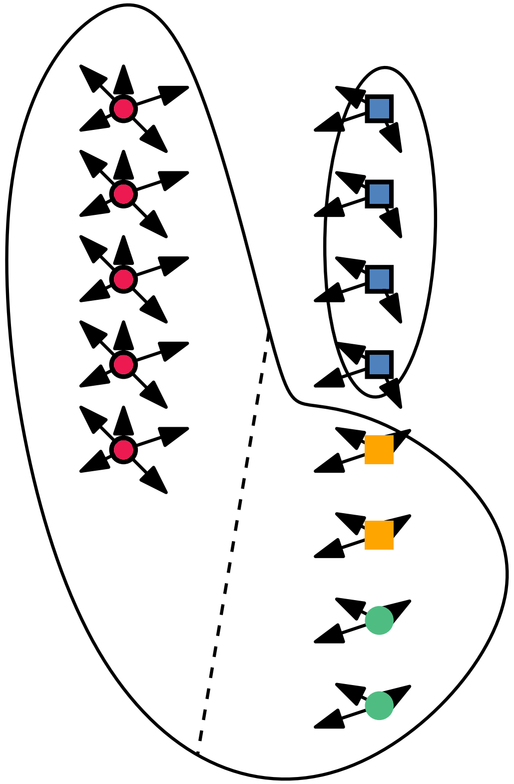
True for complete multipartite graphs.



Use bipartite graphs.

In each partition class at least 3 vertices: CE $\implies$ RS

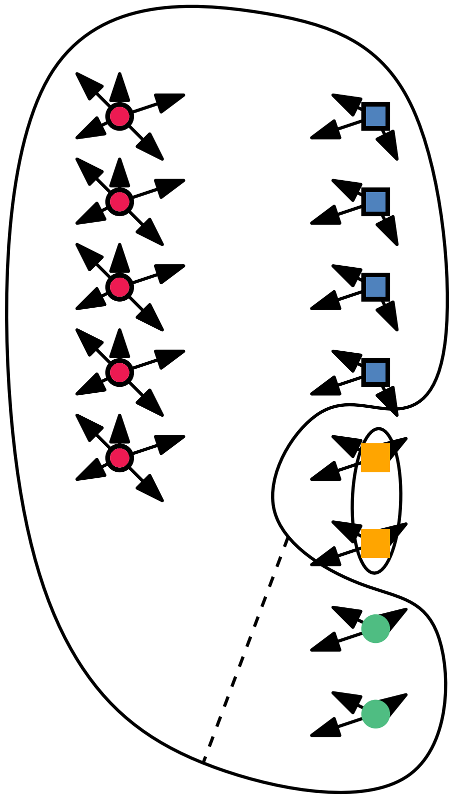
True for complete multipartite graphs.



Use bipartite graphs.

In each partition class at least 3 vertices: CE $\implies$ RS

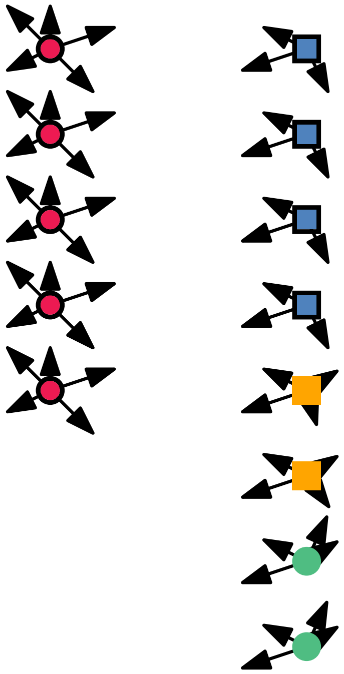
True for complete multipartite graphs.



Use bipartite graphs.

In each partition class at least 3 vertices: CE $\implies$ RS

True for complete multipartite graphs.



Use bipartite graphs.

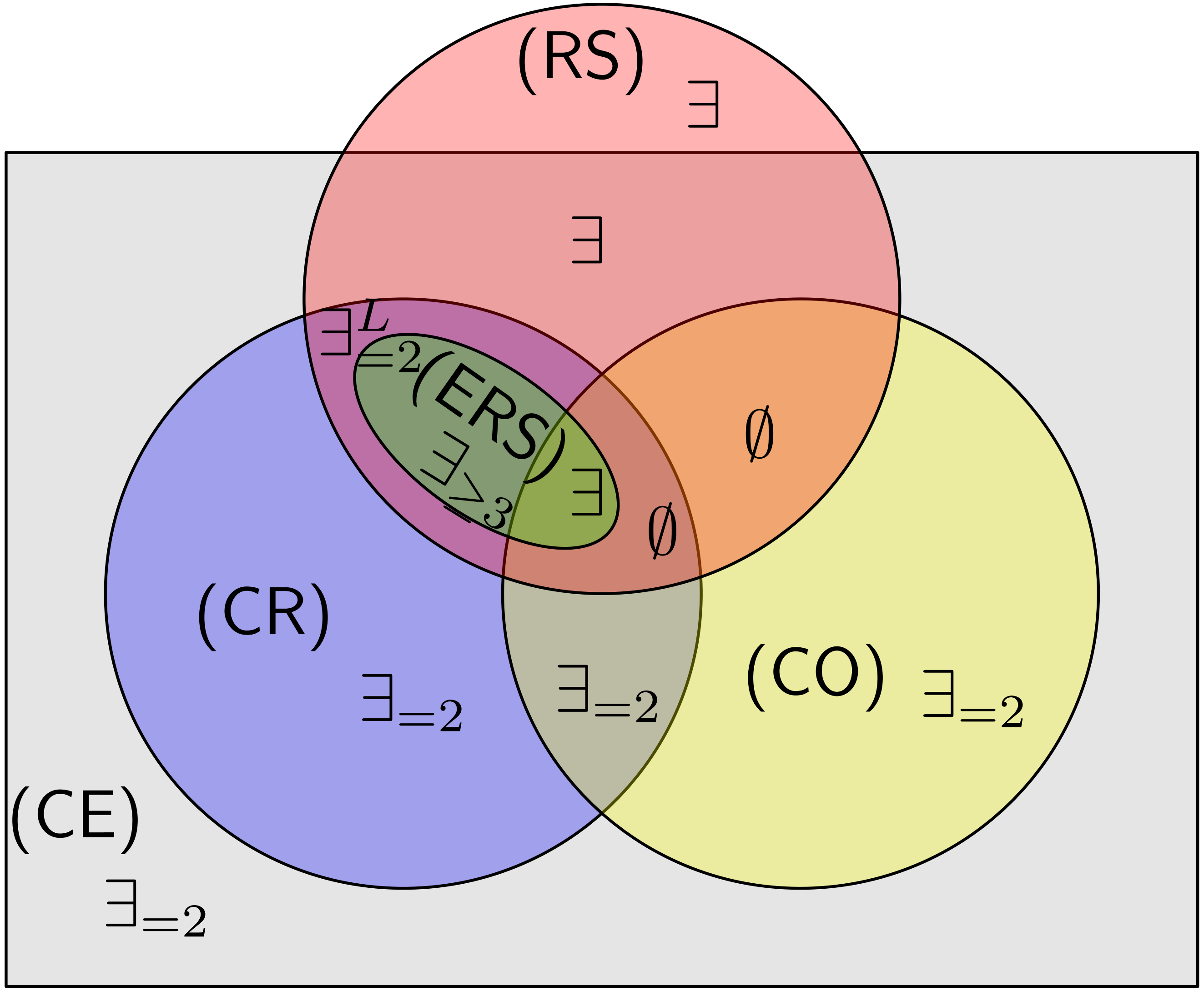
Conclusion (Which isomorphisms are relevant?)

For complete multipartite graphs:

RS + CO => strong iso.

If each partition class has ≥ 3 vertices:
CE => RS
CR => ERS
CO => strong iso.

For $K_{2,n}$: ERS => strong iso.[1]



1) [O. Aichholzer, M.K. Chiu, H. Hoang, M. Hoffmann, J. Kynčl, Y. Maus, B. Vogtenhuber, A.W. 2023]

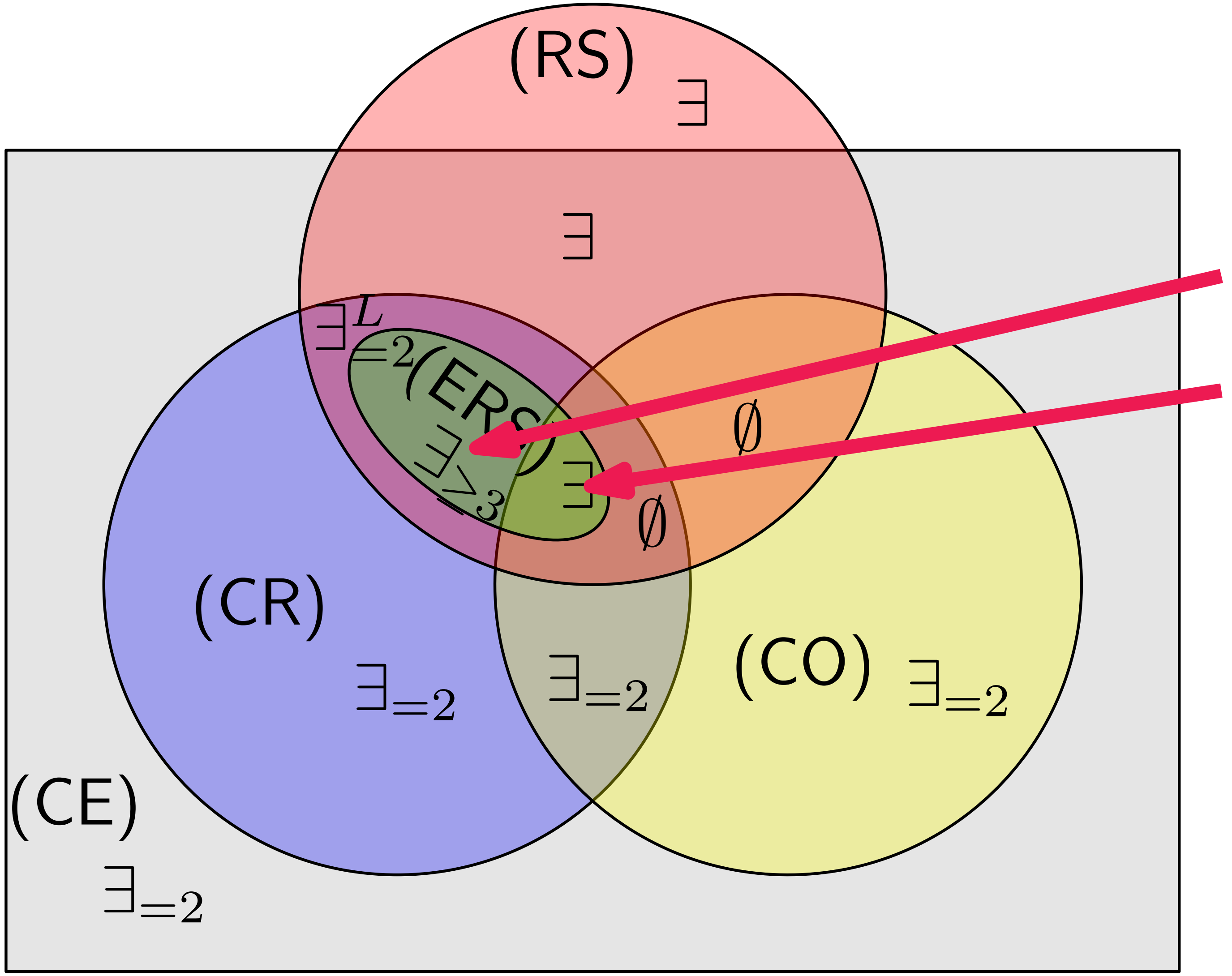
Conclusion (Which isomorphisms are relevant?)

For complete multipartite graphs:

RS + CO => strong iso.

If each partition class has ≥ 3 vertices:
CE => RS
CR => ERS
CO => strong iso.

For $K_{2,n}$: ERS => strong iso.[1]



1) [O. Aichholzer, M.K. Chiu, H. Hoang, M. Hoffmann, J. Kynčl, Y. Maus, B. Vogtenhuber, A.W. 2023]

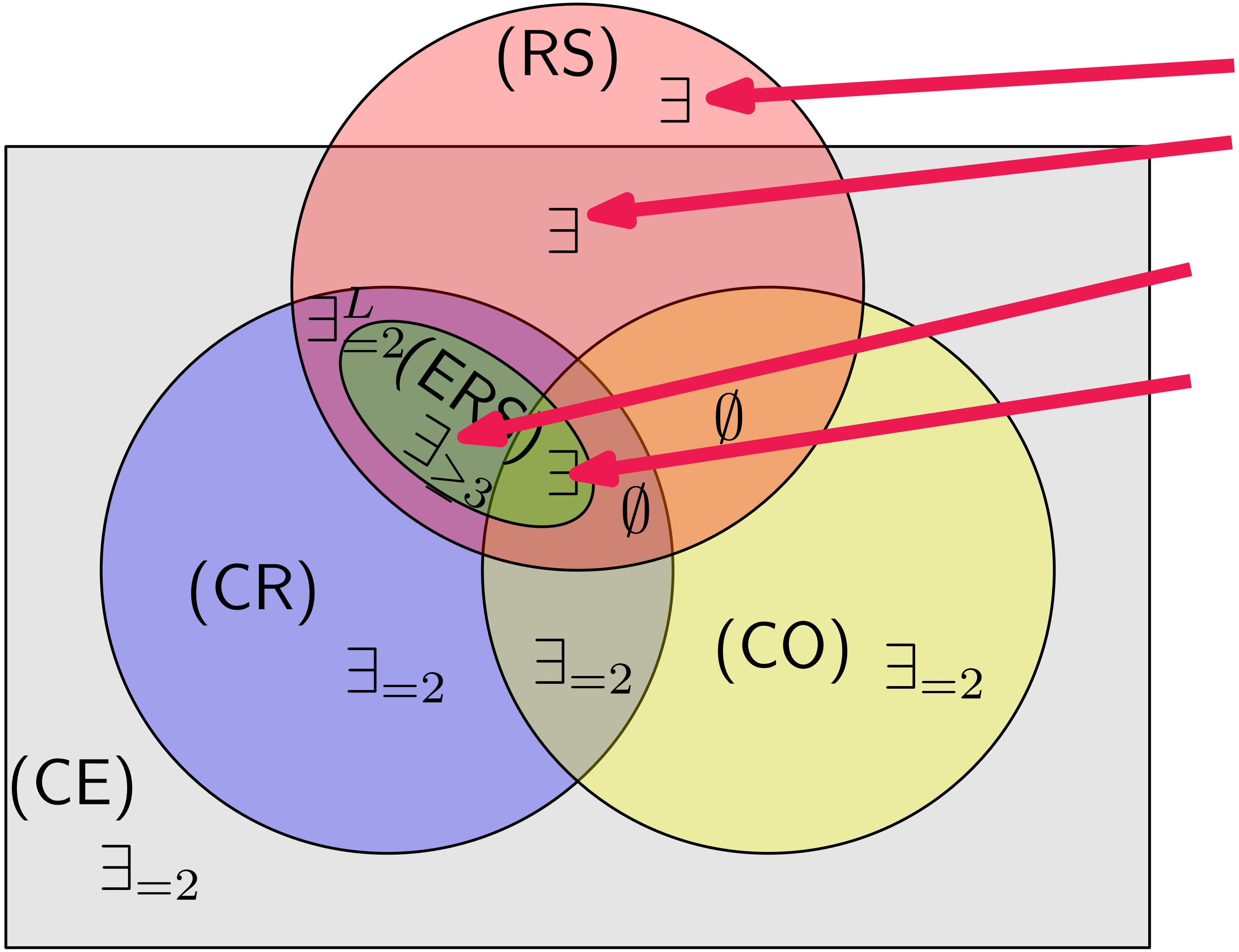
Conclusion (Which isomorphisms are relevant?)

For complete multipartite graphs:

RS + CO => strong iso.

If each partition class has ≥ 3 vertices:
CE => RS
CR => ERS
CO => strong iso.

For $K_{2,n}$: ERS => strong iso.[1]



1) [O. Aichholzer, M.K. Chiu, H. Hoang, M. Hoffmann, J. Kynčl, Y. Maus, B. Vogtenhuber, A.W. 2023]

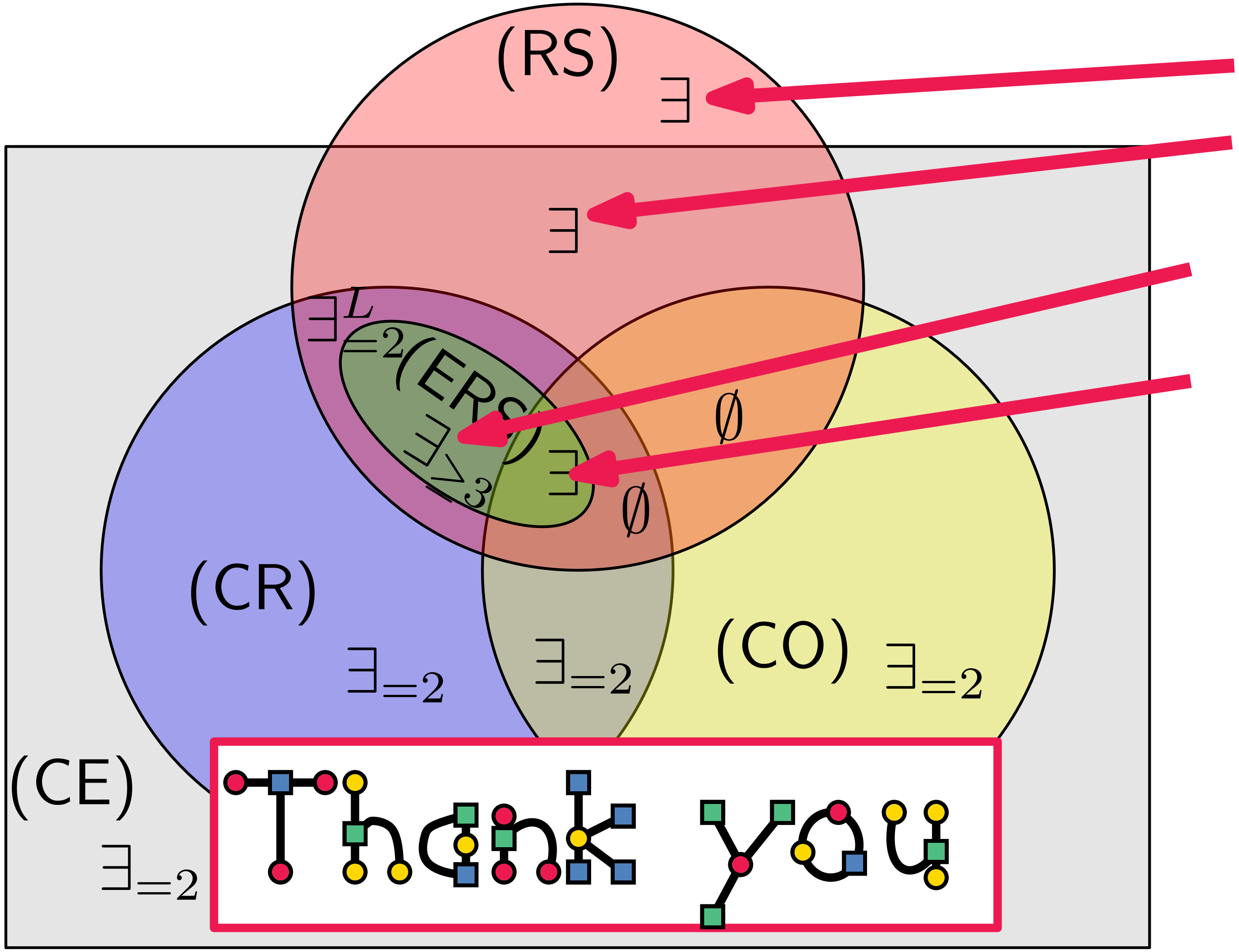
Conclusion (Which isomorphisms are relevant?)

For complete multipartite graphs:

RS + CO => strong iso.

If each partition class has ≥ 3 vertices:
CE => RS
CR => ERS
CO => strong iso.

For $K_{2,n}$: ERS => strong iso.[1]

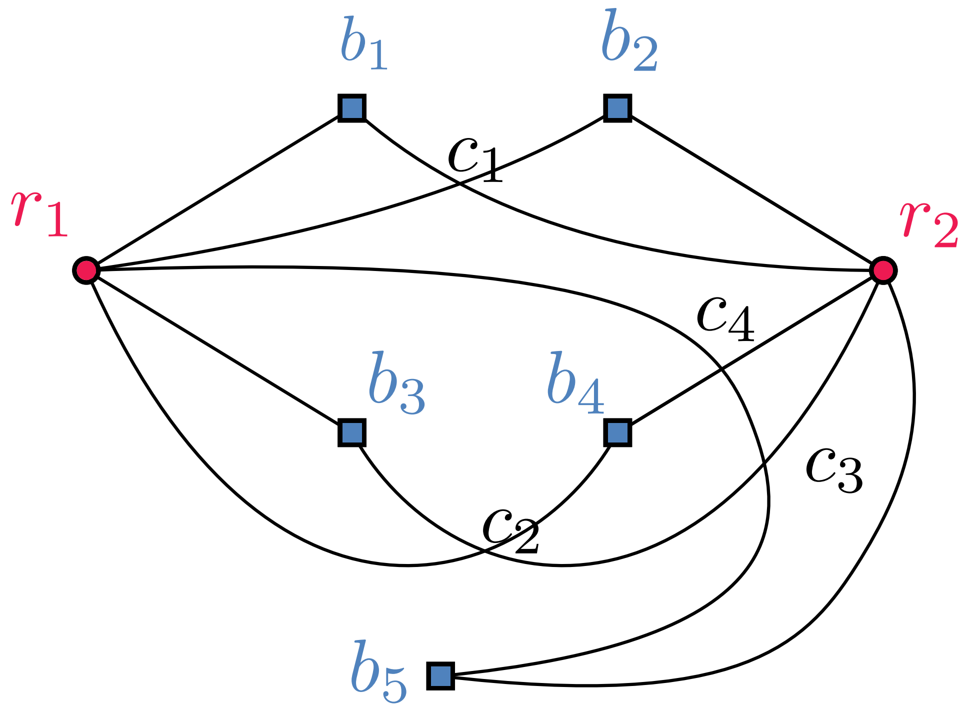


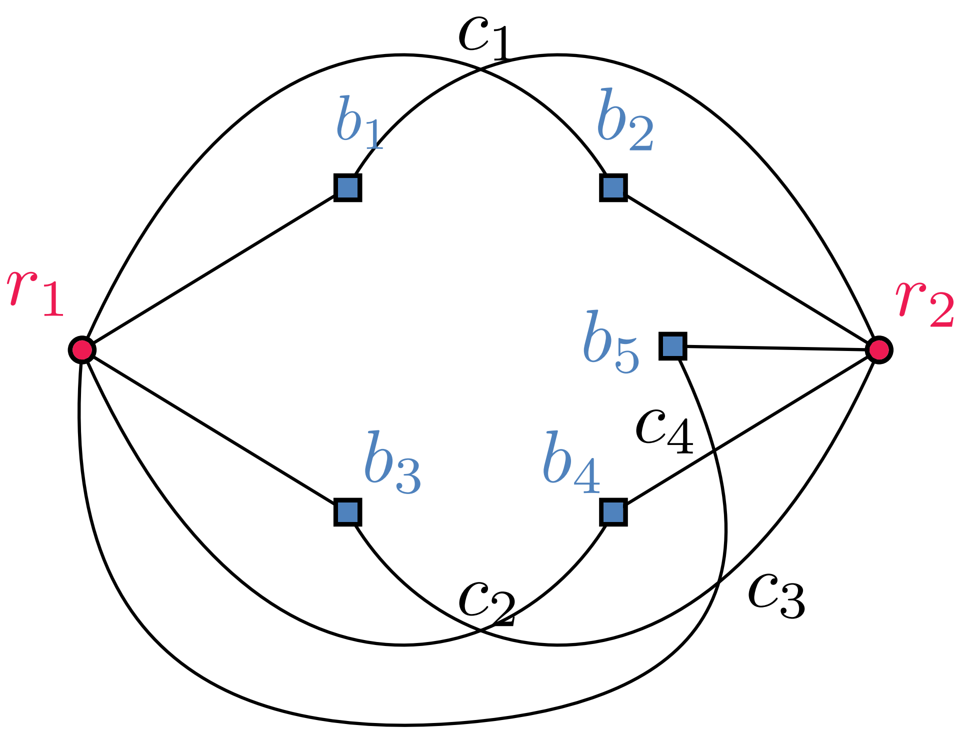
1) [O. Aichholzer, M.K. Chiu, H. Hoang, M. Hoffmann, J. Kynčl, Y. Maus, B. Vogtenhuber, A.W. 2023]

Combinations that do not imply others

Combinations that do not imply others

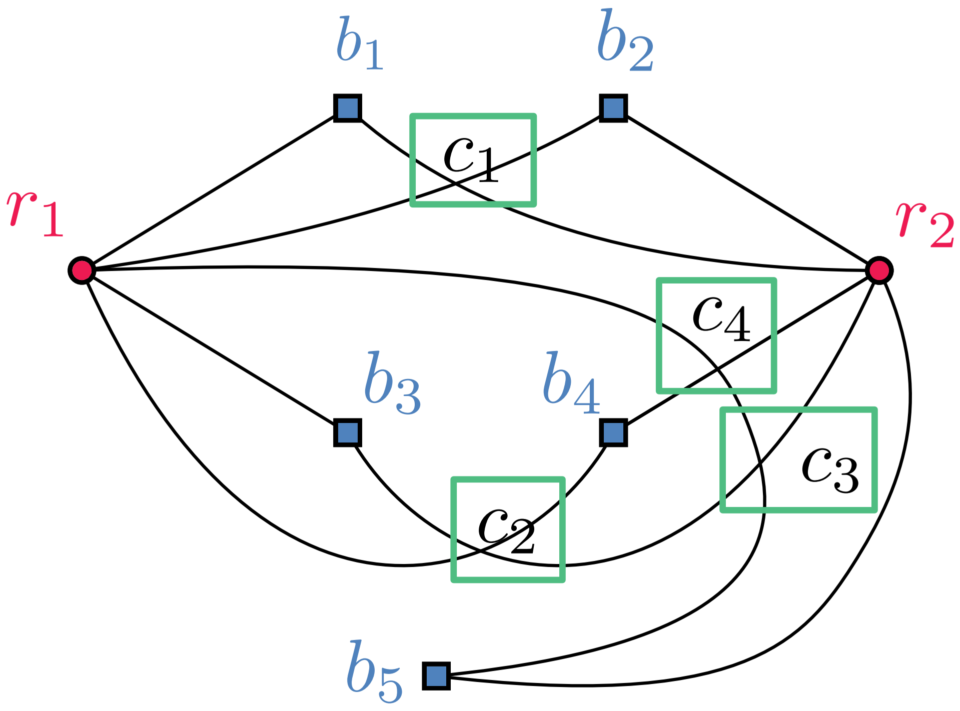
CE-iso., not RS-iso., not CO-iso. not CR-iso. – Only possible for $K_{2,n}$

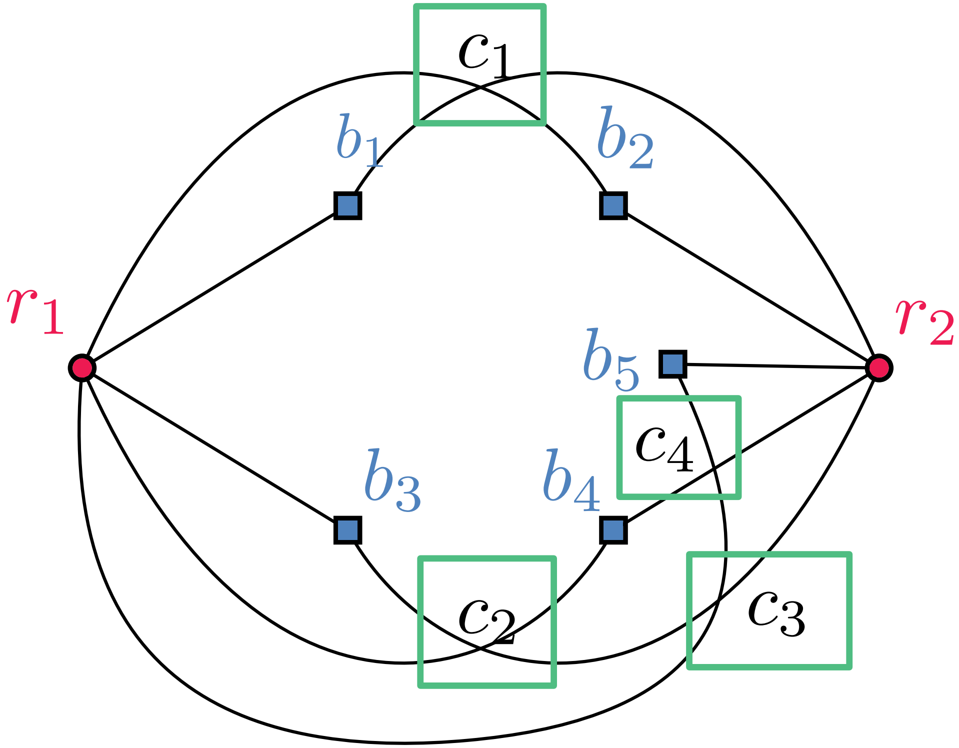




Combinations that do not imply others

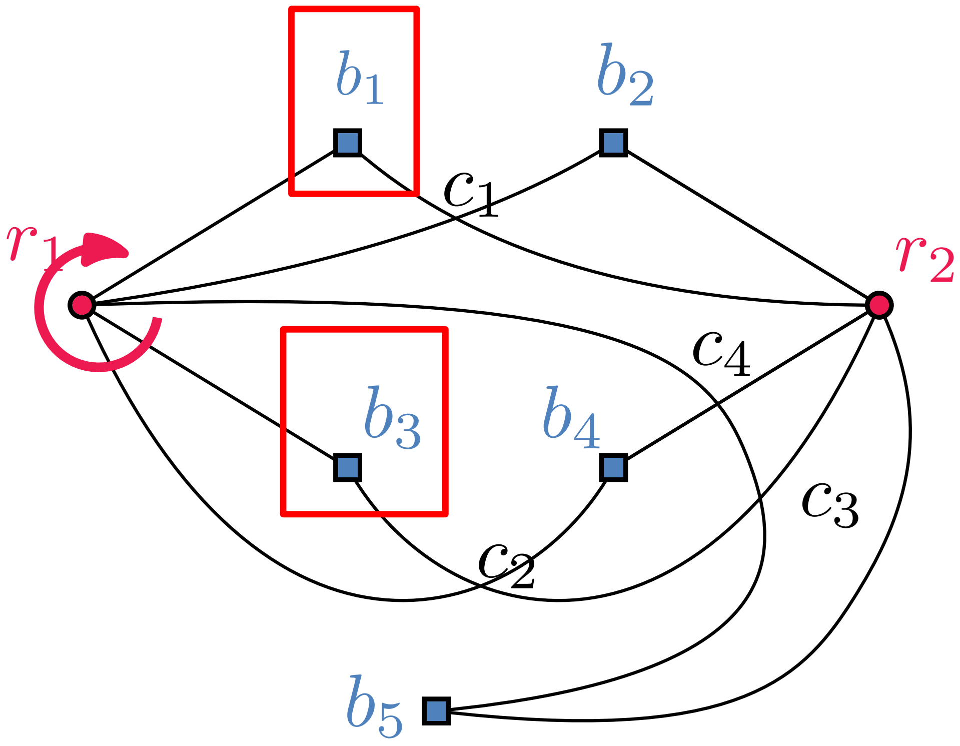
CE-iso., not RS-iso., not CO-iso. not CR-iso. – Only possible for $K_{2,n}$

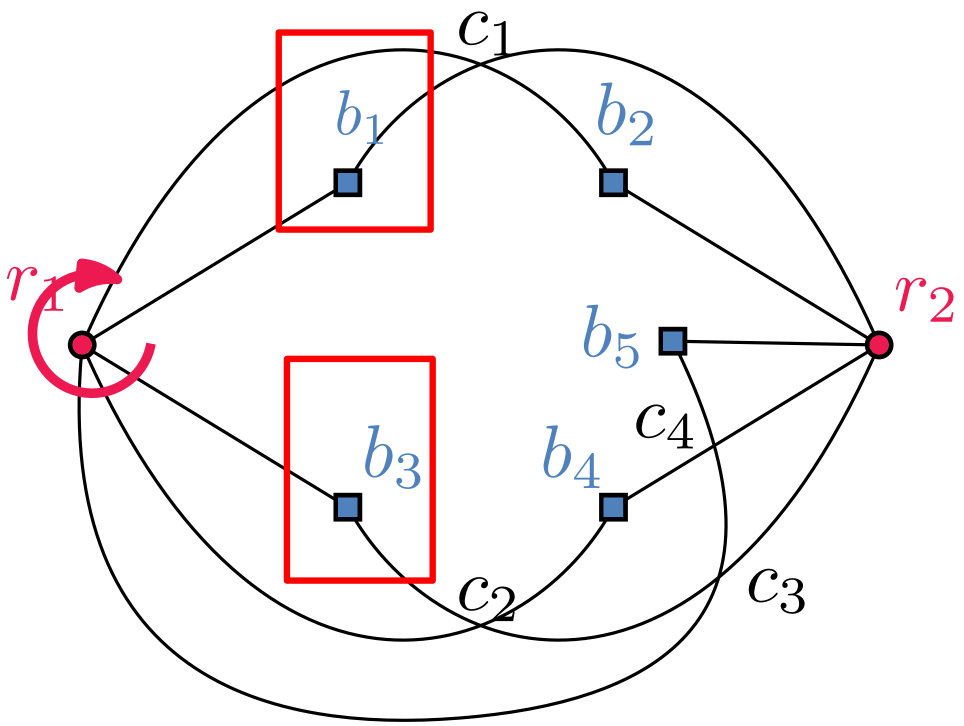




Combinations that do not imply others

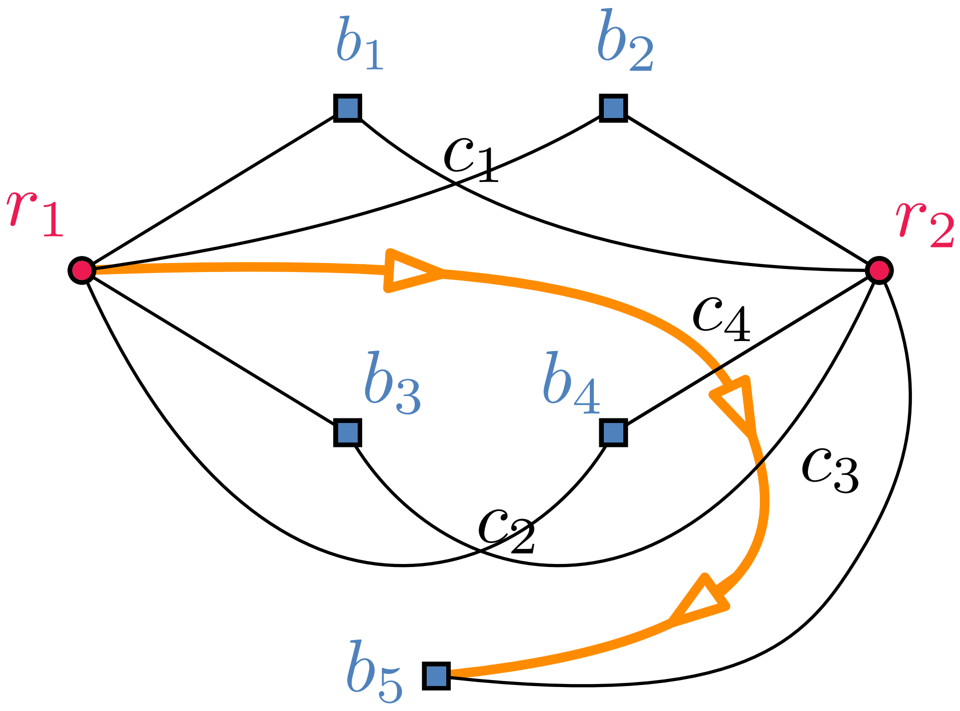
CE-iso., not RS-iso., not CO-iso. not CR-iso. – Only possible for $K_{2,n}$

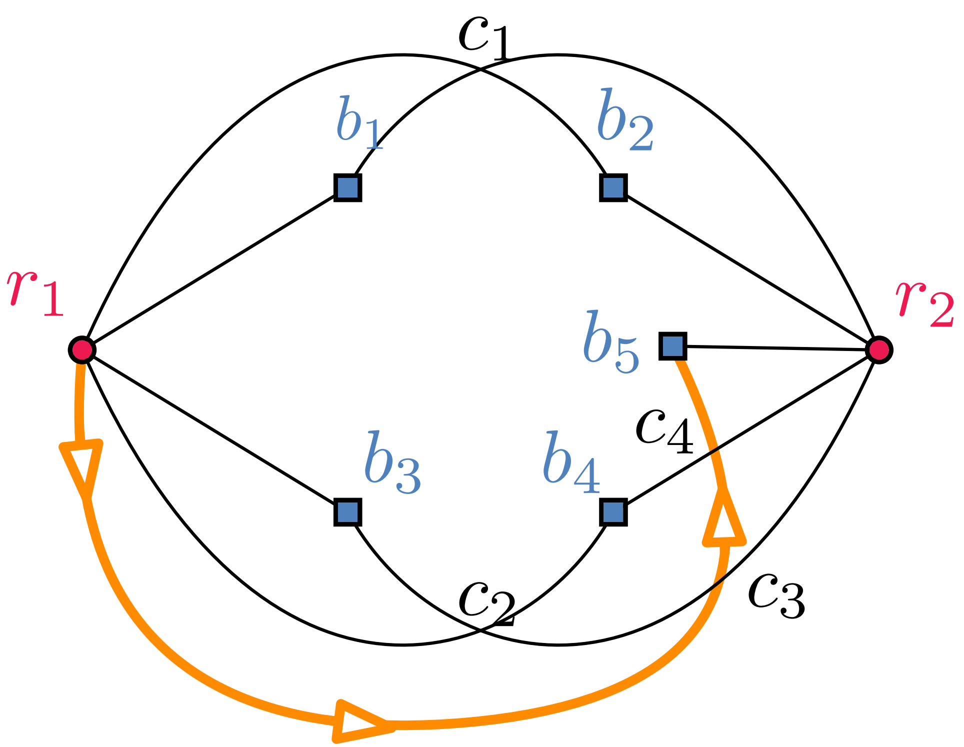




Combinations that do not imply others

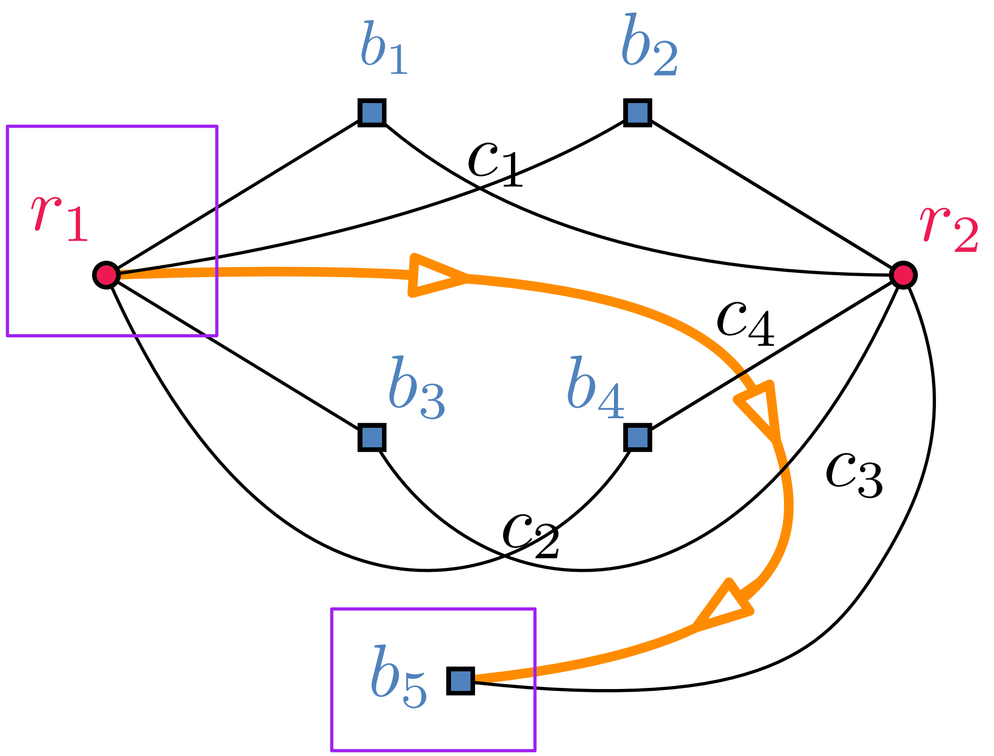
CE-iso., not RS-iso., not CO-iso. not CR-iso. – Only possible for $K_{2,n}$

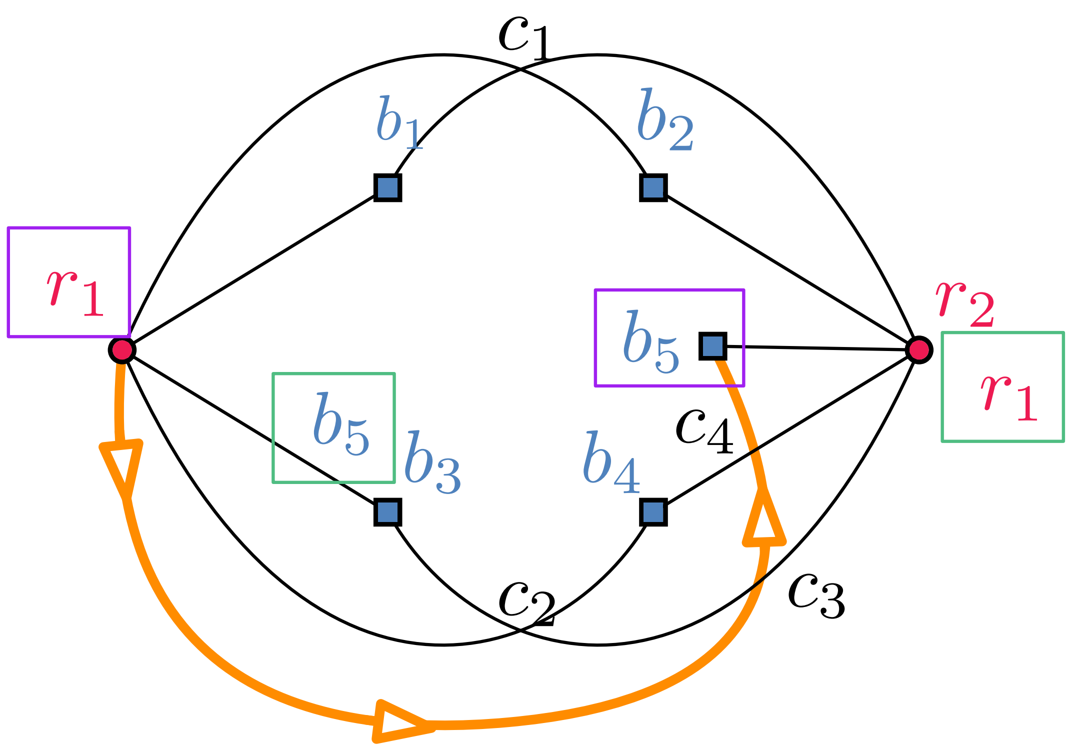




Combinations that do not imply others

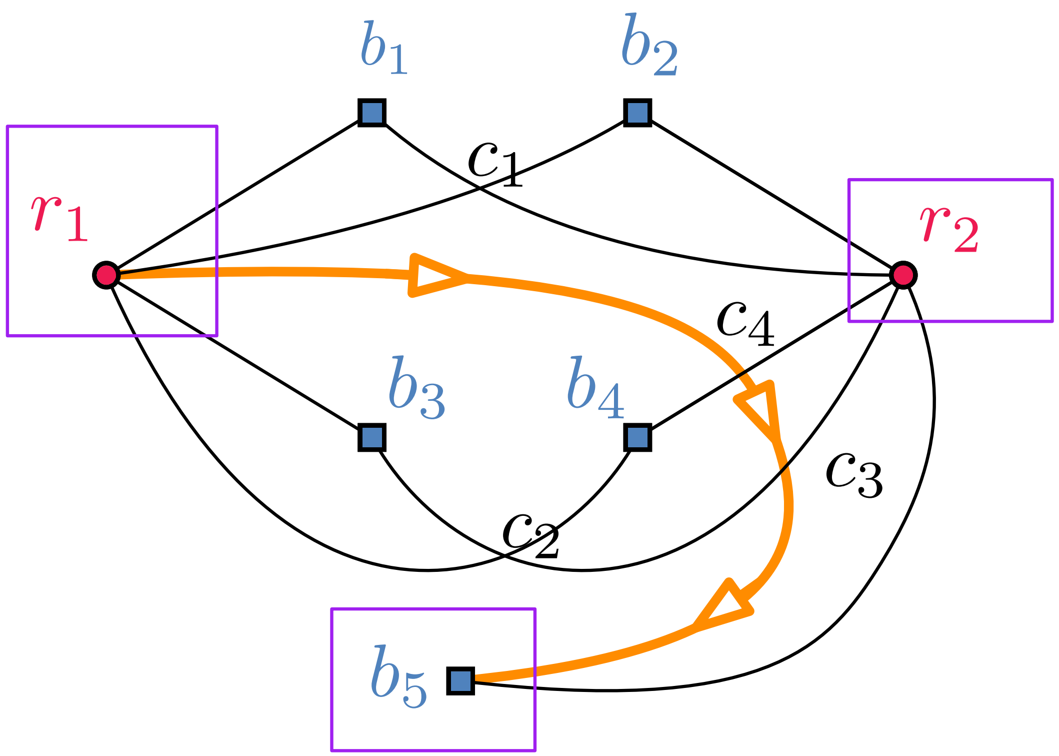
CE-iso., not RS-iso., not CO-iso. not CR-iso. – Only possible for $K_{2,n}$

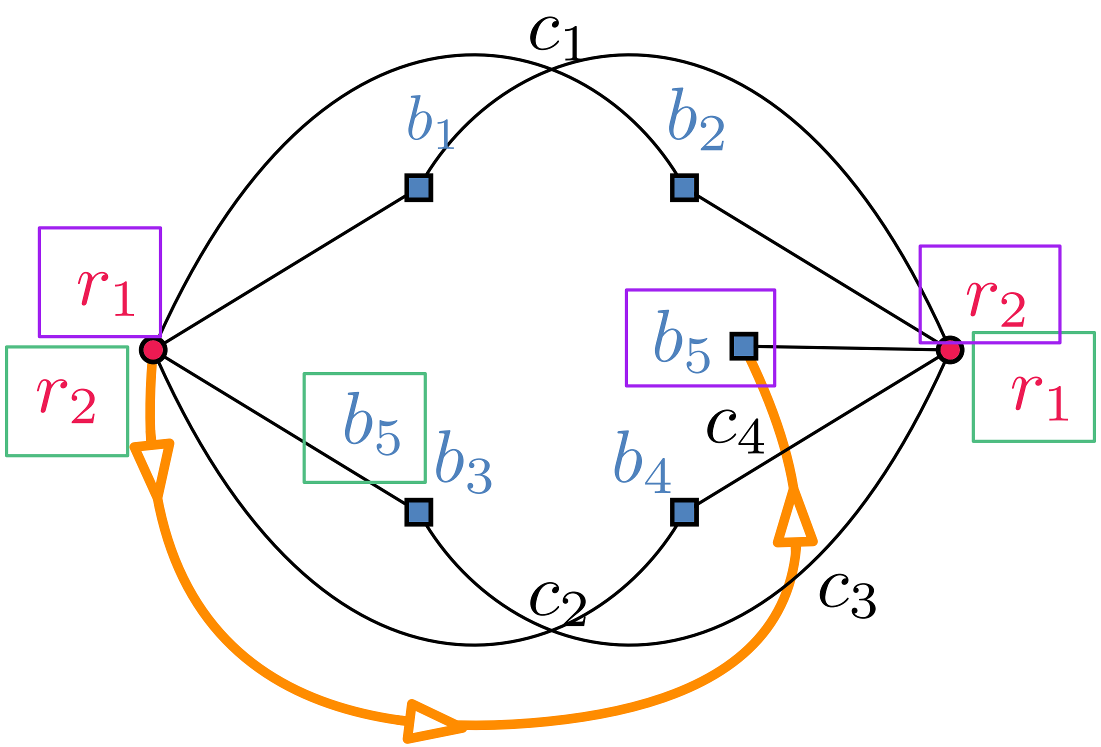




Combinations that do not imply others

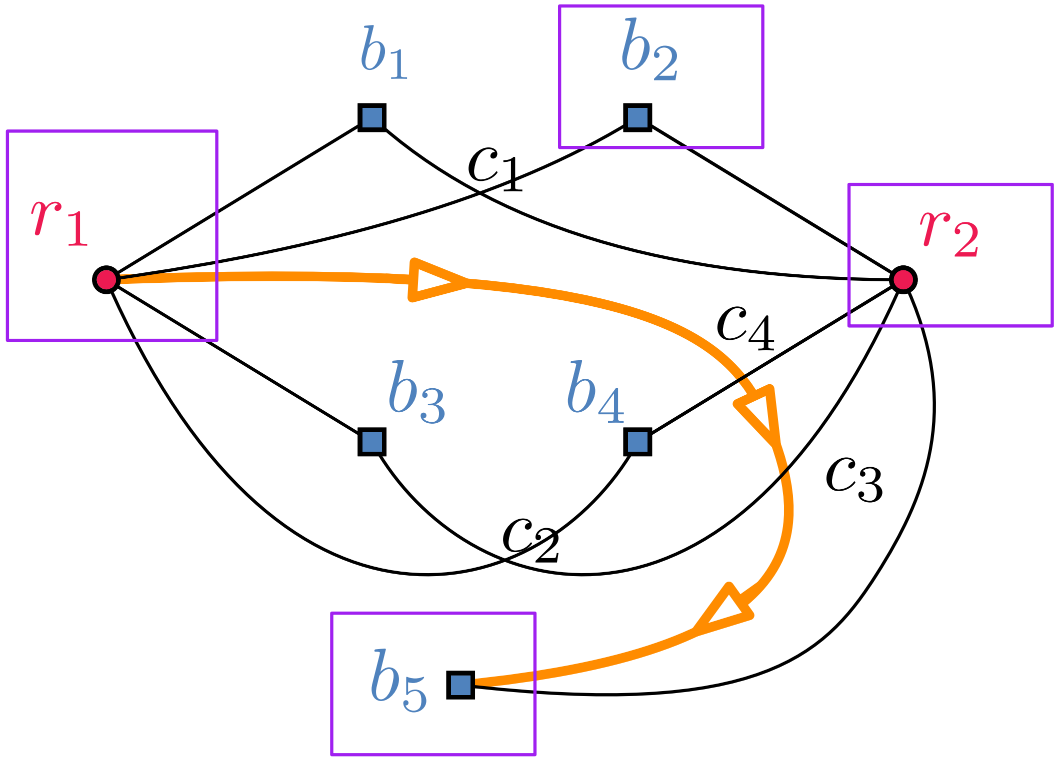
CE-iso., not RS-iso., not CO-iso. not CR-iso. – Only possible for $K_{2,n}$

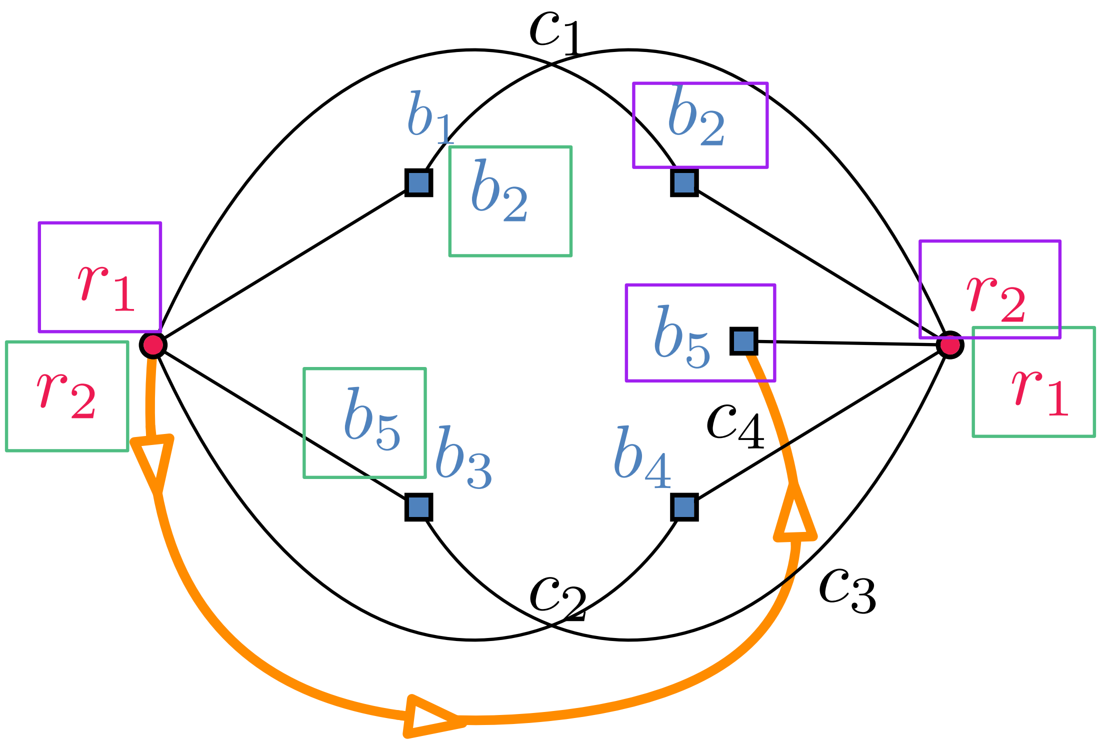




Combinations that do not imply others

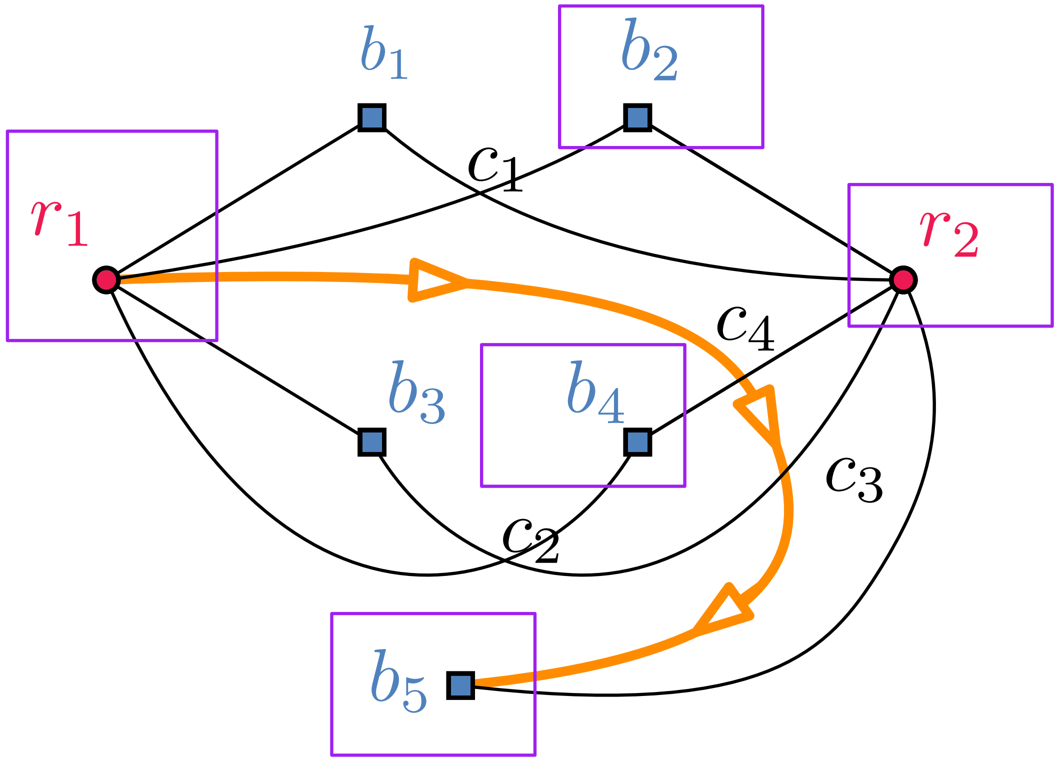
CE-iso., not RS-iso., not CO-iso. not CR-iso. – Only possible for $K_{2,n}$

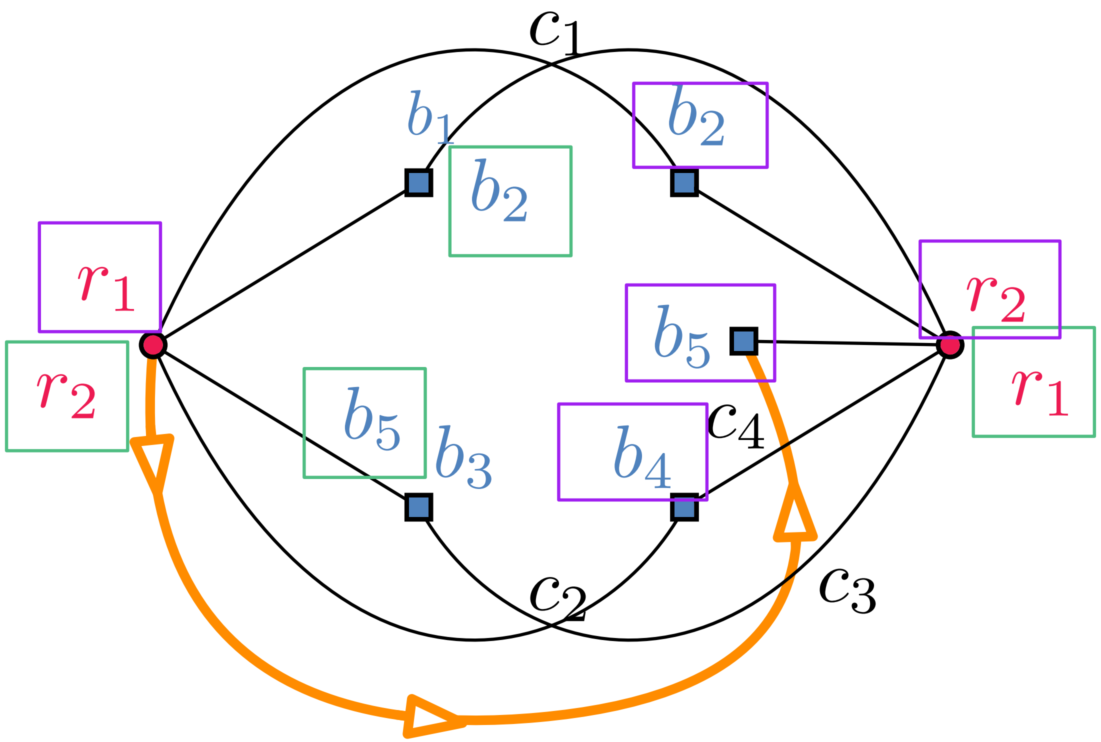




Combinations that do not imply others

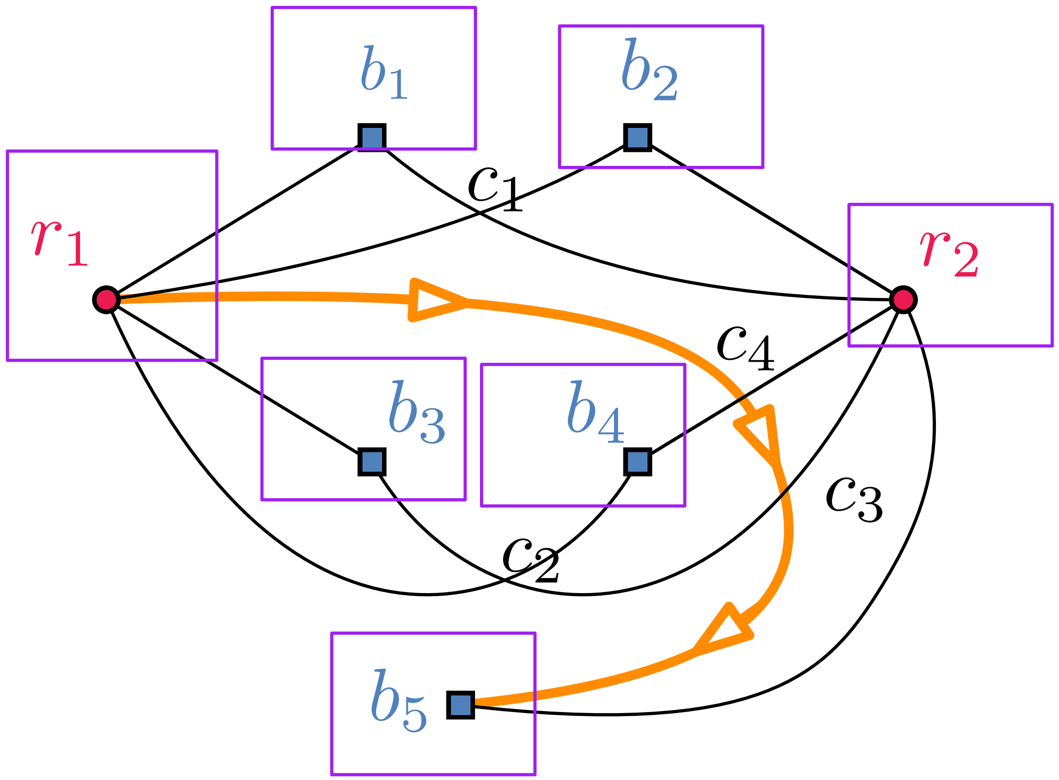
CE-iso., not RS-iso., not CO-iso. not CR-iso. – Only possible for $K_{2,n}$

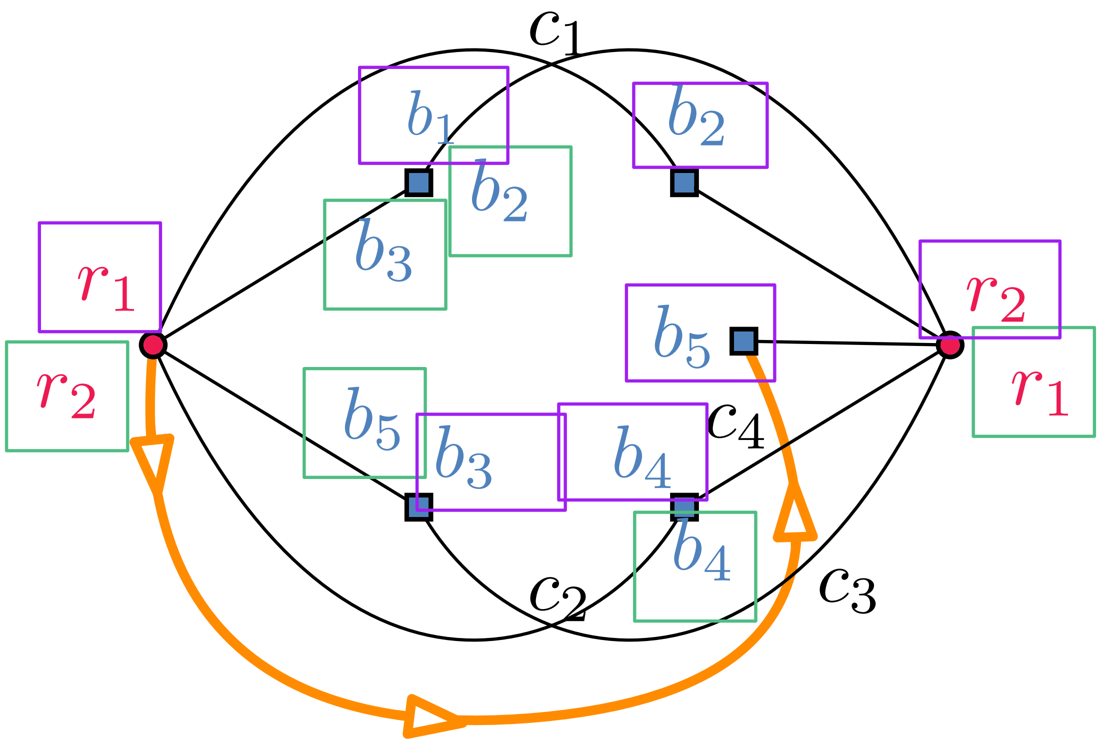




Combinations that do not imply others

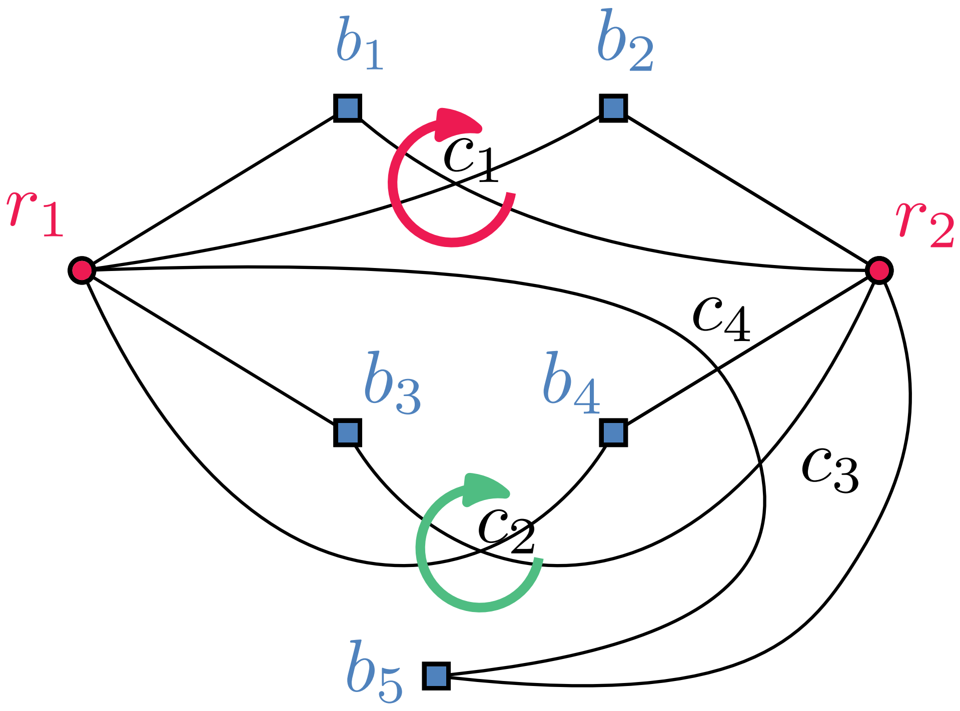
CE-iso., not RS-iso., not CO-iso. not CR-iso. – Only possible for $K_{2,n}$

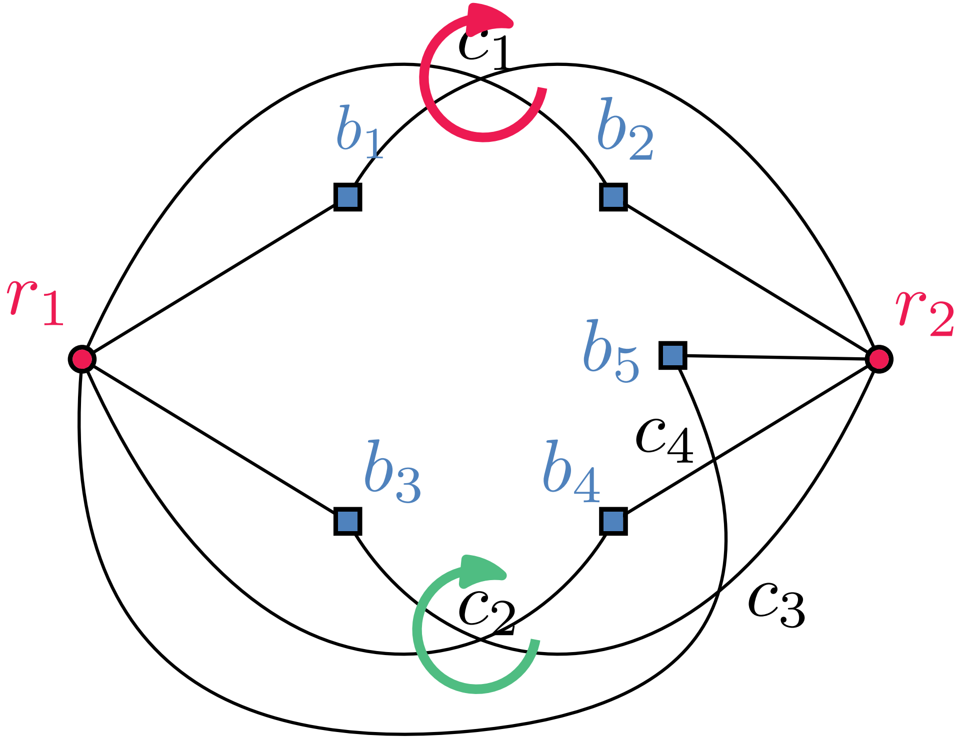




Combinations that do not imply others

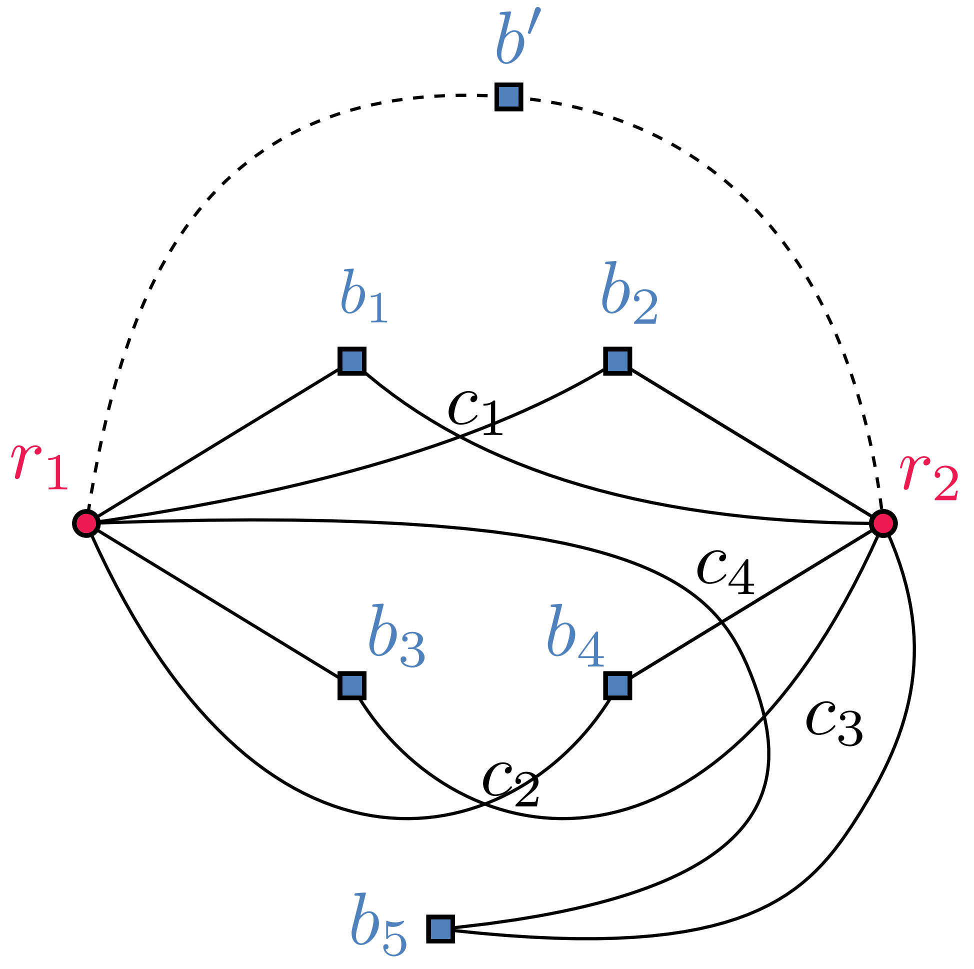
CE-iso., not RS-iso., not CO-iso. not CR-iso. – Only possible for $K_{2,n}$

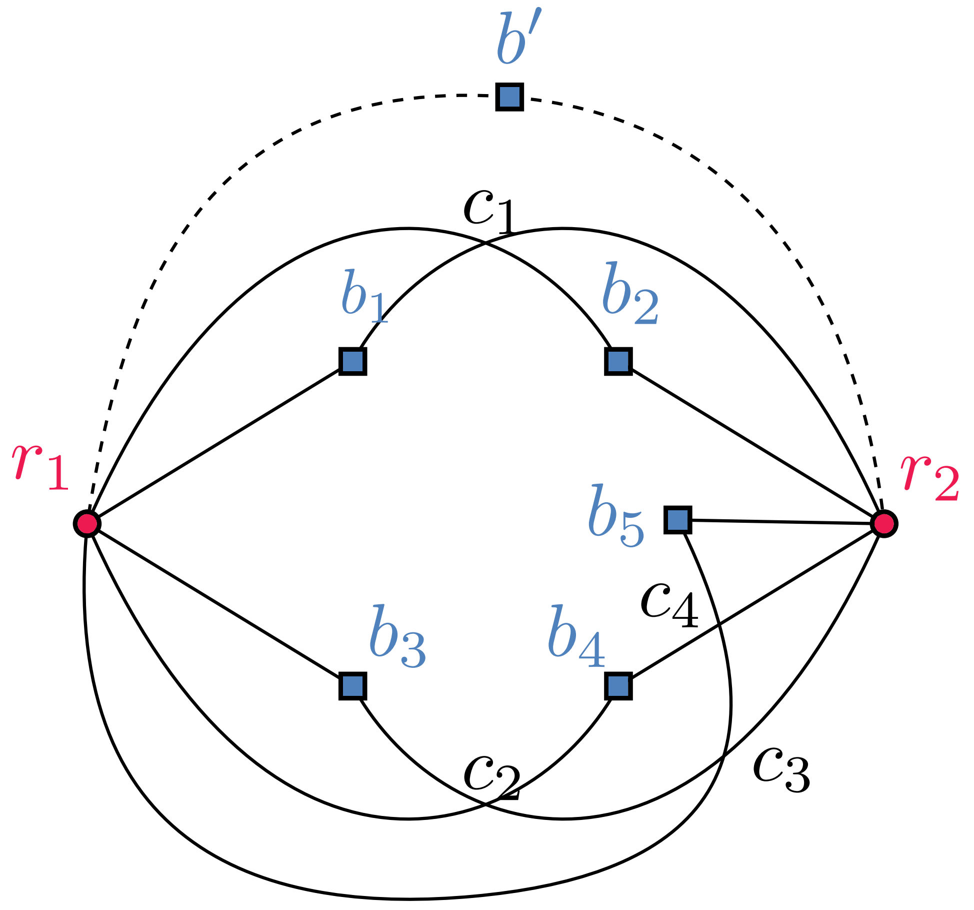




Combinations that do not imply others

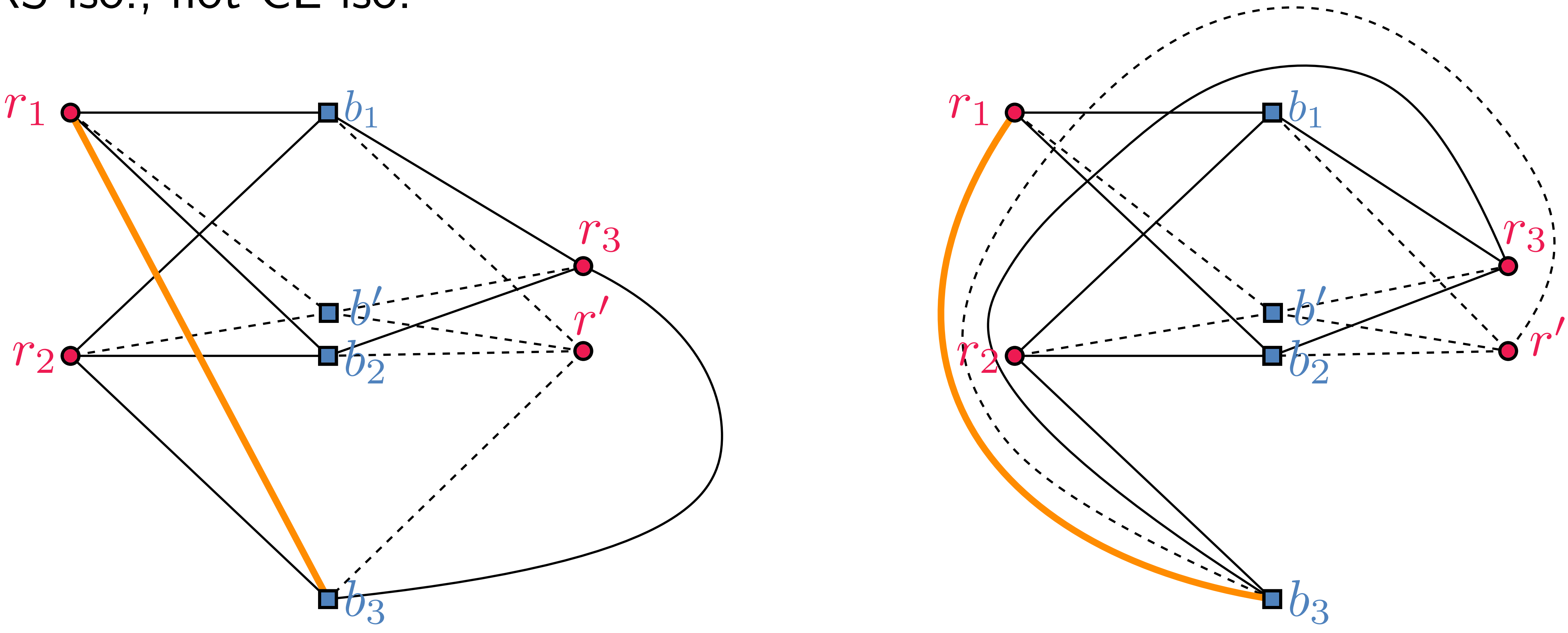
CE-iso., not RS-iso., not CO-iso. not CR-iso. – Only possible for $K_{2,n}$





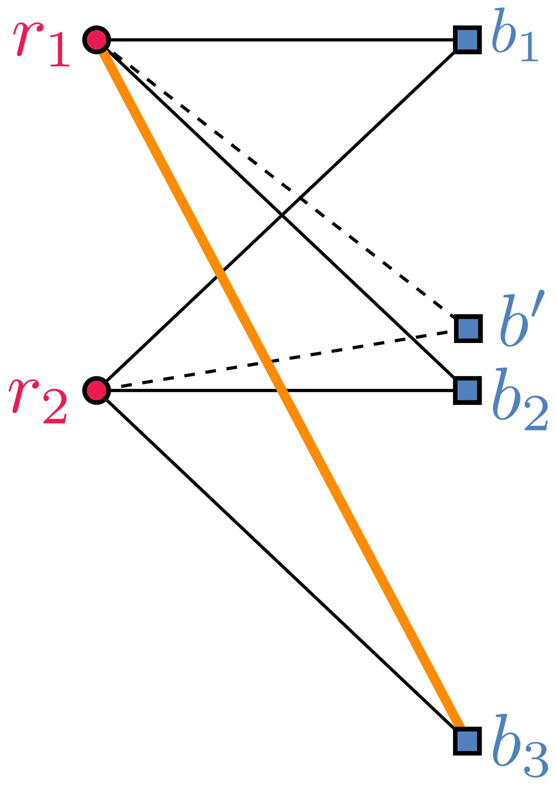
Combinations that do not imply others

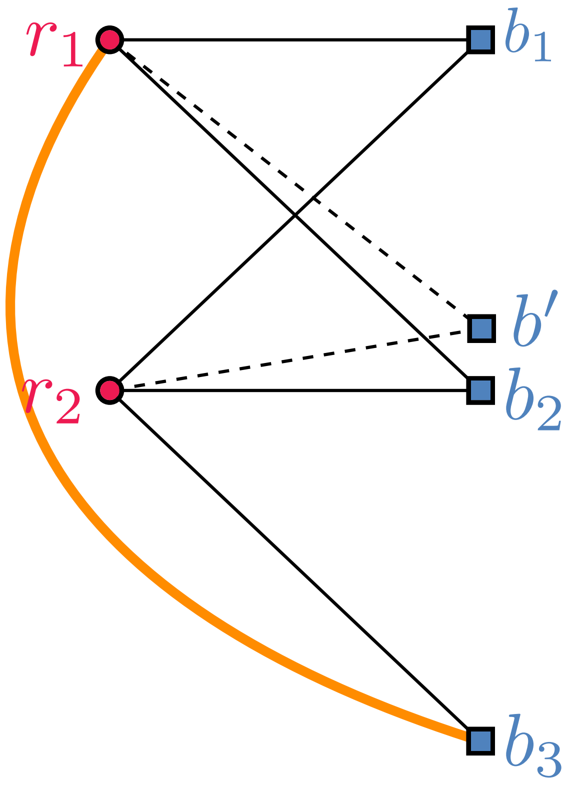
RS-iso., not CE-iso.



Combinations that do not imply others

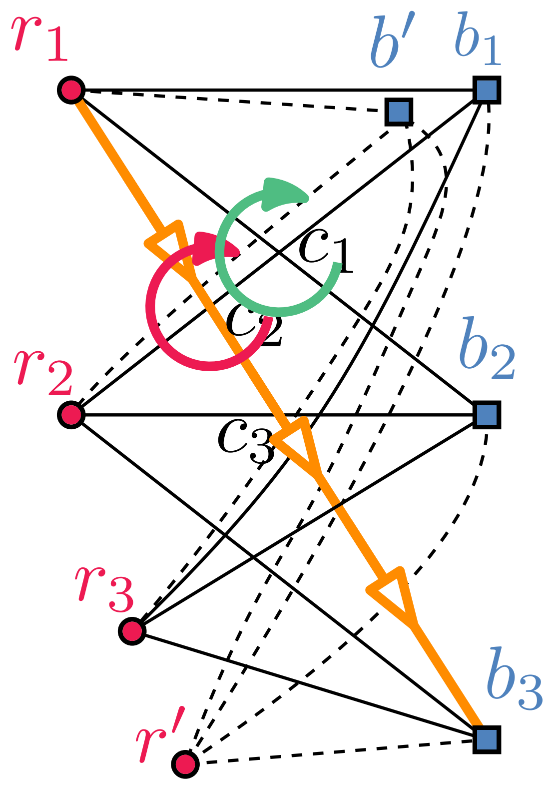
RS-iso., not CE-iso.

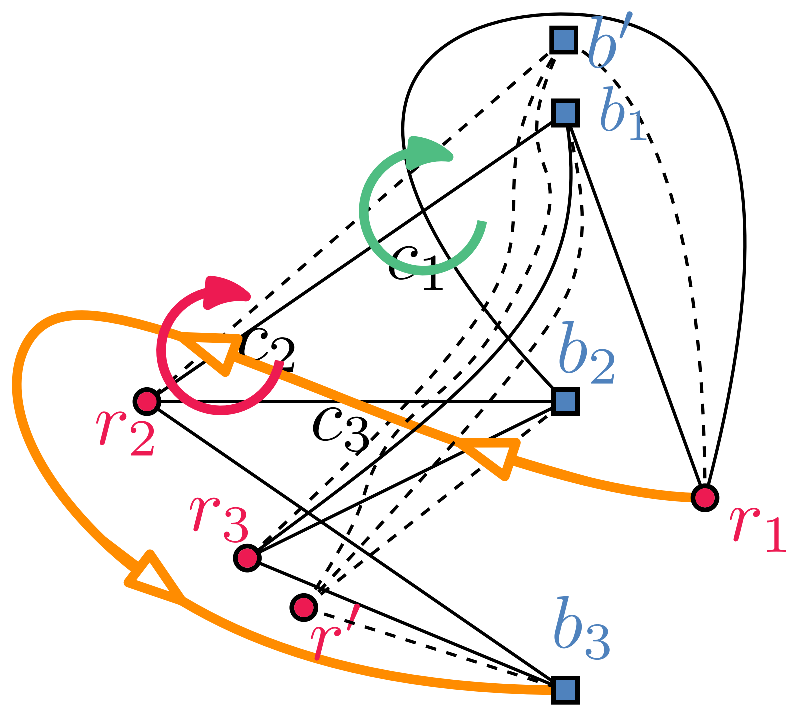




Combinations that do not imply others

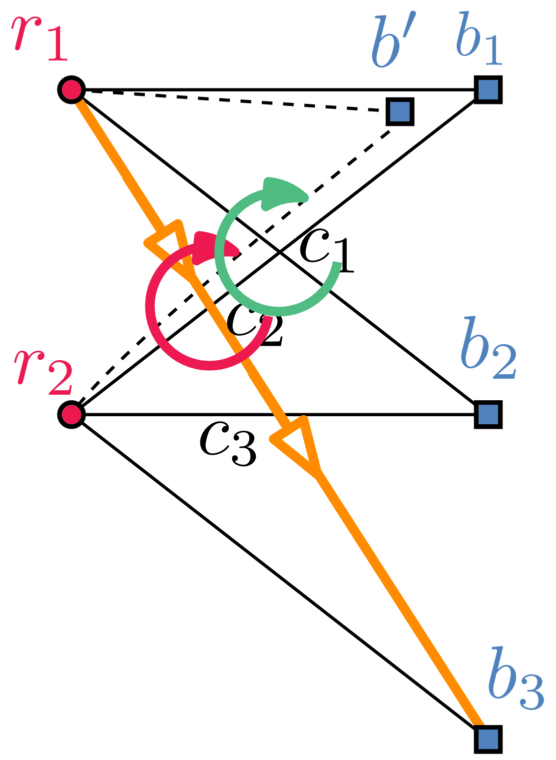
CE-iso. + RS-iso., not CR-iso. (not CO-iso.)

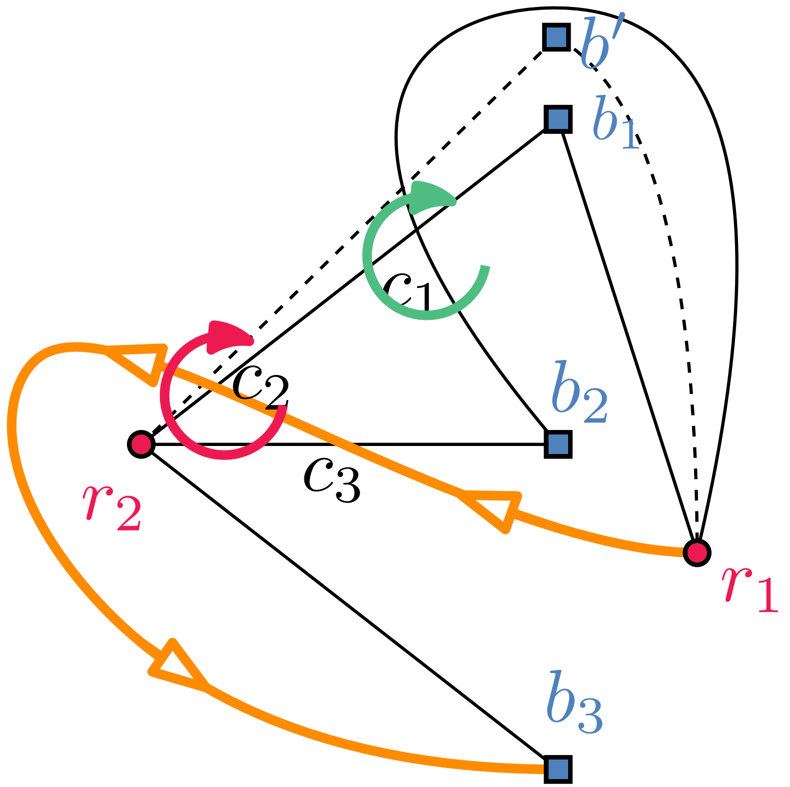




Combinations that do not imply others

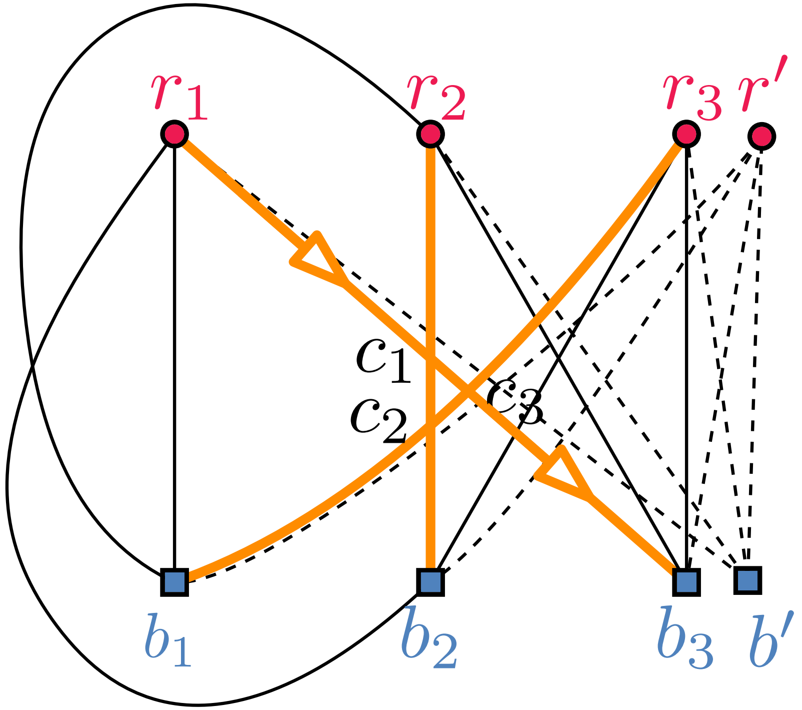
CE-iso. + RS-iso., not CR-iso. (not CO-iso.)

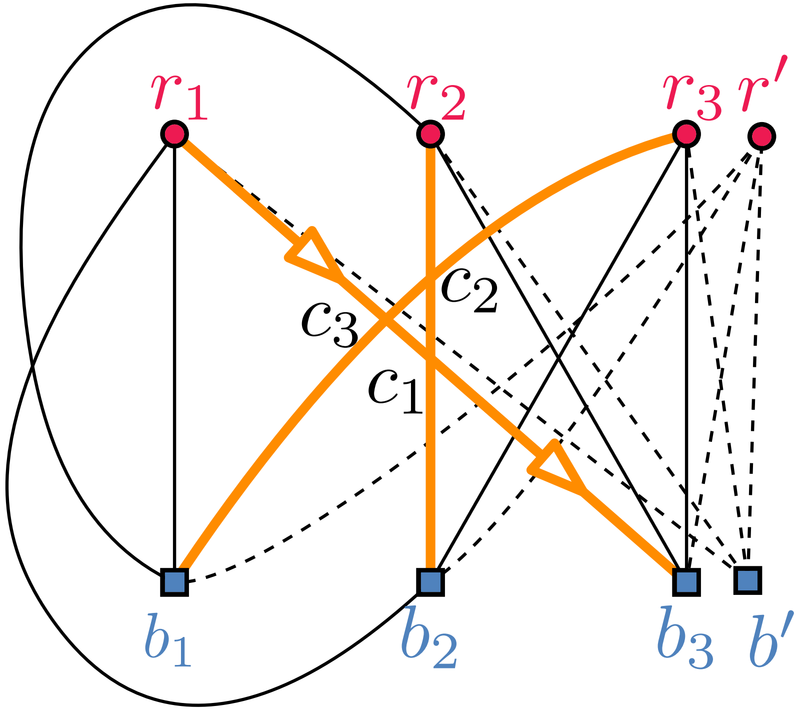




Combinations that do not imply others

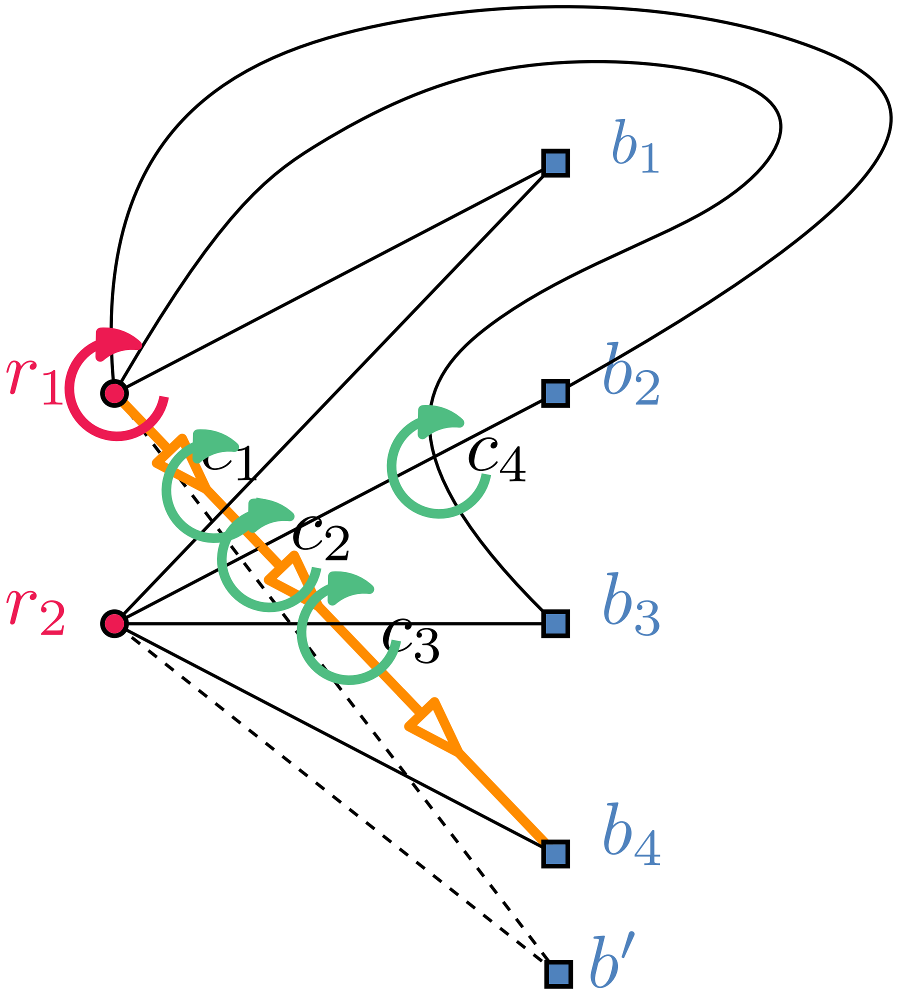
ERS-iso., not CO-iso. – Only possible if all partition classes ≥ 3

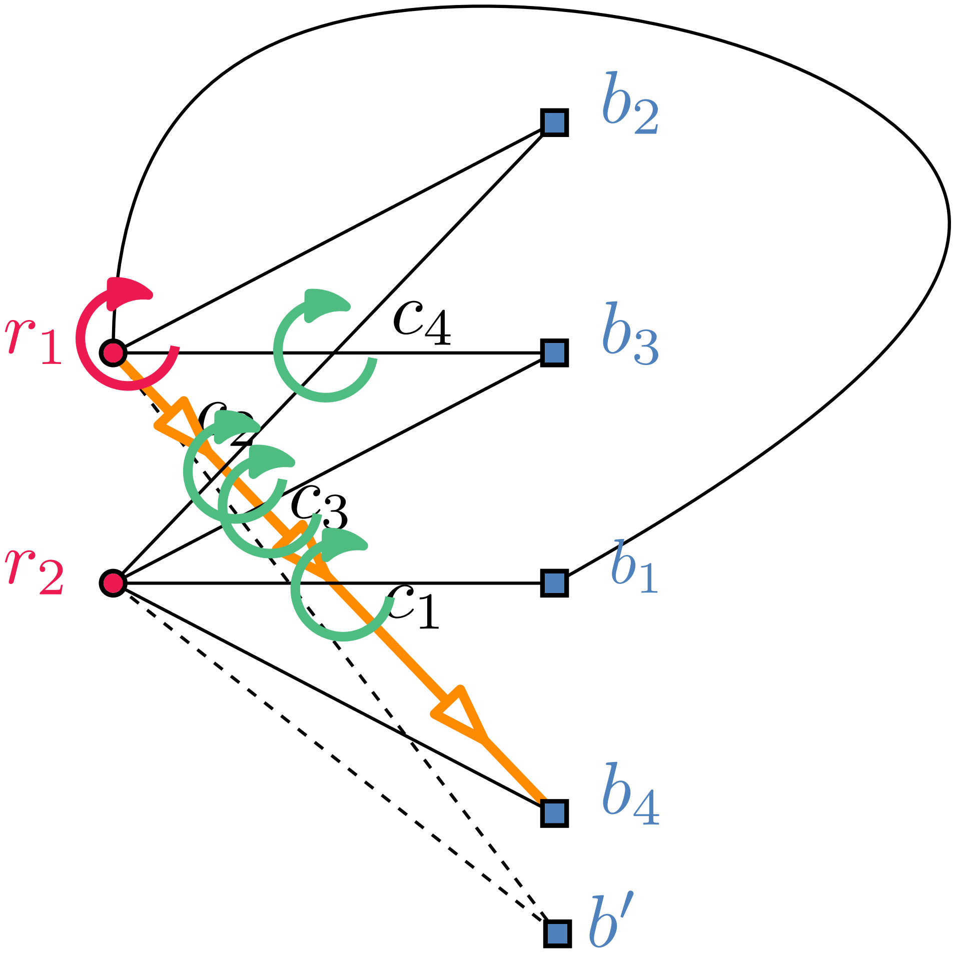




Combinations that do not imply others

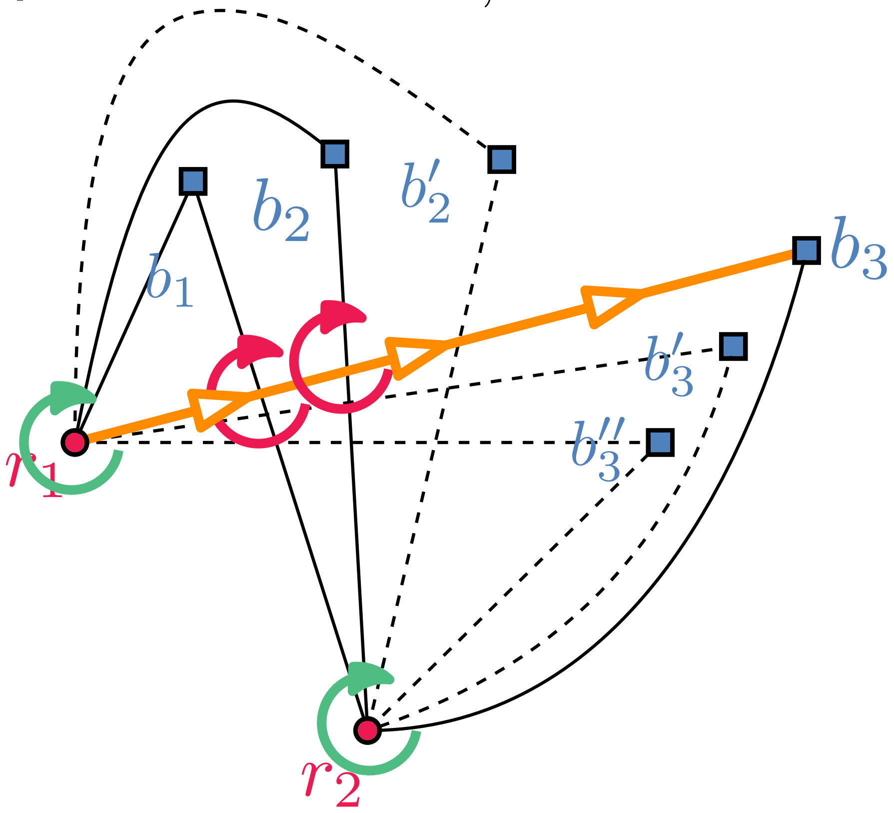
CR-iso., not RS-iso. not CO-iso. – Only possible for $K_{2,n}$

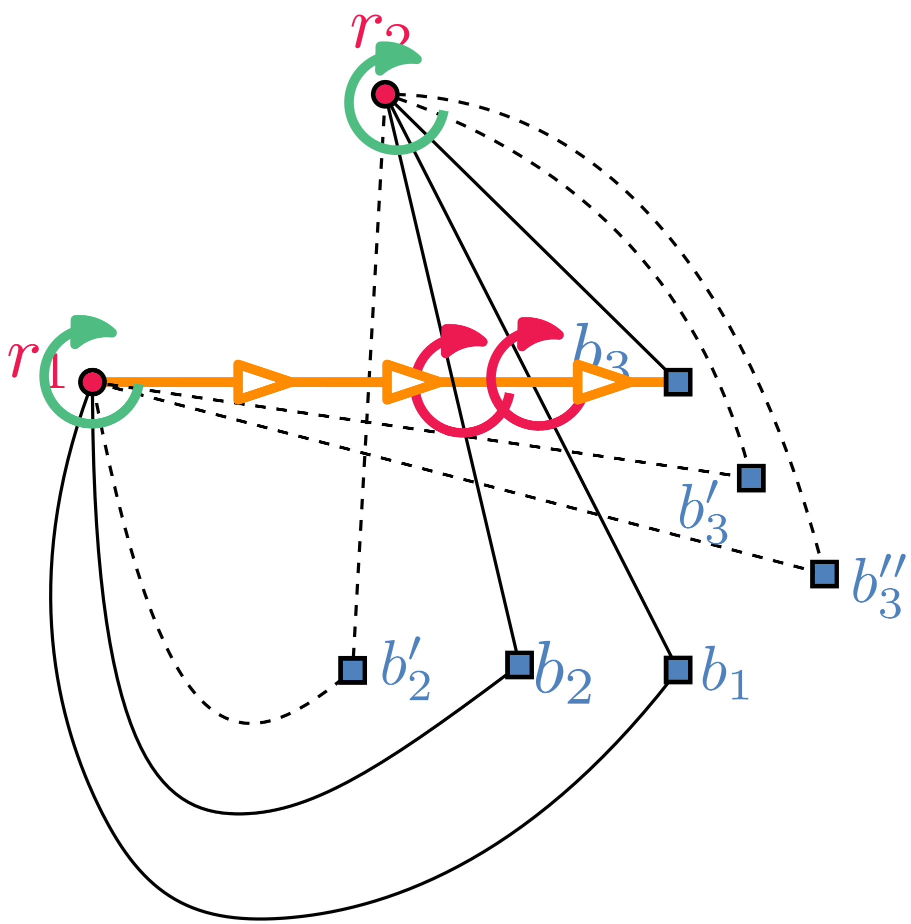




Combinations that do not imply others

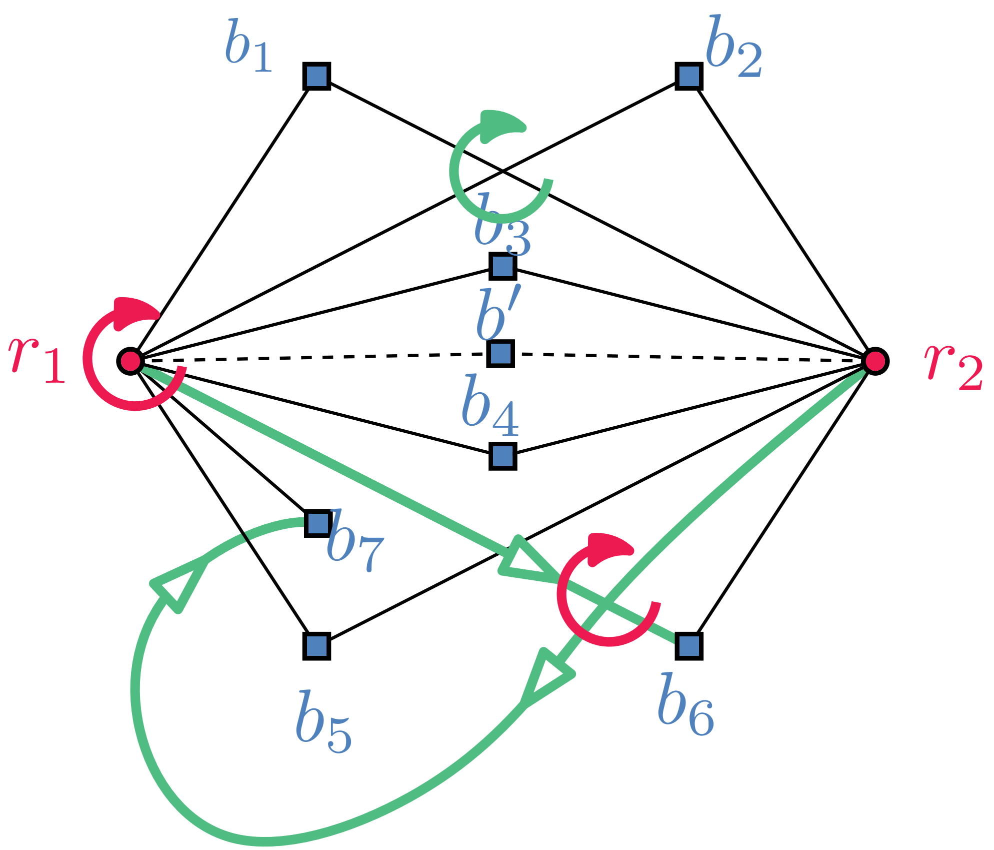
Labelled: CR-iso. and RS.-iso, not ERS-iso. not CO-iso. – Only possible for $K_{2,n}$

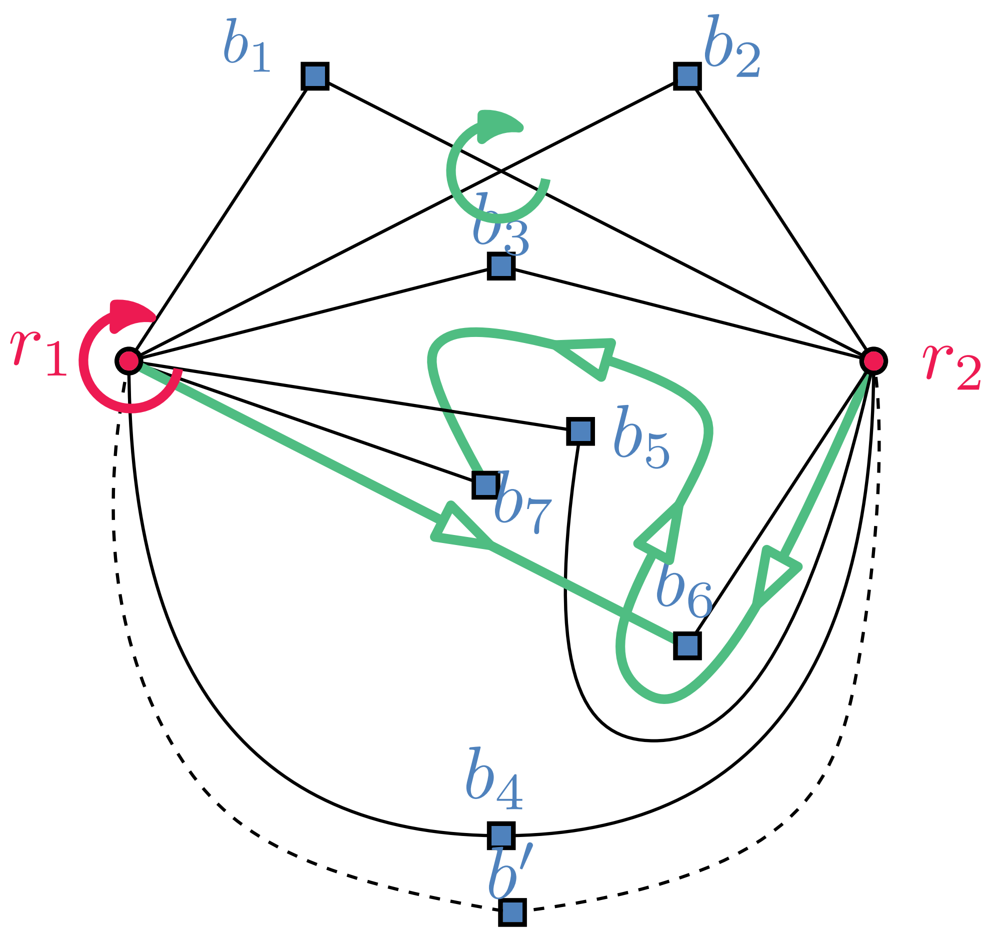




Combinations that do not imply others

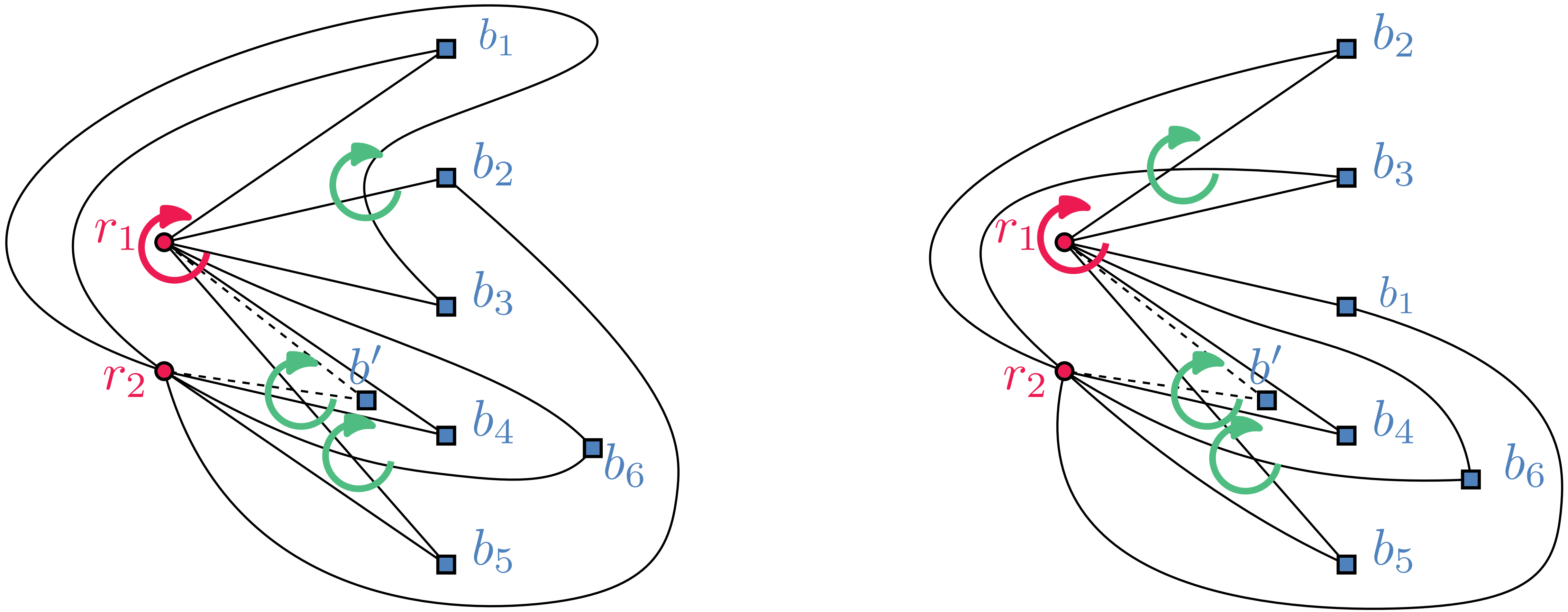
CO-iso., not RS-iso, not CR-iso. – Only possible for $K_{2,n}$





Combinations that do not imply others

CO-iso., CR-iso., not RS-iso, – Only possible for $K_{2,n}$



Proof ideas

For complete multipartite graphs:

RS + CO => strong iso.

If each partition class
has ≥ 3 vertices:
CE => RS
CR => ERS
CO => strong iso.

For $K_{2,n}$: ERS =>
strong iso.[1]

1) [O. Aichholzer, M.K. Chiu, H. Hoang, M. Hoffmann, J. Kynčl, Y. Maus, B. Vogtenhuber, A.W. 2023]

Proof ideas

For complete multipartite graphs:

RS + CO => strong iso.

Show for graphs ≤ 4 and $K_{2,3}$, then proof by contradiction (finding a 'contradicting $K_{2,3}$')

If each partition class has ≥ 3 vertices:

CE => RS

CR => ERS

CO => strong iso.

Show for $K_{3,3}$, then proof by contradiction (finding a 'contradicting $K_{2,3}$')

For $K_{2,n}$: ERS => strong iso.[1]

1) [O. Aichholzer, M.K. Chiu, H. Hoang, M. Hoffmann, J. Kynčl, Y. Maus, B. Vogtenhuber, A.W. 2023]

Proof ideas

For complete multipartite graphs:

RS + CO => strong iso.

Show for graphs ≤ 4 and $K_{2,3}$, then proof by contradiction (finding a 'contradicting $K_{2,3}$')

If each partition class has ≥ 3 vertices:

CE => RS

CR => ERS

CO => strong iso.

Show for $K_{3,3}$, then proof by contradiction (finding a 'contradicting $K_{2,3}$')

For $K_{2,n}$: ERS => strong iso.[1]

Near-direct corollary

1) [O. Aichholzer, M.K. Chiu, H. Hoang, M. Hoffmann, J. Kynčl, Y. Maus, B. Vogtenhuber, A.W. 2023]

Proof ideas

For complete multipartite graphs:

RS + CO => strong iso.

Show for graphs ≤ 4 and $K_{2,3}$, then proof by contradiction (finding a 'contradicting $K_{2,3}$')

If each partition class has ≥ 3 vertices:

CE => RS

Let's look at a sketch of this one :-)

CR => ERS

Show for $K_{3,3}$, then proof by contradiction (finding a 'contradicting $K_{2,3}$')

CO => strong iso.

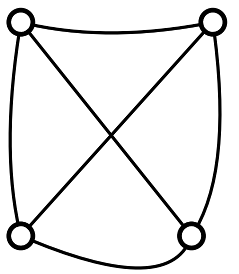
Near-direct corollary

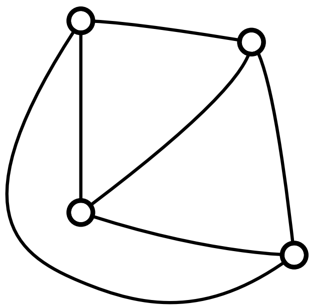
For $K_{2,n}$: ERS => strong iso.[1]

1) [O. Aichholzer, M.K. Chiu, H. Hoang, M. Hoffmann, J. Kynčl, Y. Maus, B. Vogtenhuber, A.W. 2023]

CO+RS $\Longrightarrow$ strong isomorphism

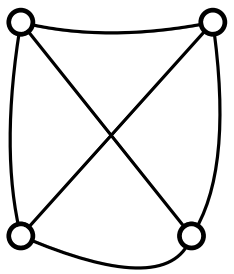
Unlabelled drawings of graphs with $n \leq 4$

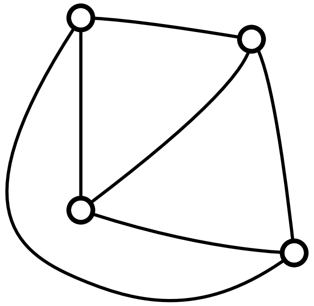


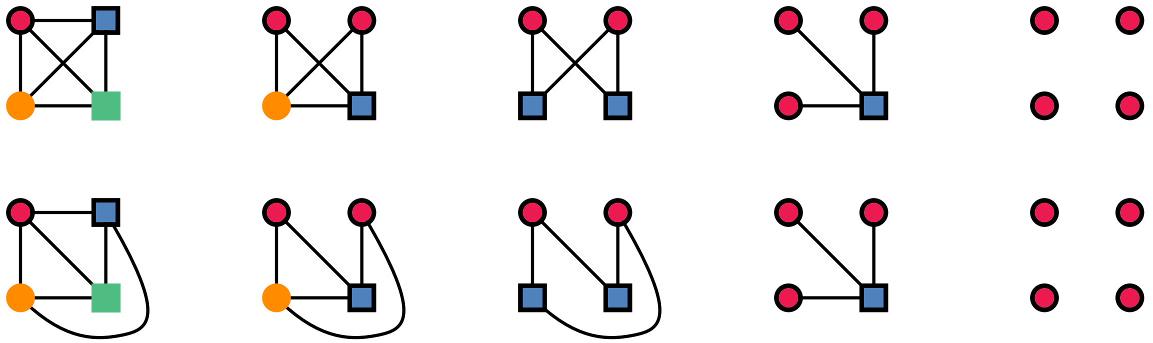


CO+RS $\implies$ strong isomorphism

Unlabelled drawings of graphs with $n \leq 4$

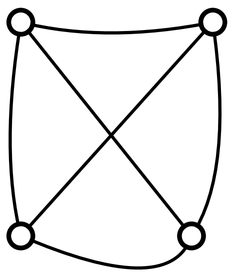


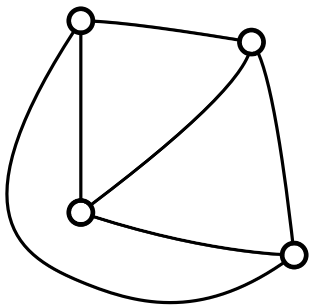




CO+RS $\implies$ strong isomorphism

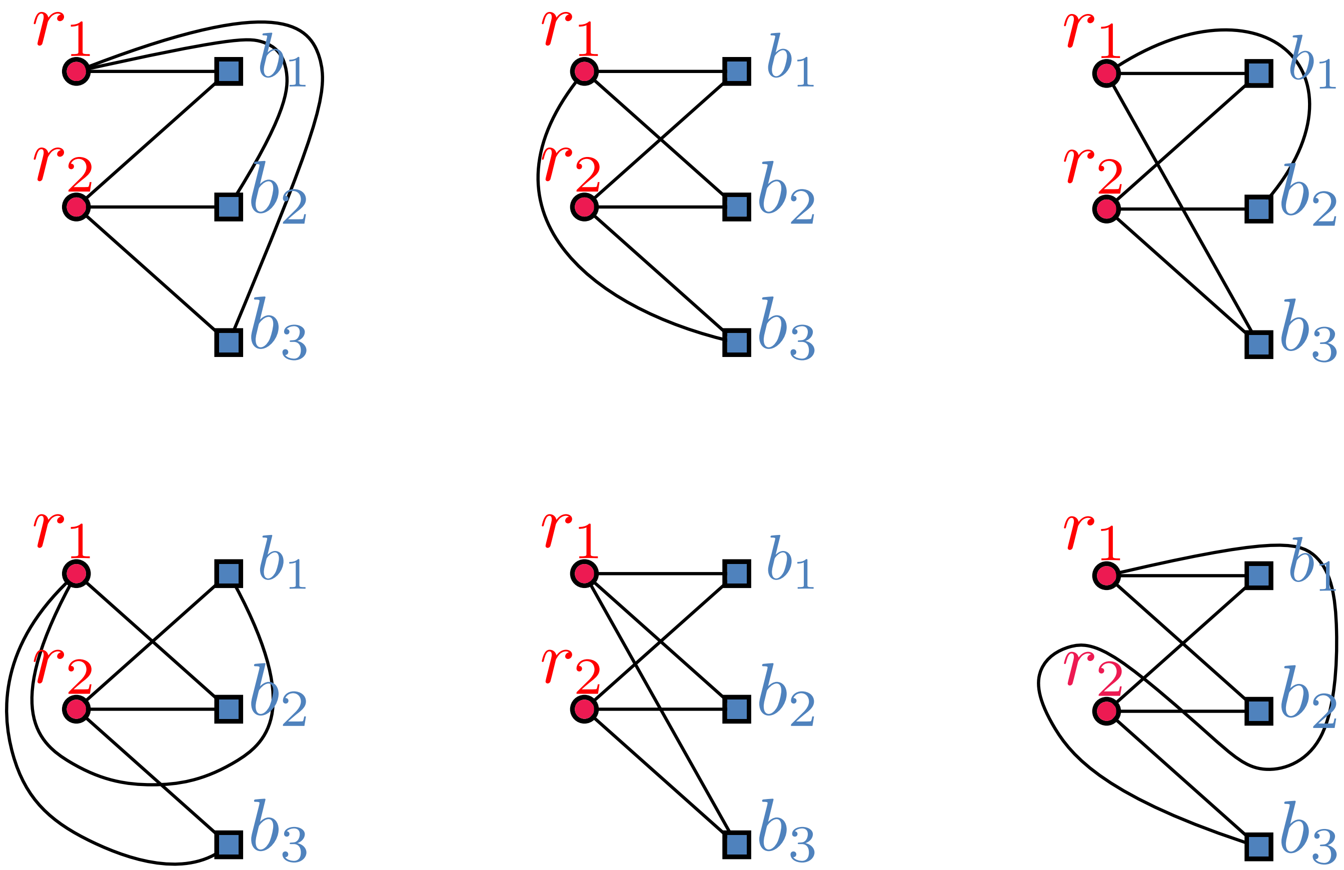
Unlabelled drawings of graphs with $n \leq 4$





CO+RS $\implies$ strong isomorphism

True for all labelled drawings of $K_{2,3}$ (and thus also all unlabelled drawings).

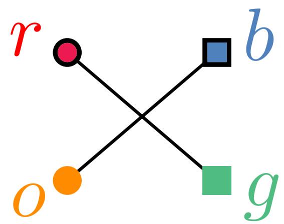


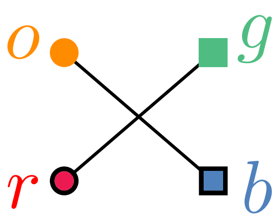
CO+RS $\implies$ strong isomorphism

Recall: ERS+CO $\implies$ strong isomorphism
What can go wrong? Crossing rotation.

CO+RS $\implies$ strong isomorphism

Recall: ERS+CO $\implies$ strong isomorphism
What can go wrong? Crossing rotation.
Assume a crossing is s.t. ERS is not the same. Look at a $K_{2,3}$ that contains that crossing.



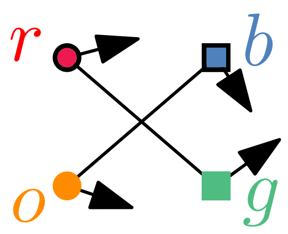


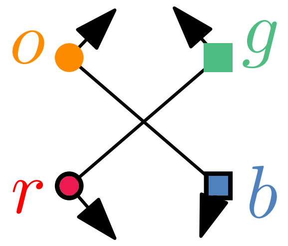
CO+RS $\implies$ strong isomorphism

Recall: ERS+CO $\implies$ strong isomorphism

What can go wrong? Crossing rotation.

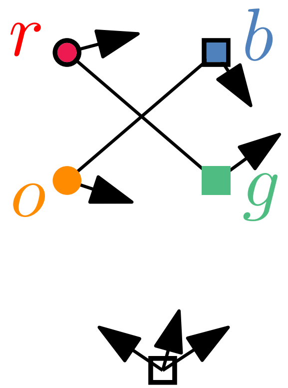
Assume a crossing is s.t. ERS is not the same. Look at a $K_{2,3}$ that contains that crossing.

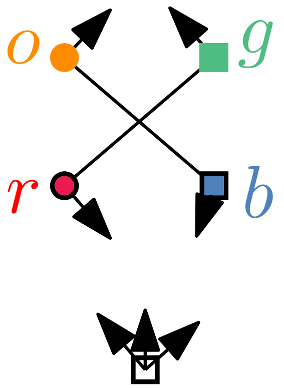




CO+RS $\implies$ strong isomorphism

Recall: ERS+CO $\implies$ strong isomorphism
What can go wrong? Crossing rotation.
Assume a crossing is s.t. ERS is not the same. Look at a $K_{2,3}$ that contains that crossing.





In each partition class at least 3 vertices: CO $\implies$ strong iso.

CO-iso. $\implies$ CE-iso. $\implies$ RS-iso.

In each partition class at least 3 vertices: CO $\Longrightarrow$ strong iso.

CO-iso. $\Longrightarrow$ CE-iso. $\Longrightarrow$ RS-iso.

CO-iso. $+$ RS-iso. $\Longrightarrow$ strong iso.

In each class at least 3: CR $\implies$ ERS

True for (labelled) drawings of $K_{3,3}$.

In each class at least 3: CR $\implies$ ERS

True for (labelled) drawings of $K_{3,3}$.

CR-iso. $\implies$ CE-iso. $\implies$ RS-iso.

In each class at least 3: CR $\implies$ ERS

True for (labelled) drawings of $K_{3,3}$.

CR-iso. $\implies$ CE-iso. $\implies$ RS-iso.

What could go wrong? Crossing rotations are inverse RS is same (or CR the same, RS inverse)

In each class at least 3: CR $\implies$ ERS

True for (labelled) drawings of $K_{3,3}$.

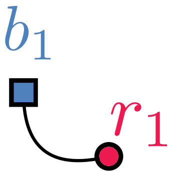
CR-iso. $\implies$ CE-iso. $\implies$ RS-iso.

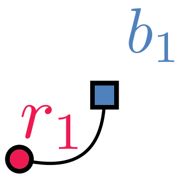
What could go wrong? Crossing rotations are inverse RS is same (or CR the same, RS inverse)

If CR same, RS inverse $\implies$ also in every $K_{3,3}$-subdrawing.
Contradiction!

Remark on connection of rotations and degrees

r_1: b_1
r_1: b_1





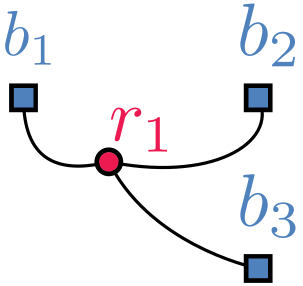
Remark on connection of rotations and degrees

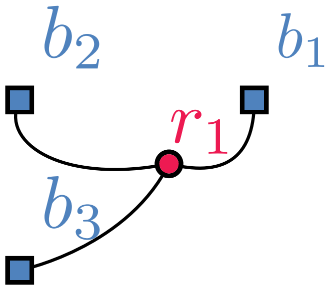
r_1: b_1 b_2
r_1: b_1 b_2

Remark on connection of rotations and degrees

r_1: b_1 b_2 b_3

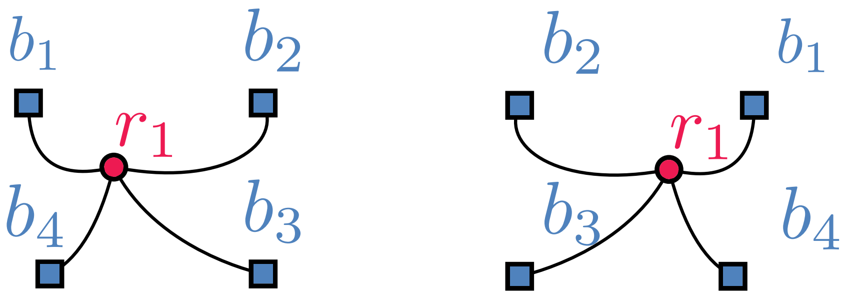
r_1: b_1 b_3 b_2



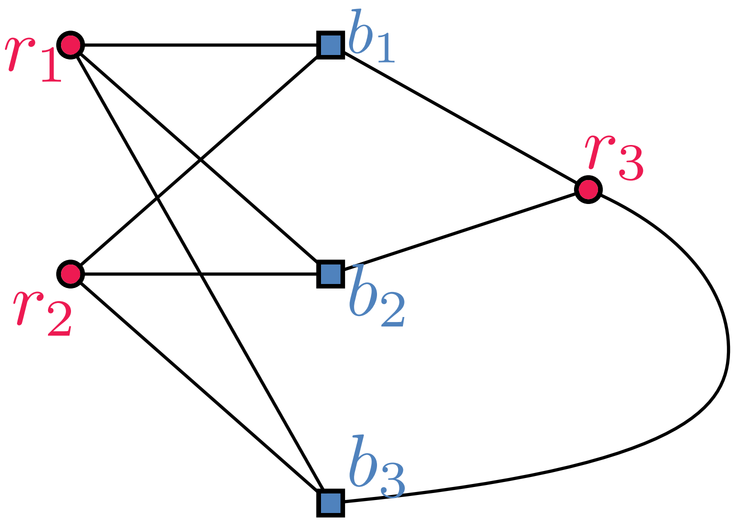


Remark on connection of rotations and degrees

r_1: b_1 b_2 b_3 b_4
r_1: b_1 b_4 b_3 b_2

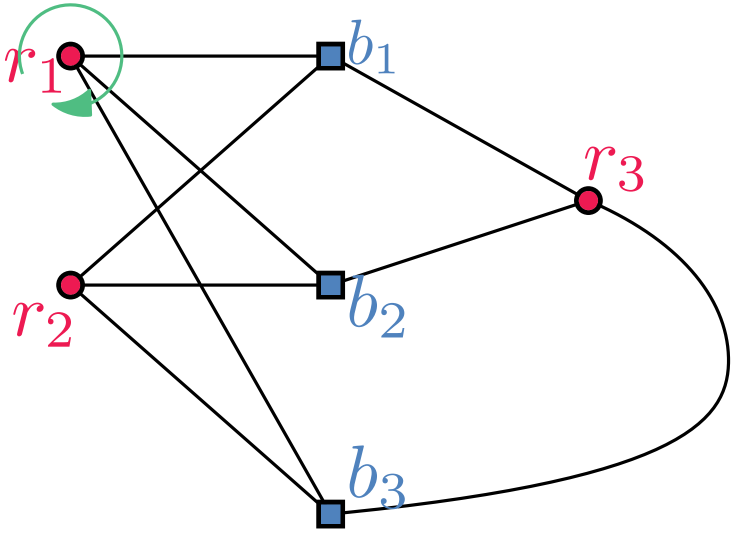


Describing simple drawings – Types of isomorphism



Rotation ... Cyclical order of incident edges

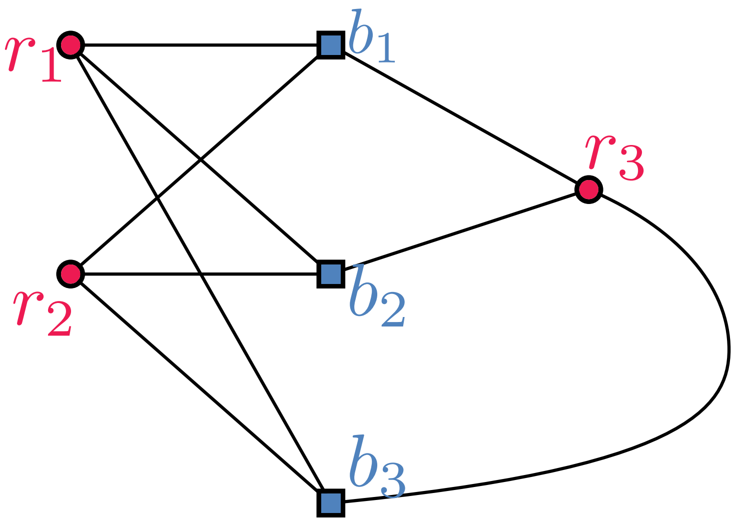
Describing simple drawings – Types of isomorphism



Rotation ... Cyclical order of incident edges

Rotation around r_1: $b_1\ b_2\ b_3$

Describing simple drawings – Types of isomorphism



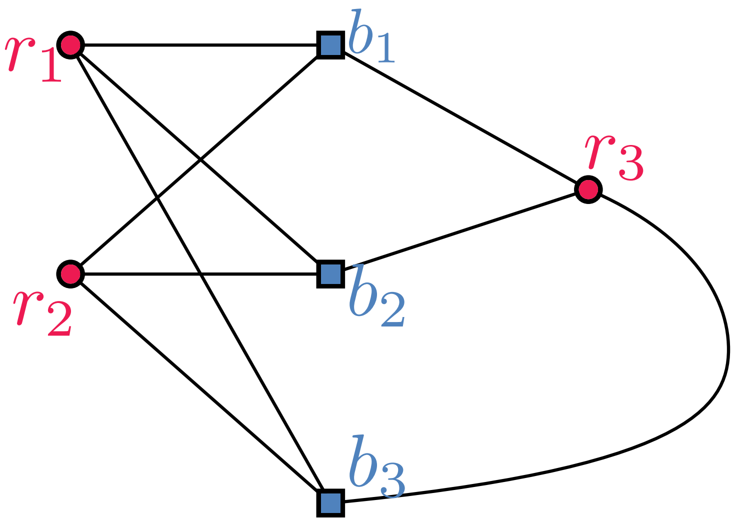
Rotation ... Cyclical order of incident edges

Rotation around r_1: $b_1\ b_2\ b_3$

Rotation System ... Collection of the rotations of all vertices.

r_1 : b_1 b_2 b_3
r_2 : b_1 b_2 b_3
r_3 : b_1 b_3 b_2
b_1 : r_1 r_3 r_2
b_2 : r_1 r_3 r_2
b_3 : r_1 r_3 r_2

Describing simple drawings – Types of isomorphism

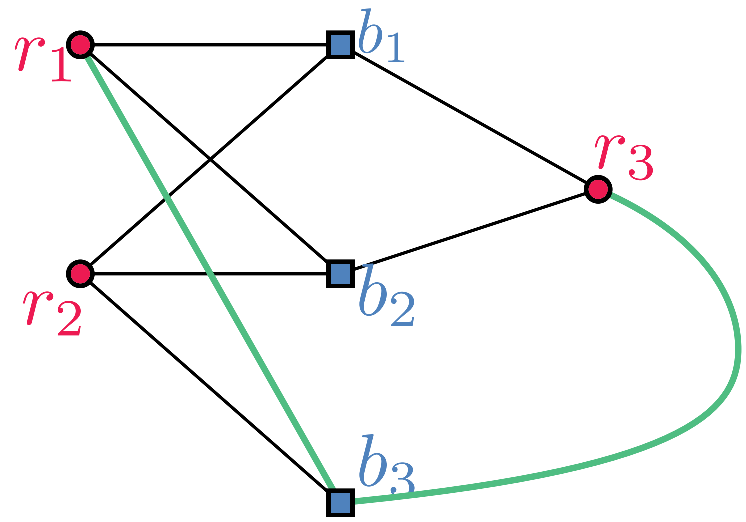


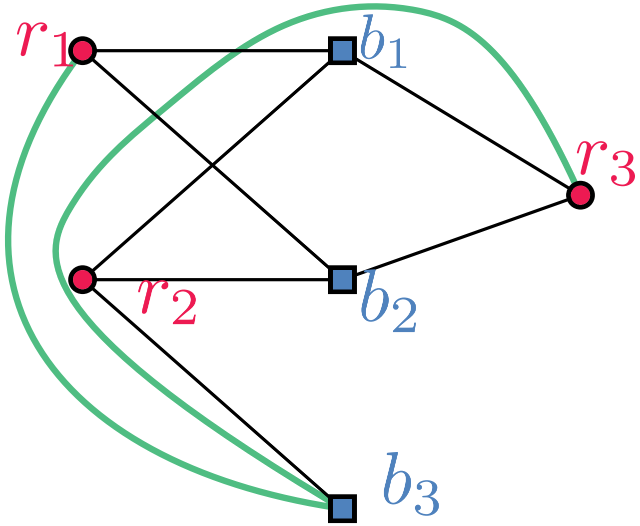
Rotation ... Cyclical order of incident edges
Rotation around r_1: b_1 b_2 b_3

Rotation System ... Collection of the rotations of all vertices.

Two labelled simple drawings are RS-isomorphic iff they have the same or inverse rotation systems.

Describing simple drawings – Types of isomorphism





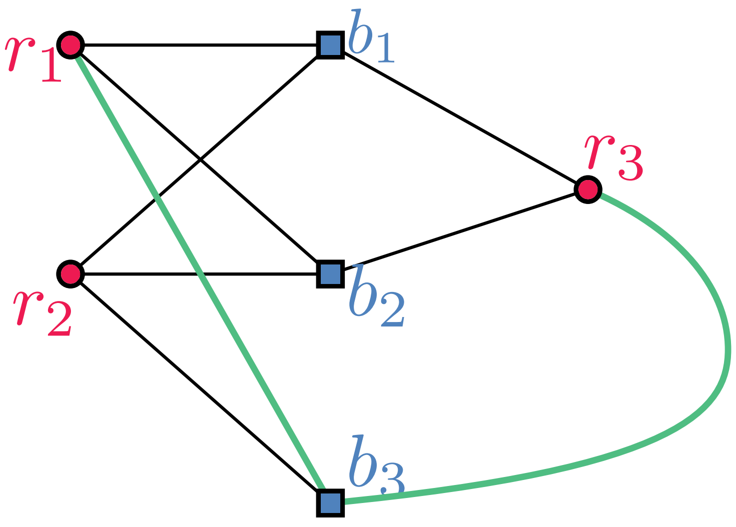
Rotation ... Cyclical order of incident edges

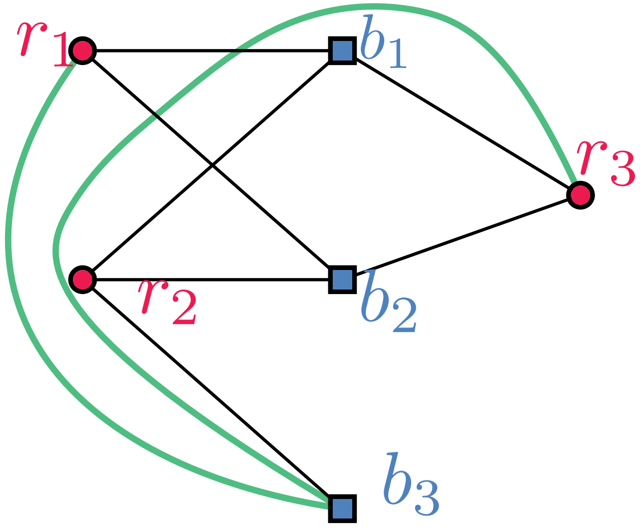
Rotation around r_1: $b_1\ b_2\ b_3$

Rotation System ... Collection of the rotations of all vertices.

Two labelled simple drawings are RS-isomorphic iff they have the same or inverse rotation systems.

Describing simple drawings – Types of isomorphism

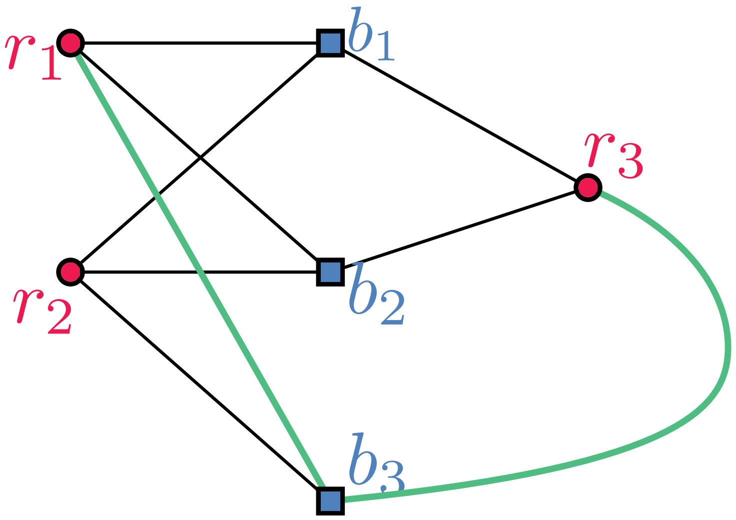


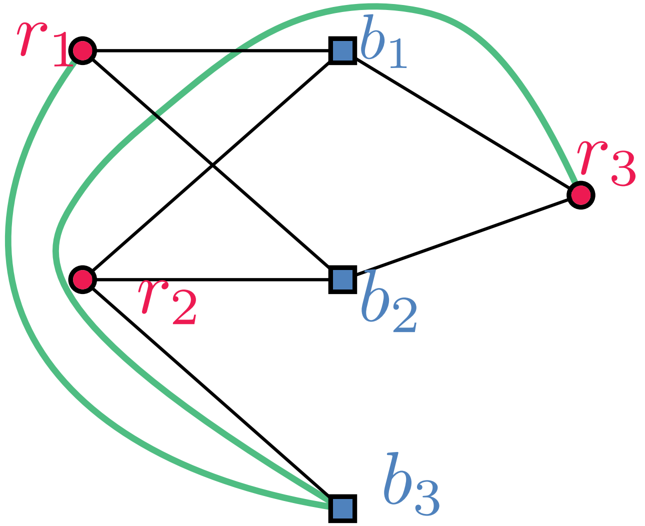


Two labelled simple drawings are RS-isomorphic iff they have the same or inverse rotation systems.

r_1 : b_1 b_2 b_3
r_2 : b_1 b_2 b_3
r_3 : b_1 b_3 b_2
b_1 : r_1 r_3 r_2
b_2 : r_1 r_3 r_2
b_3 : r_1 r_3 r_2

Describing simple drawings – Types of isomorphism



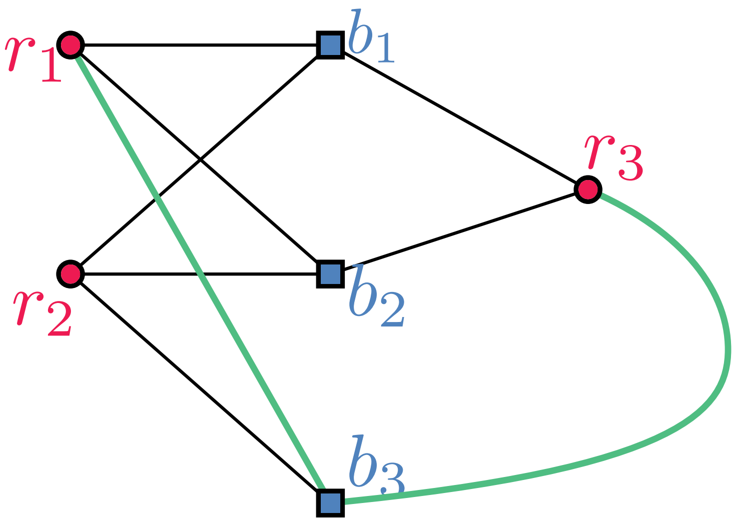


Two labelled simple drawings are RS-isomorphic iff they have the same or inverse rotation systems.

r_1 :	b_1	b_2	b_3
r_2 :	b_1	b_2	b_3
r_3 :	b_1	b_3	b_2
b_1 :	r_1	r_3	r_2
b_2 :	r_1	r_3	r_2
b_3 :	r_1	r_3	r_2

r_1 :	b_1	b_3	b_2
r_2 :	b_1	b_3	b_2
r_3 :	b_1	b_2	b_3
b_1 :	r_1	r_2	r_3
b_2 :	r_1	r_2	r_3
b_3 :	r_1	r_2	r_3

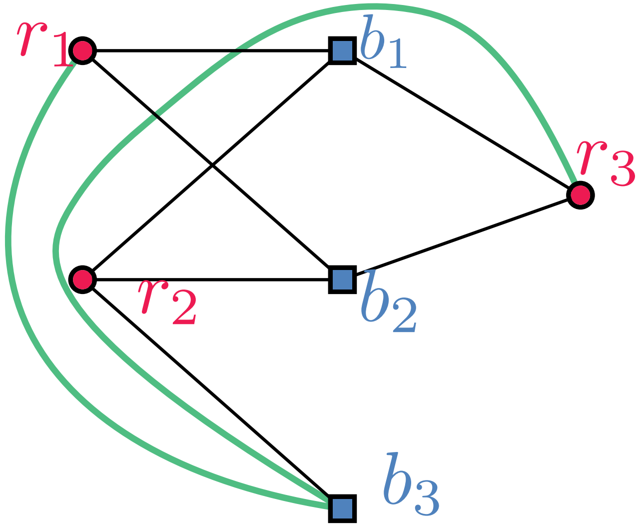
Describing simple drawings – Types of isomorphism

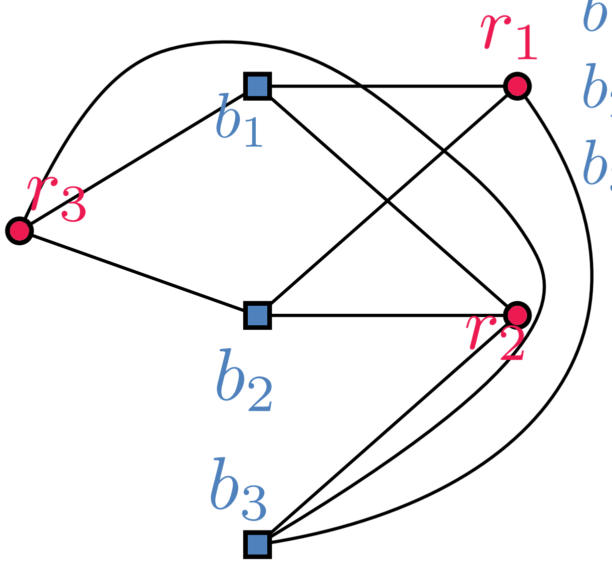


Two labelled simple drawings are RS-isomorphic iff they have the same or inverse rotation systems.

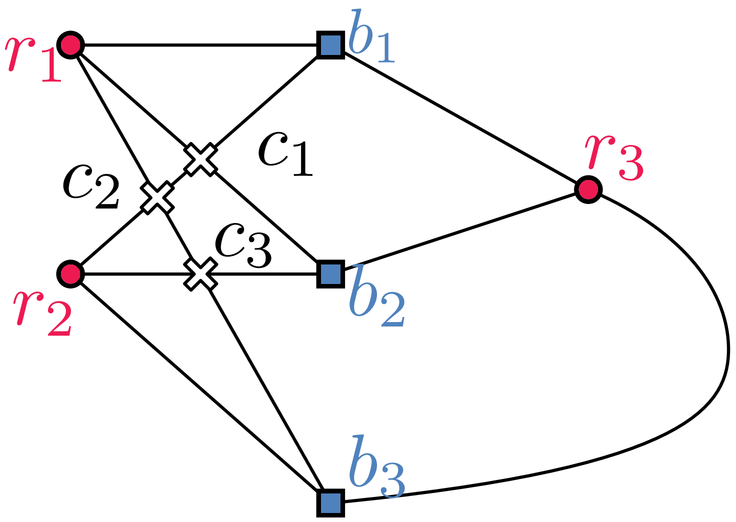
r_1 :	b_1	b_2	b_3
r_2 :	b_1	b_2	b_3
r_3 :	b_1	b_3	b_2
b_1 :	r_1	r_3	r_2
b_2 :	r_1	r_3	r_2
b_3 :	r_1	r_3	r_2

r_1 :	b_1	b_3	b_2
r_2 :	b_1	b_3	b_2
r_3 :	b_1	b_2	b_3
b_1 :	r_1	r_2	r_3
b_2 :	r_1	r_2	r_3
b_3 :	r_1	r_2	r_3





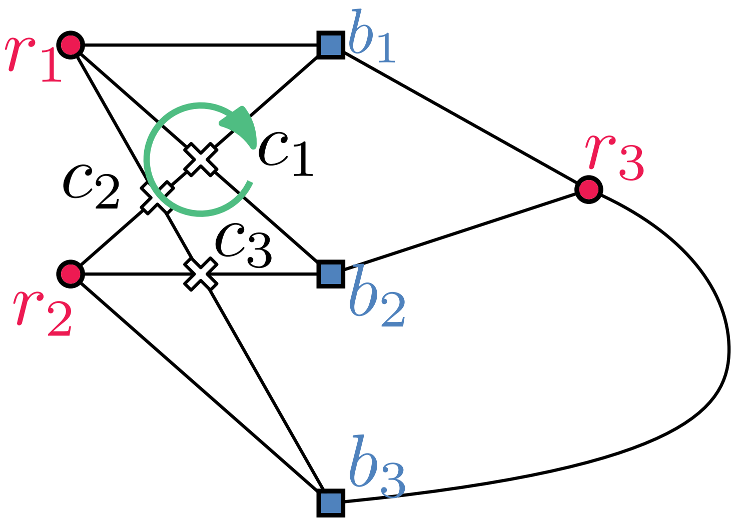
Describing simple drawings – Types of isomorphism



Rotation ... Cyclical order of incident edges

Rotation around r_1: b_1 b_2 b_3

Describing simple drawings – Types of isomorphism

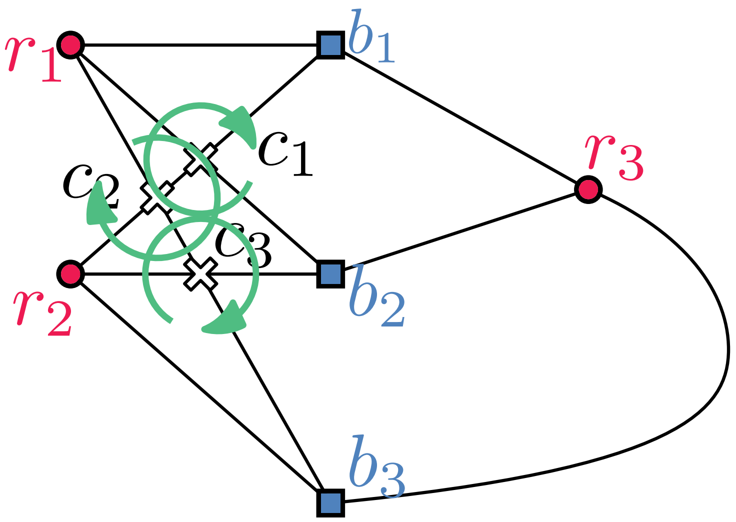


Rotation ... Cyclical order of incident edges

Rotation around r_1: b_1 b_2 b_3

Rotation around c_1: r_1 b_1 b_2 r_2

Describing simple drawings – Types of isomorphism



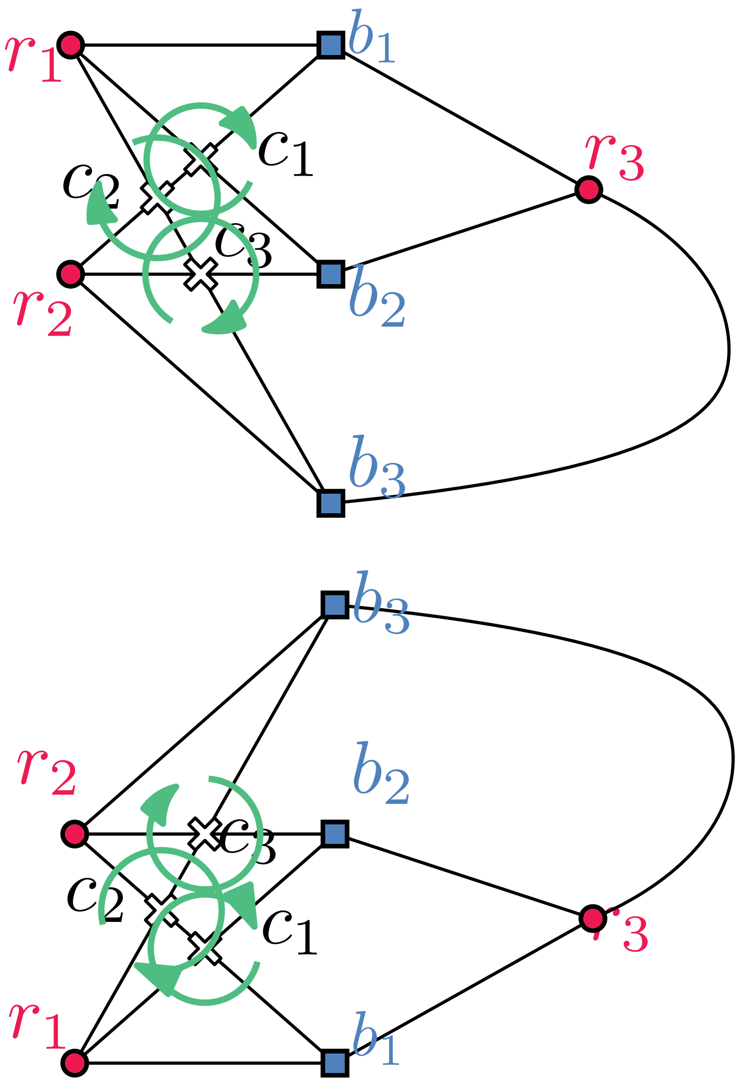
Rotation ... Cyclical order of incident edges

Rotation around r_1: b_1 b_2 b_3

Rotation around c_1: r_1 b_1 b_2 r_2

Two labelled simple drawings are CR-isomorphic iff either all crossings have the same rotations or all crossings have inverse rotations.

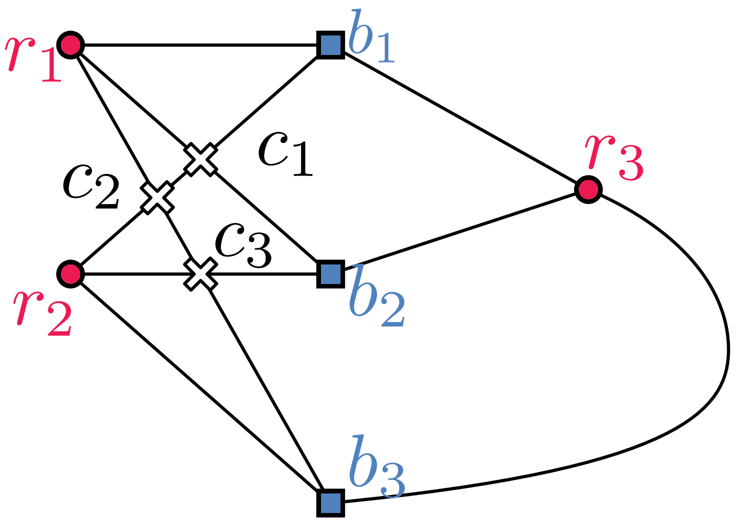
Describing simple drawings – Types of isomorphism



Rotation ... Cyclical order of incident edges
Rotation around r_1: b_1 b_2 b_3
Rotation around c_1: r_1 b_1 b_2 r_2

Two labelled simple drawings are CR-isomorphic iff either all crossings have the same rotations or all crossings have inverse rotations.

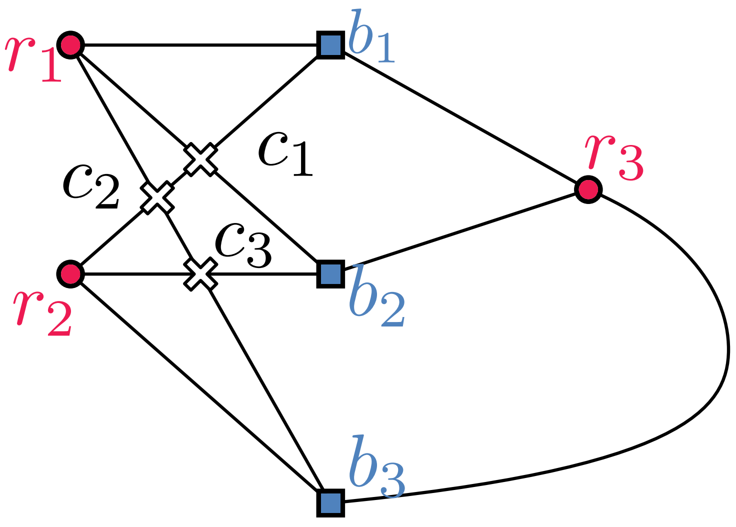
Describing simple drawings – Types of isomorphism



Rotation ... Cyclical order of incident edges
Rotation around r_1: b_1 b_2 b_3
Rotation around c_1: r_1 b_1 b_2 r_2

Extended rotation system ... Collection of the rotations of all vertices and crossings.

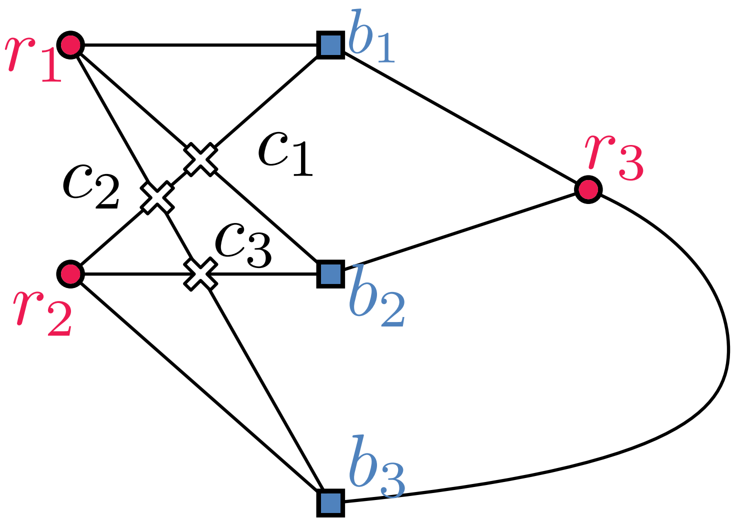
Describing simple drawings – Types of isomorphism



Extended rotation system ...
Collection of the rotations of all vertices and crossings.

r_1 :	b_1	b_2	b_3	
r_2 :	b_1	b_2	b_3	
r_3 :	b_1	b_3	b_2	
b_1 :	r_1	r_3	r_2	
b_2 :	r_1	r_3	r_2	
b_3 :	r_1	r_3	r_2	
c_1 :	r_1	b_1	b_2	r_2
c_2 :	r_1	b_1	b_3	r_2
c_3 :	r_1	b_2	b_3	r_2

Describing simple drawings – Types of isomorphism



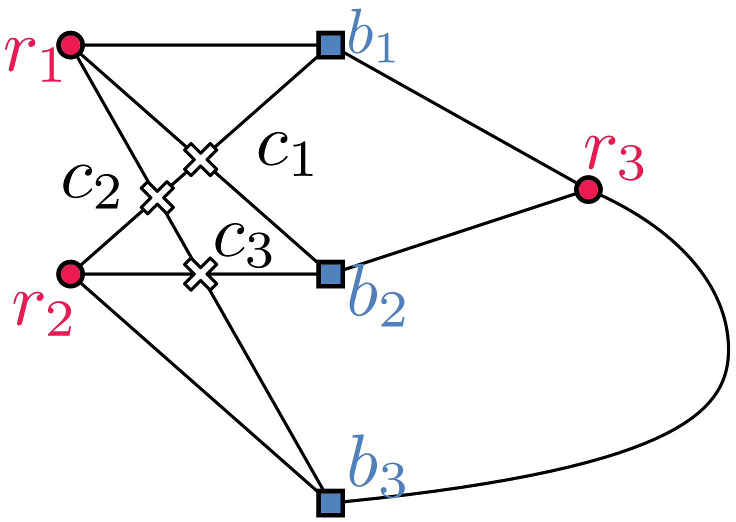
Rotation ... Cyclical order of incident edges

Rotation around r_1: $b_1\ b_2\ b_3$

Rotation around c_1: $r_1\ b_1\ b_2\ r_2$

Extended rotation system ... Collection of the rotations of all vertices and crossings.

Describing simple drawings – Types of isomorphism



Rotation ... Cyclical order of incident edges

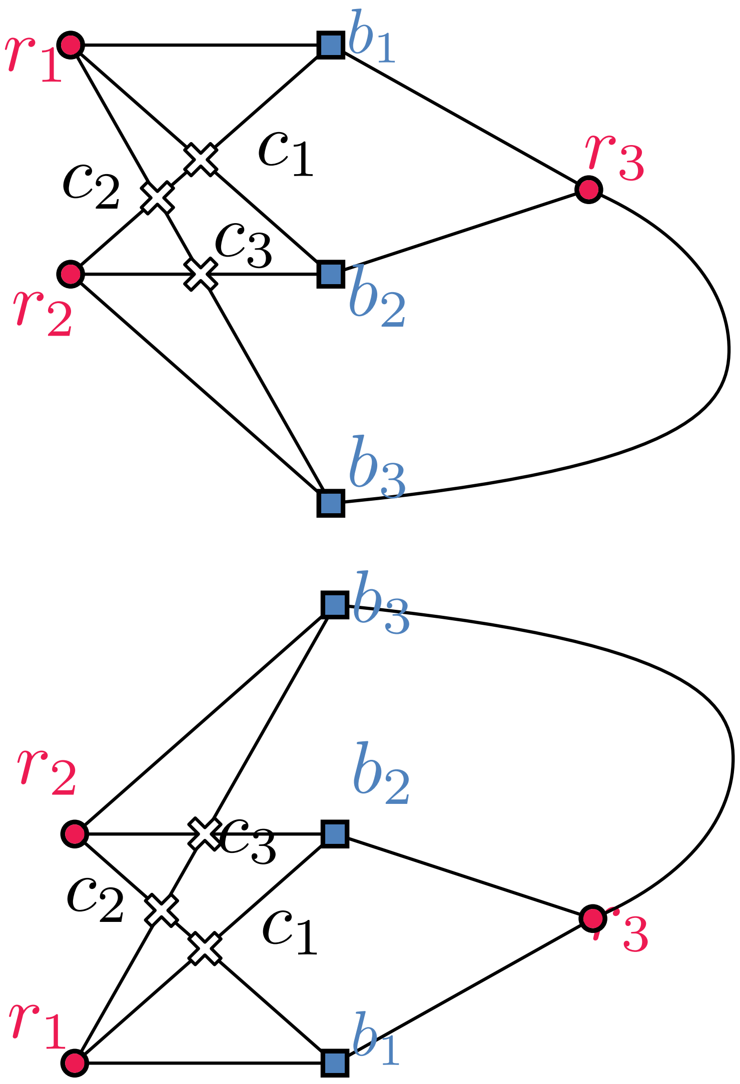
Rotation around r_1: b_1 b_2 b_3

Rotation around c_1: r_1 b_1 b_2 r_2

Extended rotation system ... Collection of the rotations of all vertices and crossings.

Two labelled simple drawings are ERS-isomorphic iff they have the same or inverse extended rotation systems.

Describing simple drawings – Types of isomorphism



Rotation … Cyclical order of incident edges

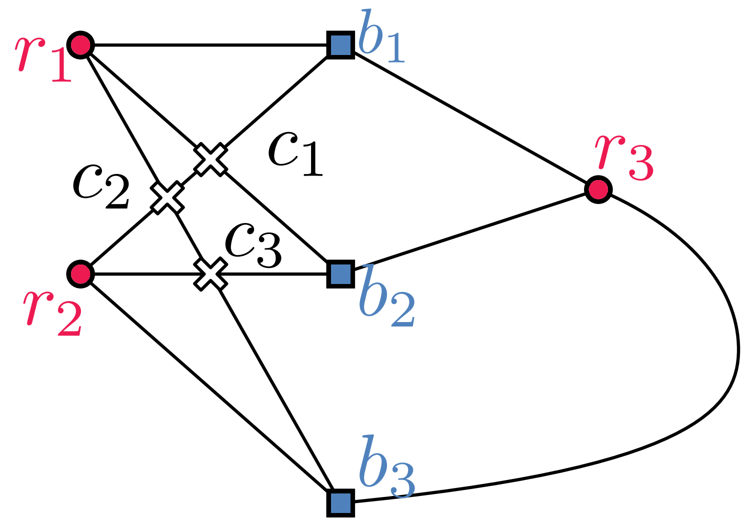
Rotation around r_1: $b_1\ b_2\ b_3$

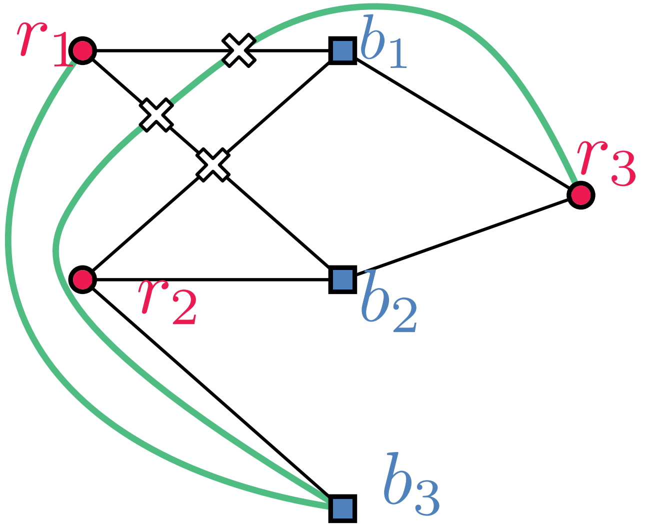
Rotation around c_1: $r_1\ b_1\ b_2\ r_2$

Extended rotation system … Collection of the rotations of all vertices and crossings.

Two labelled simple drawings are ERS-isomorphic iff they have the same or inverse extended rotation systems.

Describing simple drawings – Types of isomorphism





Rotation … Cyclical order of incident edges

Rotation around r_1: b_1 b_2 b_3

Rotation around c_1: r_1 b_1 b_2 r_2

Extended rotation system … Collection of the rotations of all vertices and crossings.

Two labelled simple drawings are ERS-isomorphic iff they have the same or inverse extended rotation systems.

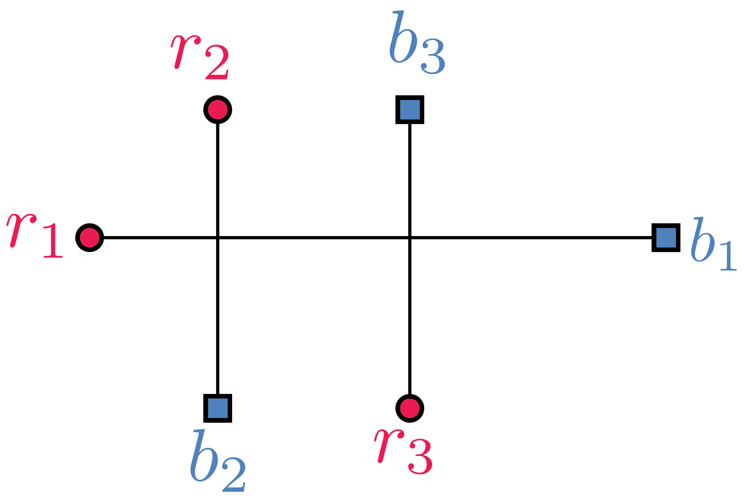
Describing simple drawings – Types of isomorphism

Two labelled simple drawings are CE-isomorphic (a.k.a. weakly isomorphic) iff they have the same crossing edge pairs.

Describing simple drawings – Types of isomorphism

Two labelled simple drawings are CE-isomorphic (a.k.a. weakly isomorphic) iff they have the same crossing edge pairs.

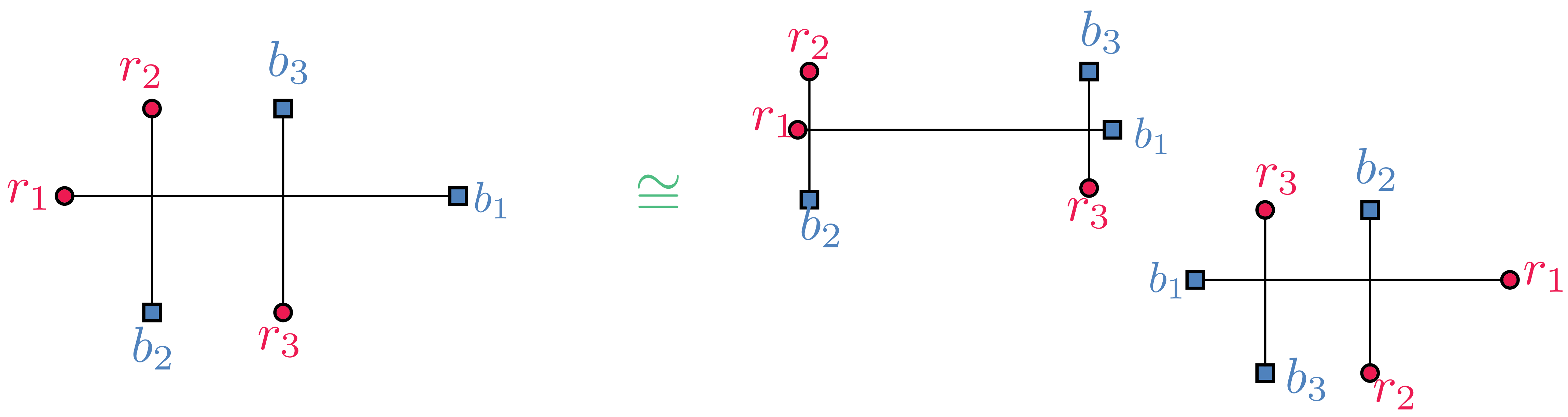
Two labelled simple drawings are CO-isomorphic iff for each edge the order in which it crosses other edges is the same.



Describing simple drawings – Types of isomorphism

Two labelled simple drawings are CE-isomorphic (a.k.a. weakly isomorphic) iff they have the same crossing edge pairs.

Two labelled simple drawings are CO-isomorphic iff for each edge the order in which it crosses other edges is the same.



Describing simple drawings – Types of isomorphism

Two labelled simple drawings are CE-isomorphic (a.k.a. weakly isomorphic) iff they have the same crossing edge pairs.

Two labelled simple drawings are CO-isomorphic iff for each edge the order in which it crosses other edges is the same.

Describing simple drawings – Types of isomorphism

Two labelled simple drawings are strongly isomorphic iff
there exists a homeomorphism of the sphere such that one drawing is mapped to the other.

Describing simple drawings – Types of isomorphism

Two labelled simple drawings are strongly isomorphic iff there exists a homeomorphism of the sphere such that one drawing is mapped to the other.

Two labelled simple drawings of connected graphs on the sphere are strongly isomorphic ('the same') iff

1. They are ERS-isomorphic and
2. CO-isomorphic.[1)]

1) [J. Kynčl 2011]

Describing simple drawings – Types of isomorphism

Two labelled simple drawings are strongly isomorphic iff there exists a homeomorphism of the sphere such that one drawing is mapped to the other.

Two labelled simple drawings of connected graphs on the sphere are strongly isomorphic ('the same') iff

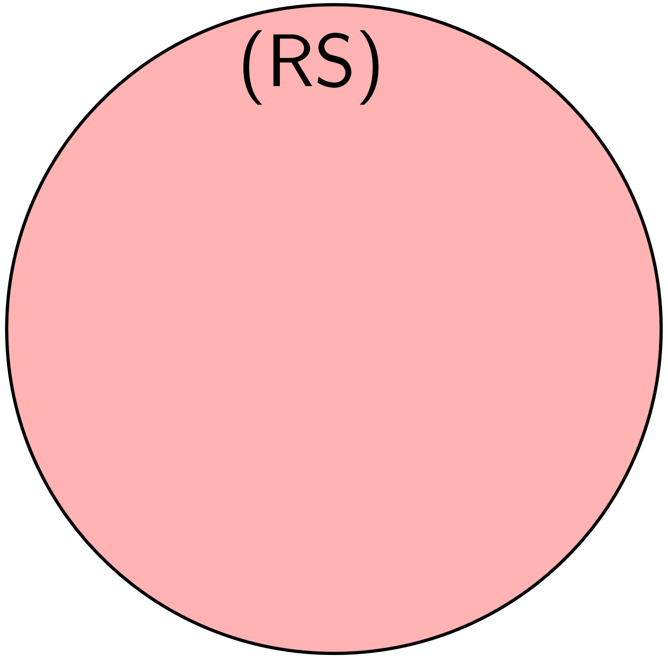
1. They are ERS-isomorphic and
2. CO-isomorphic.[1)]

Unlabelled simple drawings are isomorphic w.r.t. some type of isomorphism iff $\exists$ labeling s.t. labelled drawings are isomorphic w.r.t. that type.

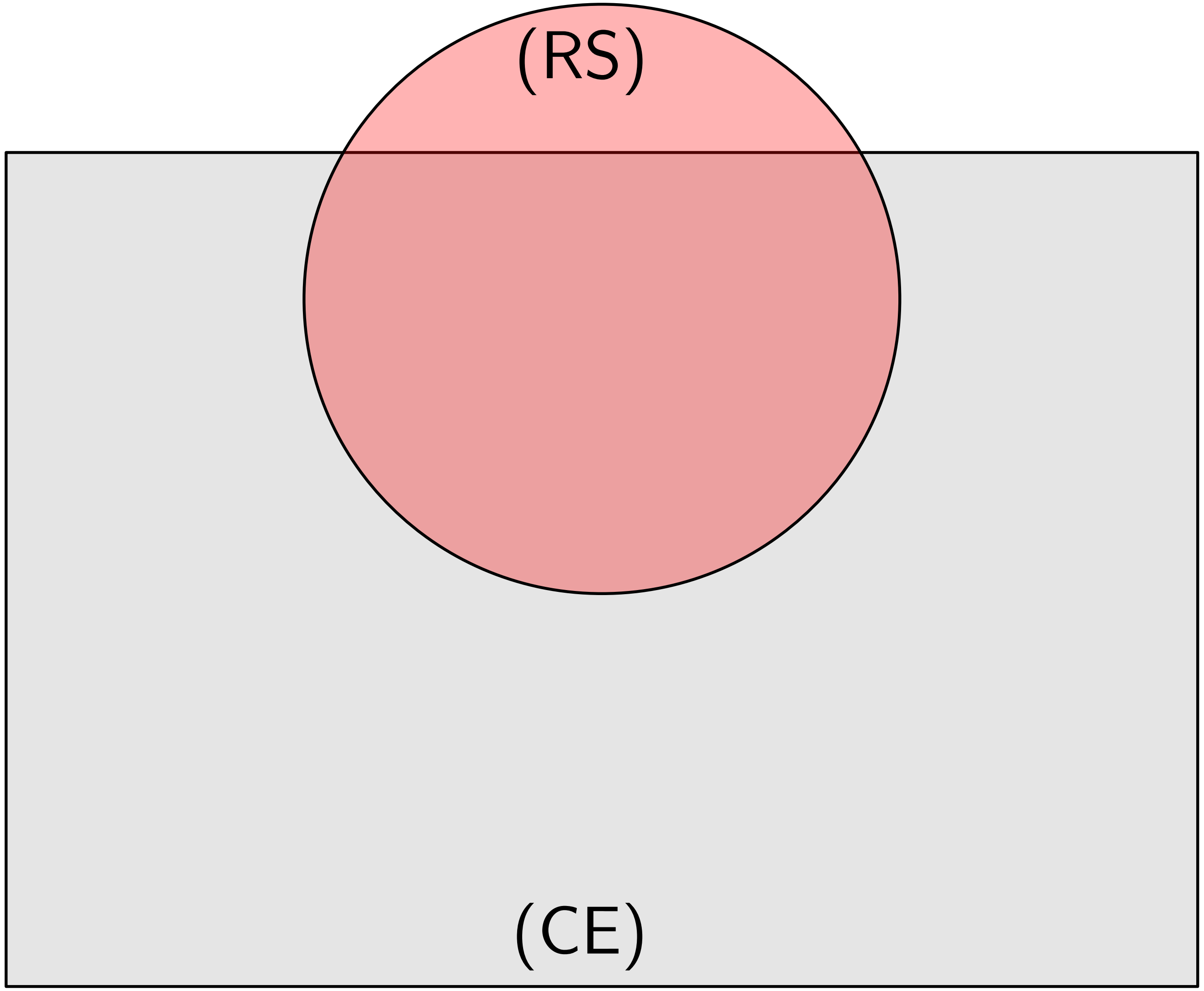
1) [J. Kynčl 2011]

Implications between isomorphisms

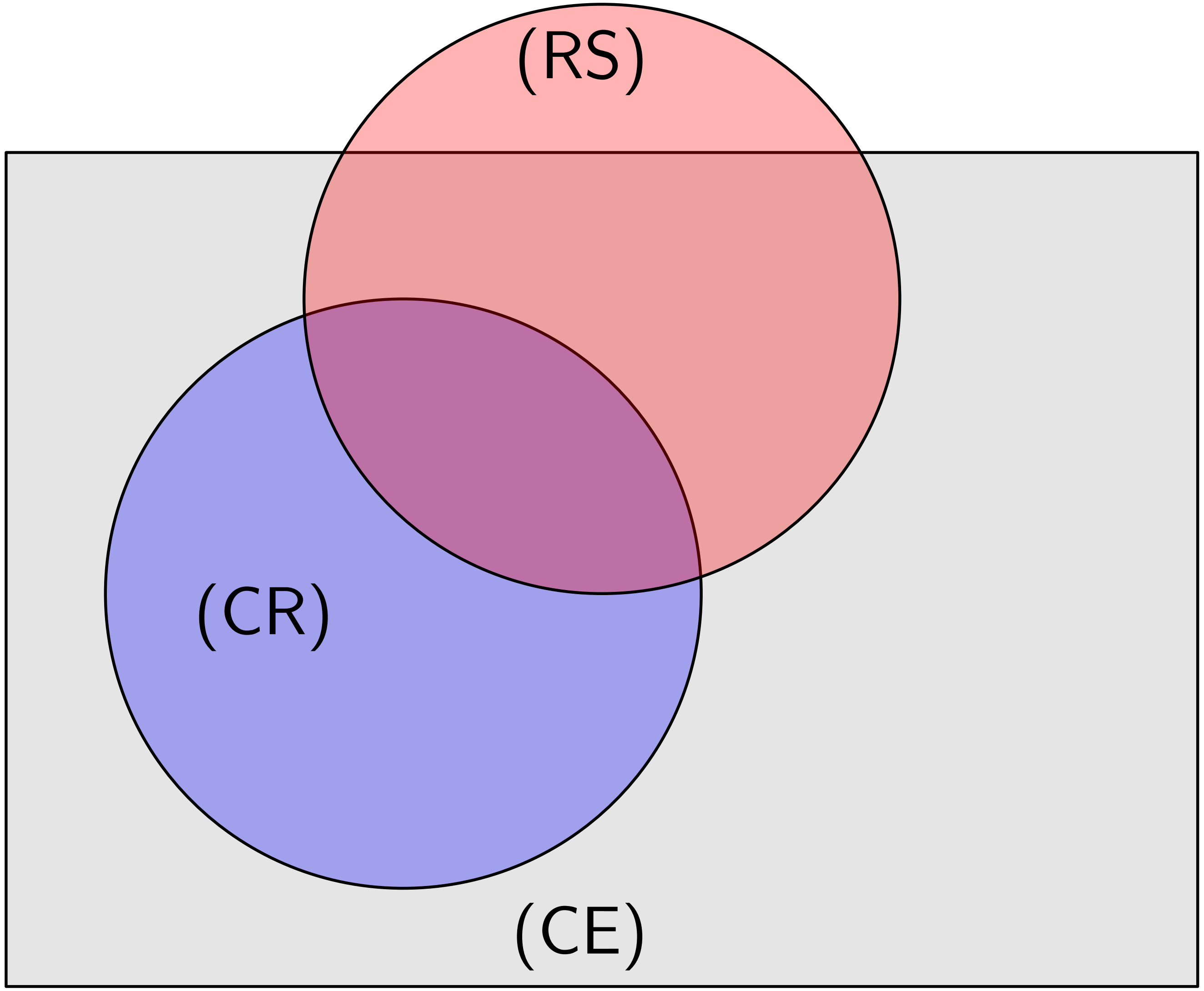
Implications between isomorphisms



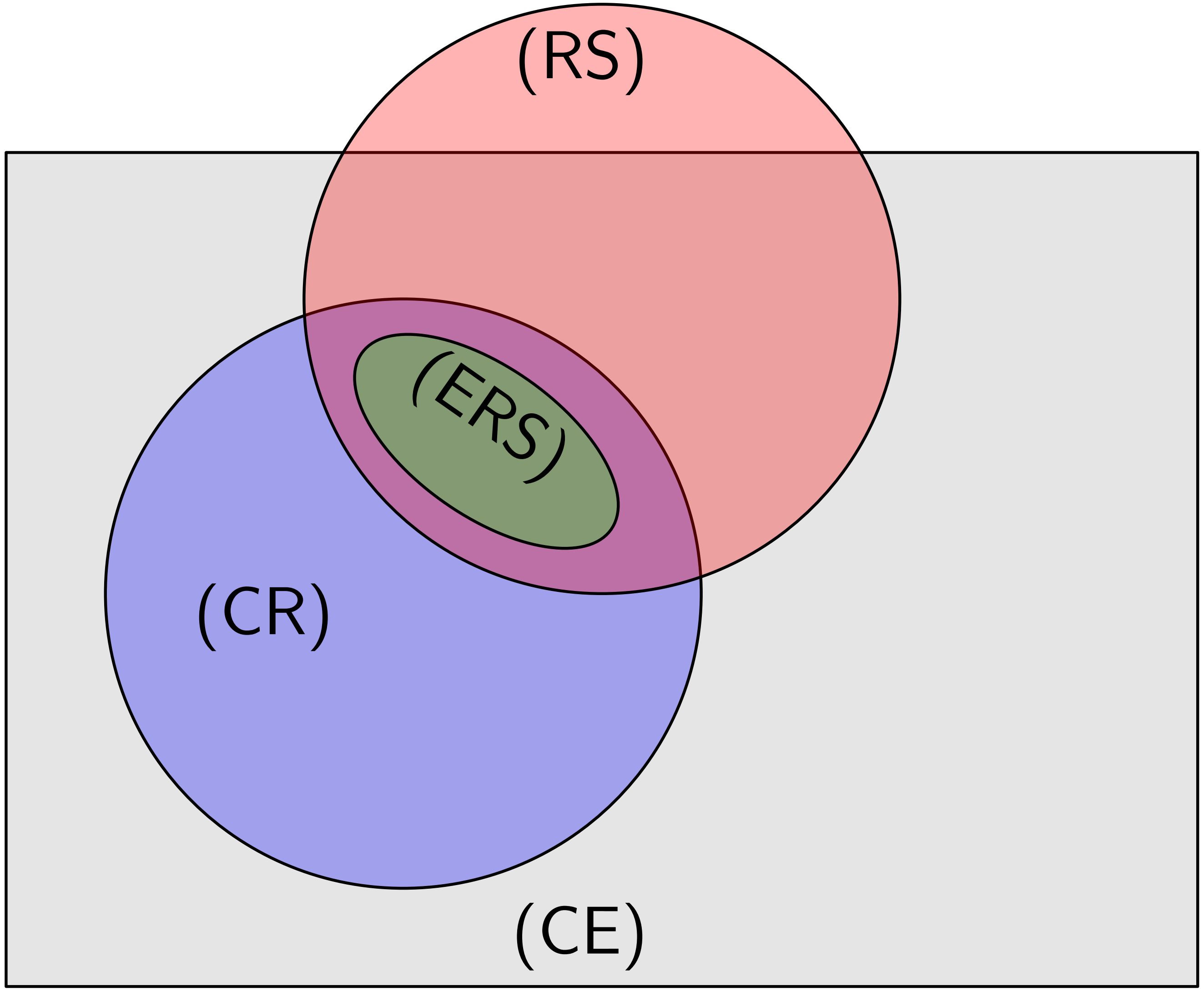
Implications between isomorphisms



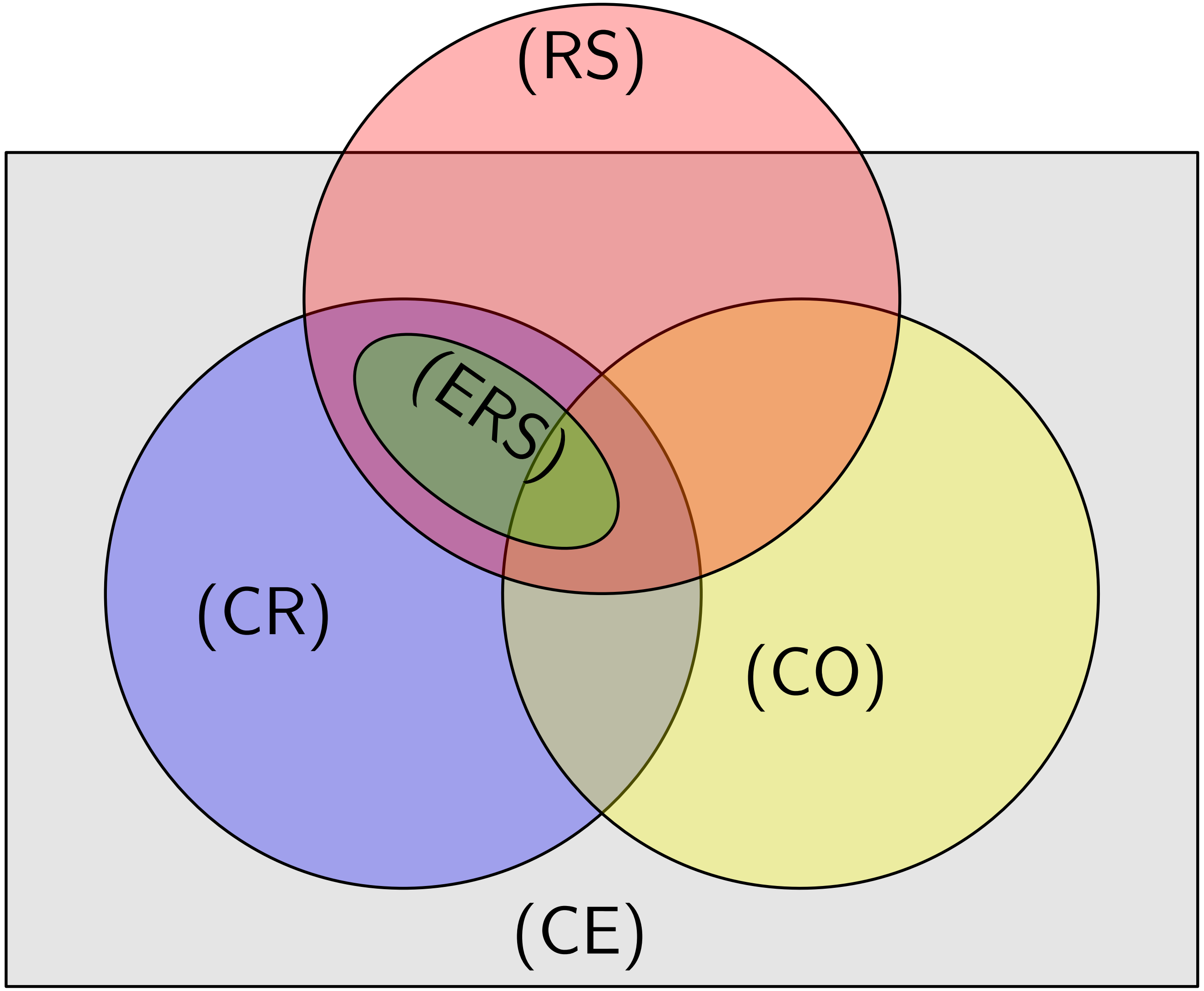
Implications between isomorphisms



Implications between isomorphisms

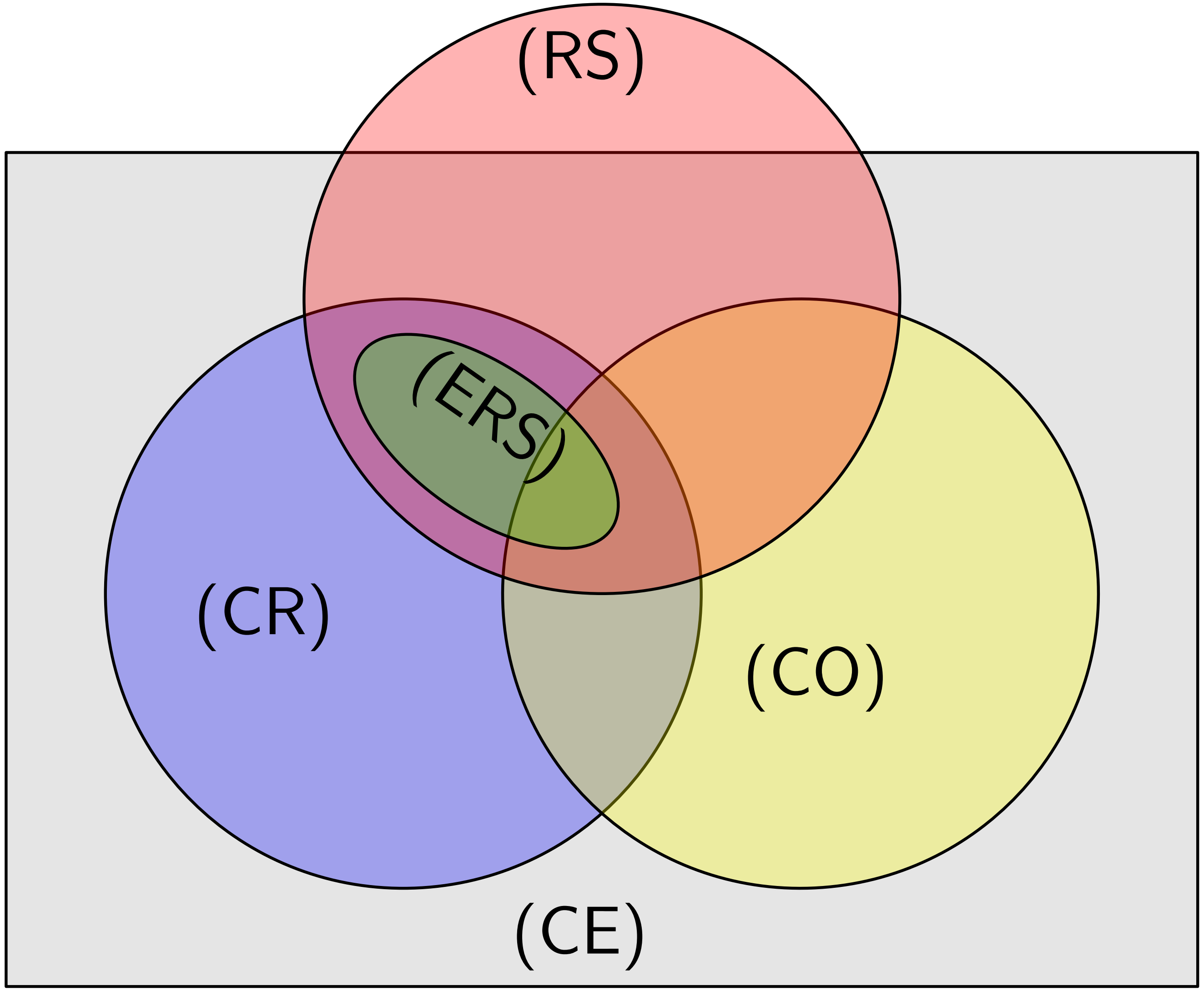


Implications between isomorphisms



Implications between isomorphisms

For the complete graph:

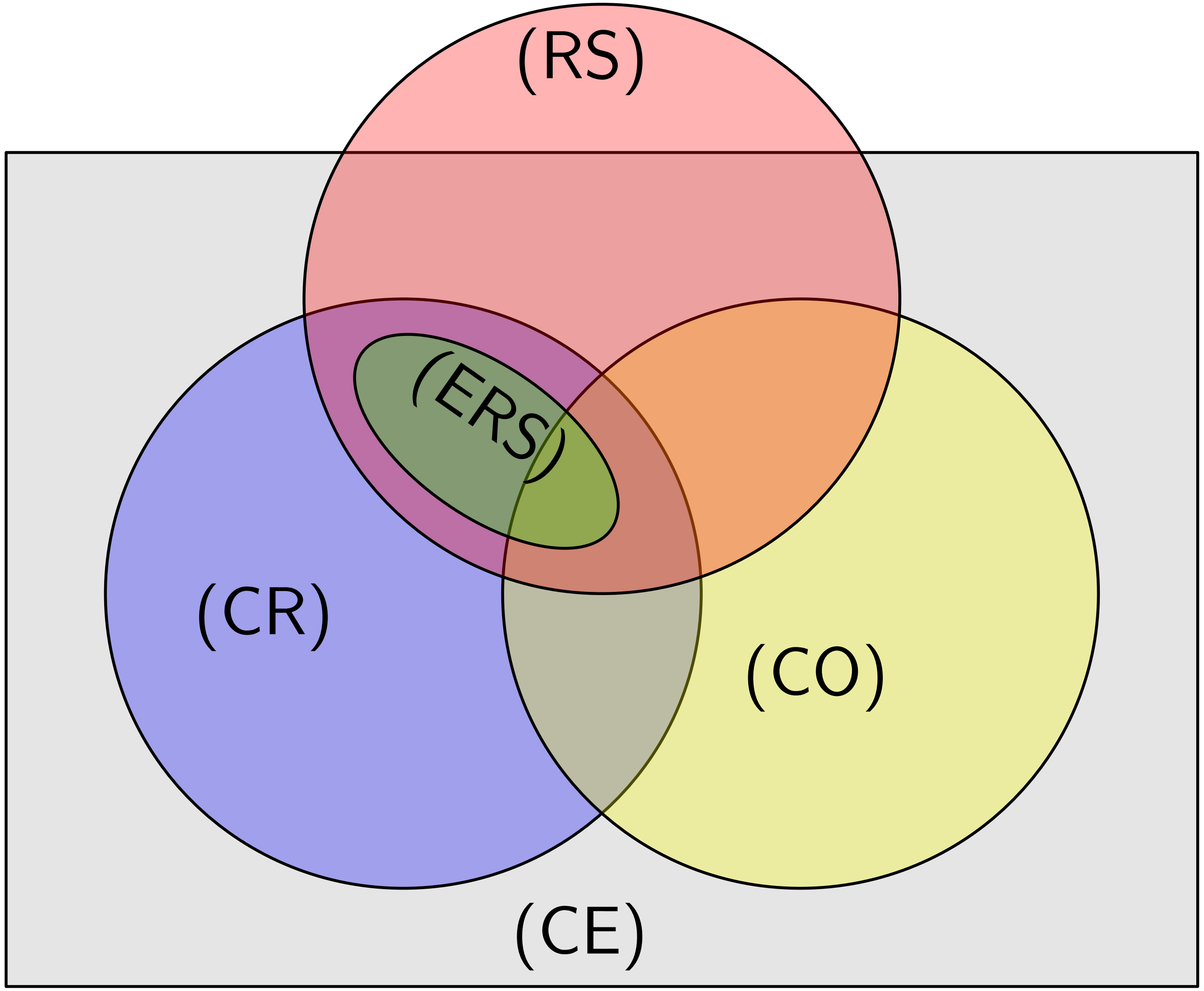


Implications between isomorphisms

For the complete graph:

ERS <=> RS <=> CE

[E. Gioan 2005, 2022], [J. Kynčl 2013]

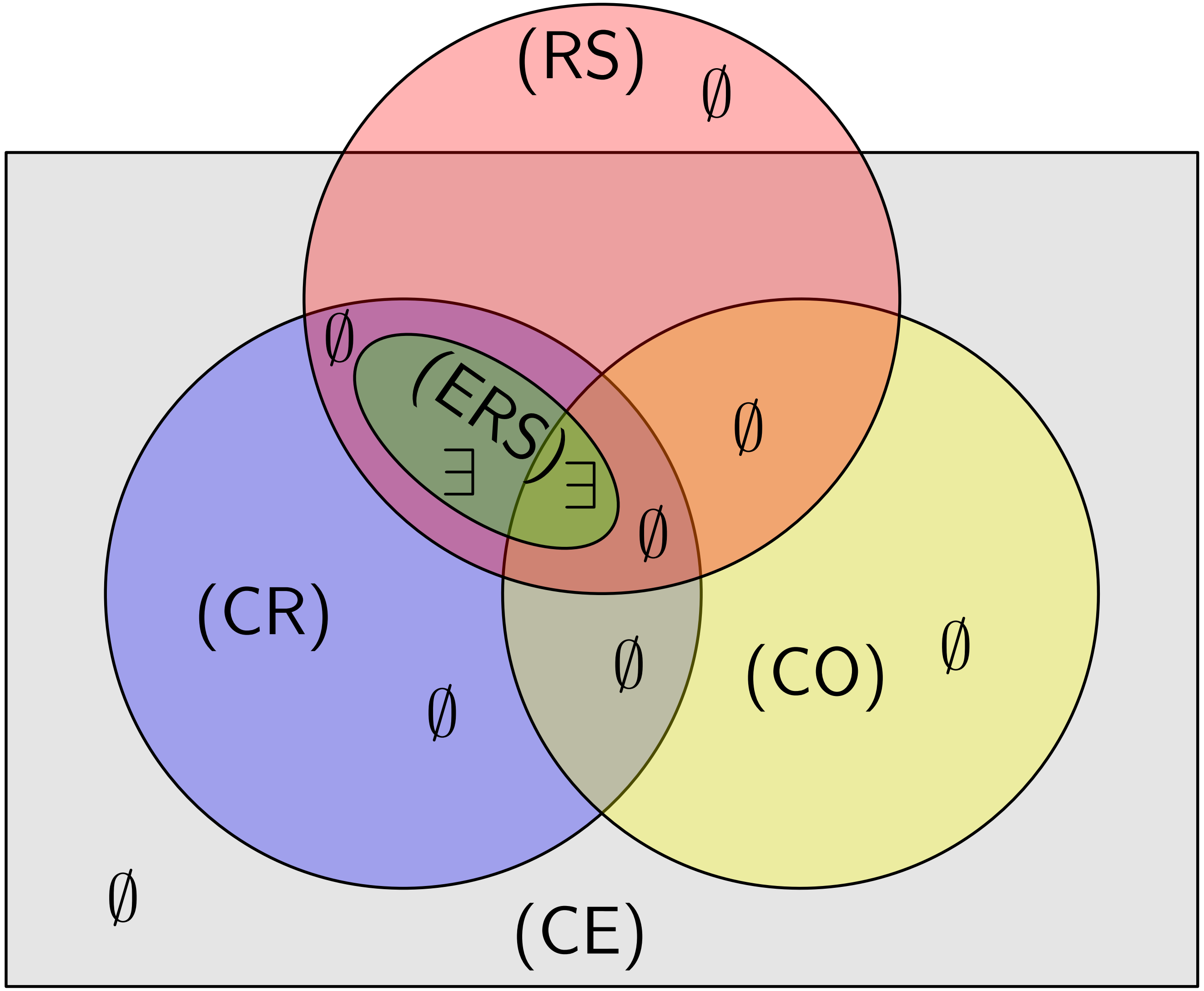


Implications between isomorphisms

For the complete graph:

ERS <=> RS <=> CE

[E. Gioan 2005, 2022], [J. Kynčl 2013]

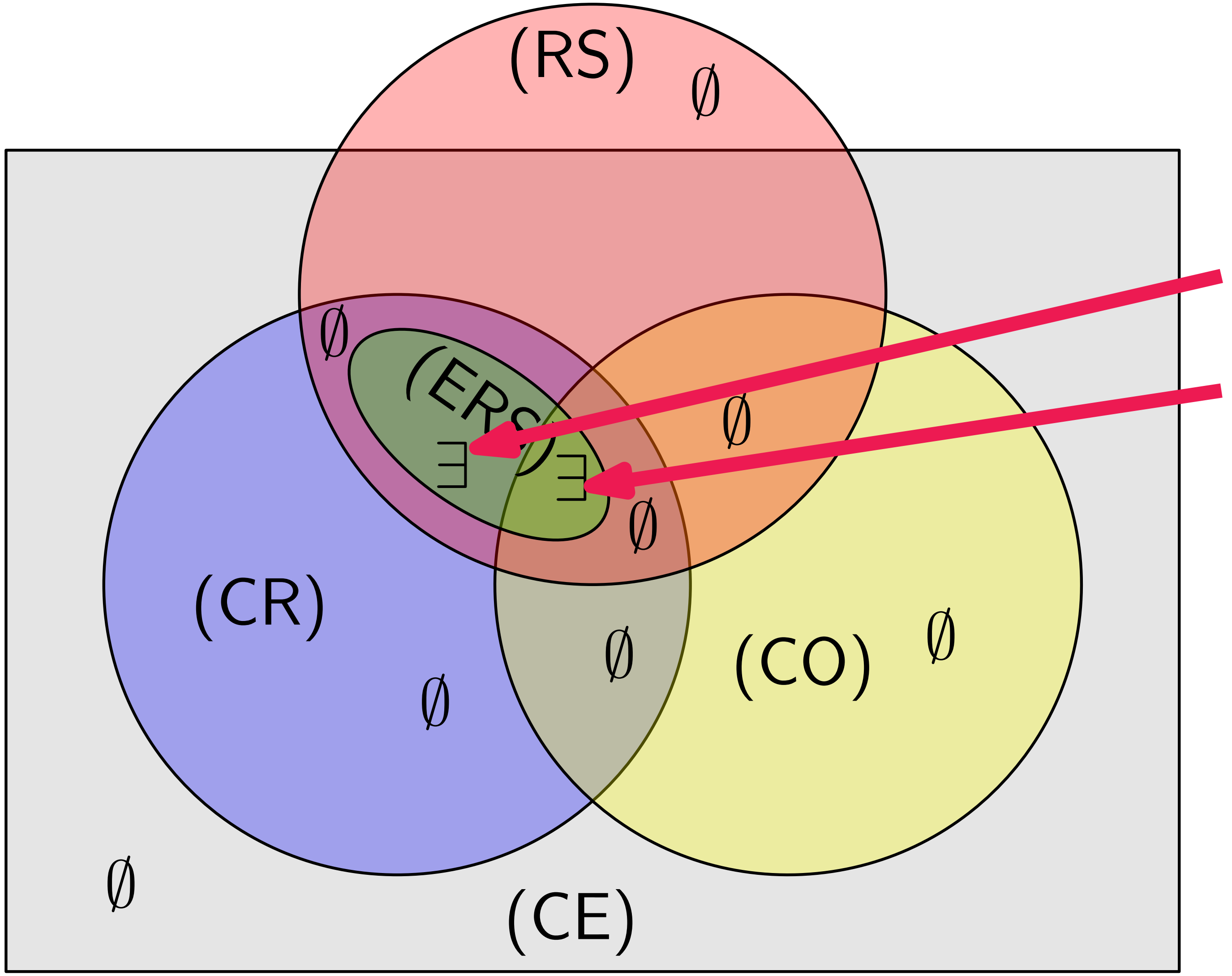


Implications between isomorphisms

For the complete graph:

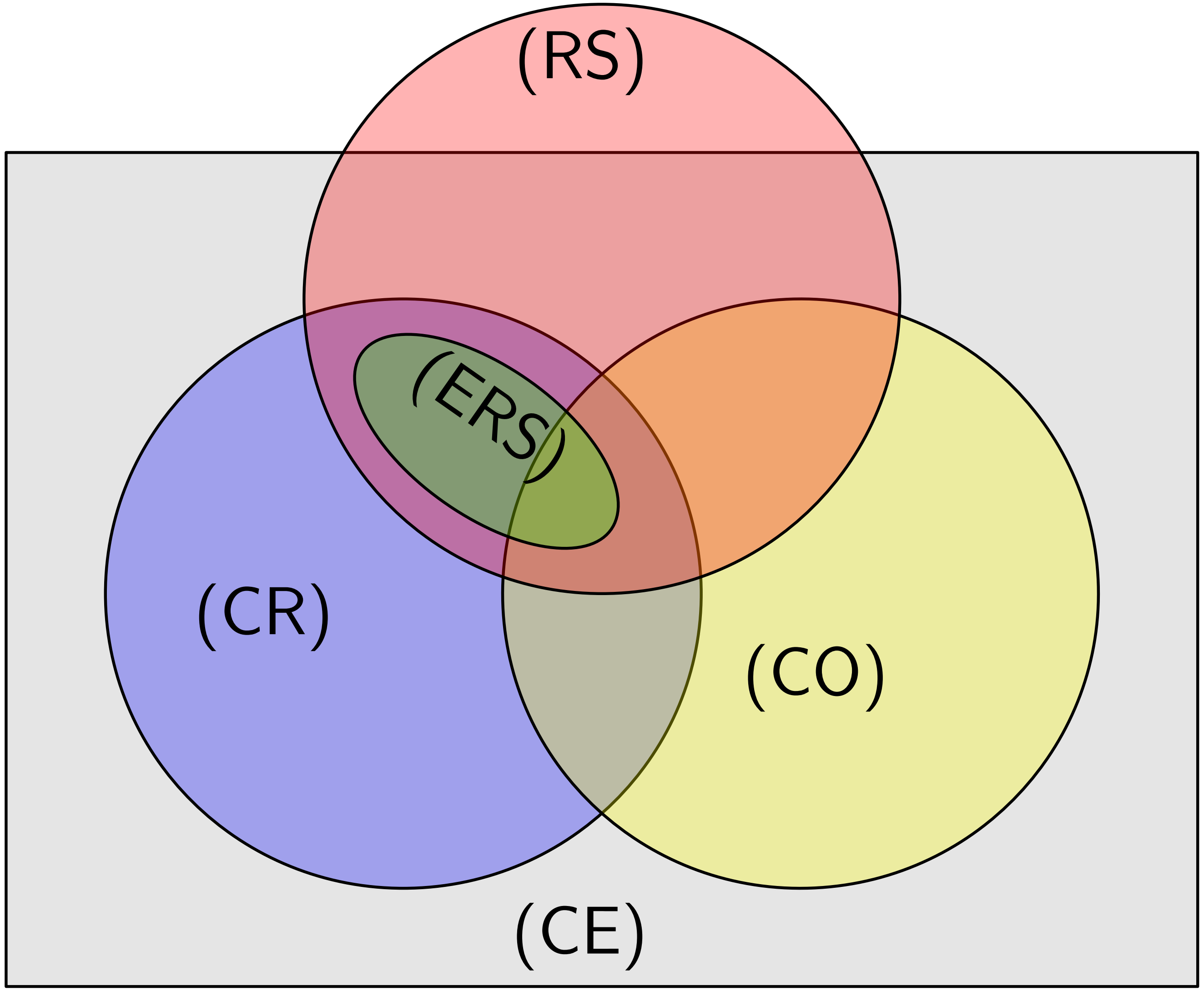
ERS <=> RS <=> CE

[E. Gioan 2005, 2022], [J. Kynčl 2013]



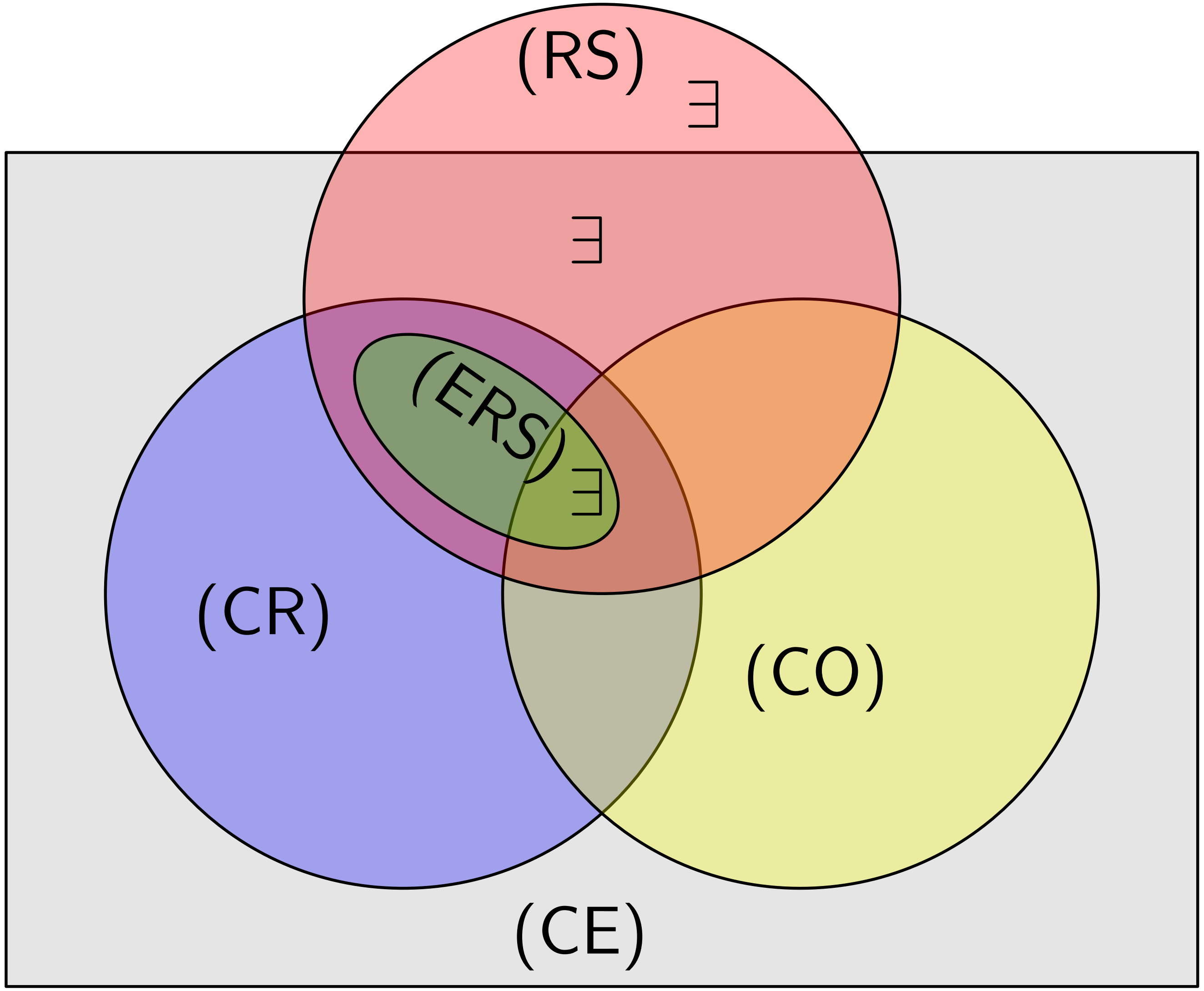
Implications between isomorphisms

For complete multipartite graphs:



Implications between isomorphisms

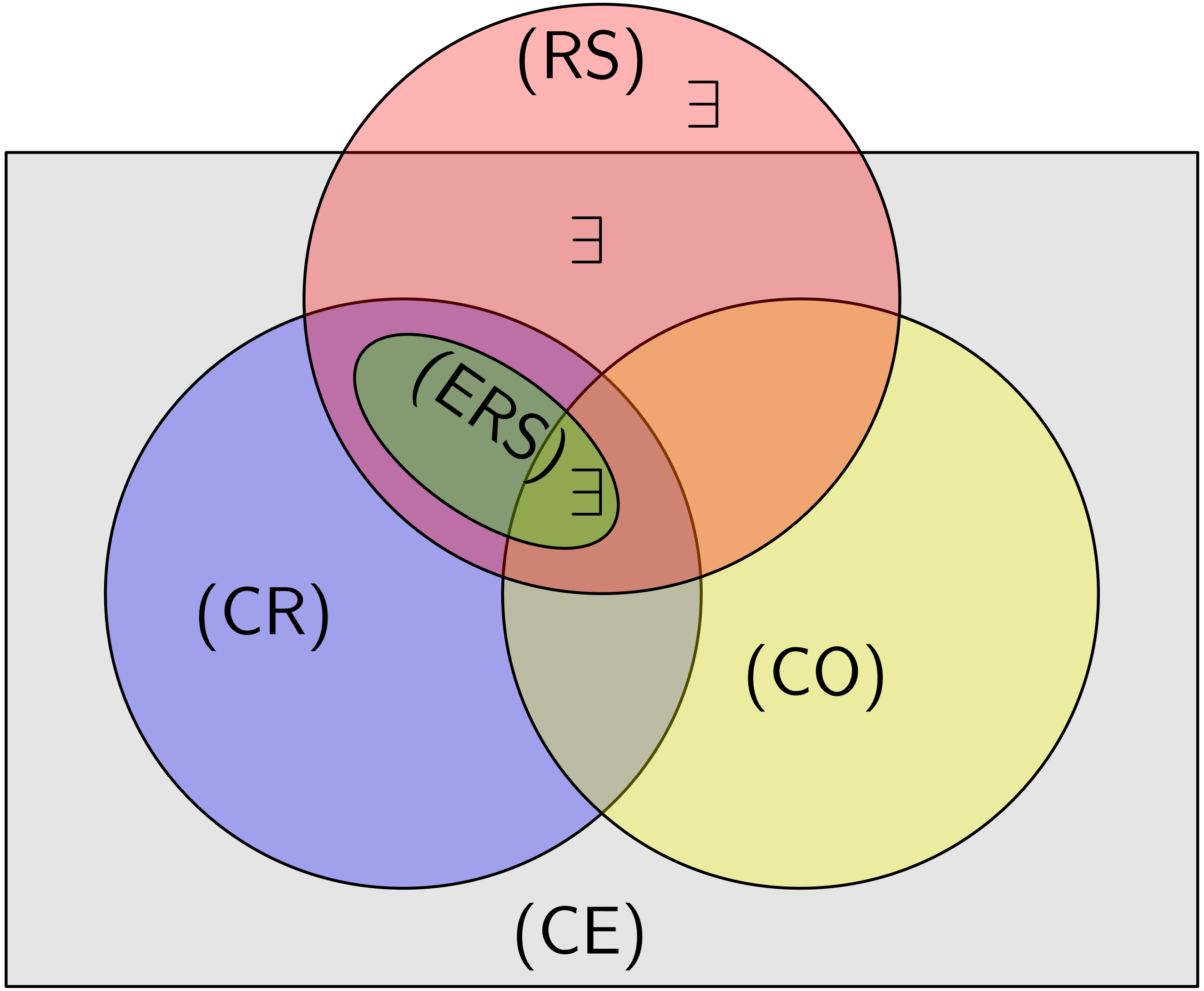
For complete
multipartite graphs:



Implications between isomorphisms

For complete multipartite graphs:

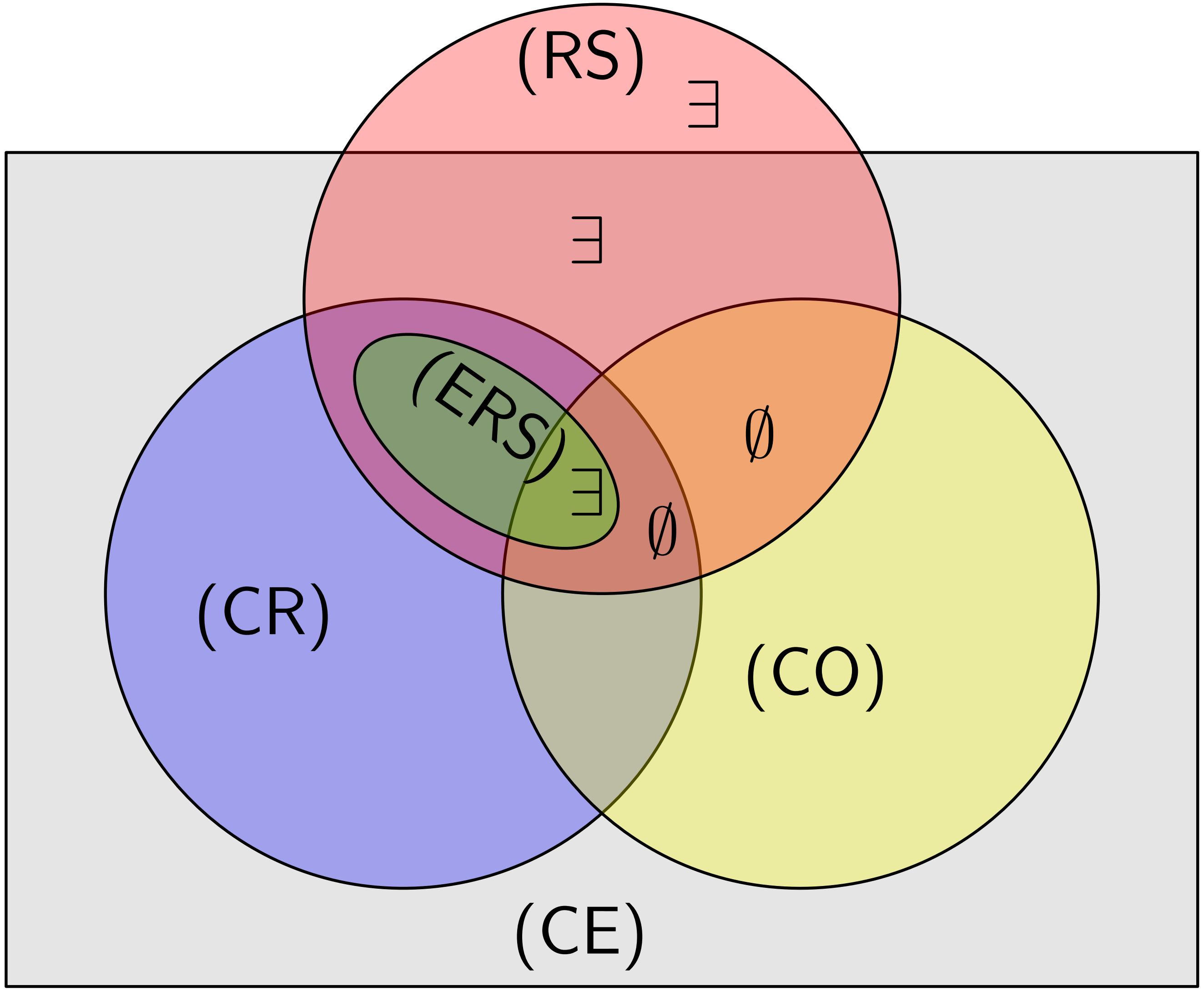
RS + CO => strong iso.



Implications between isomorphisms

For complete multipartite graphs:

RS + CO => strong iso.

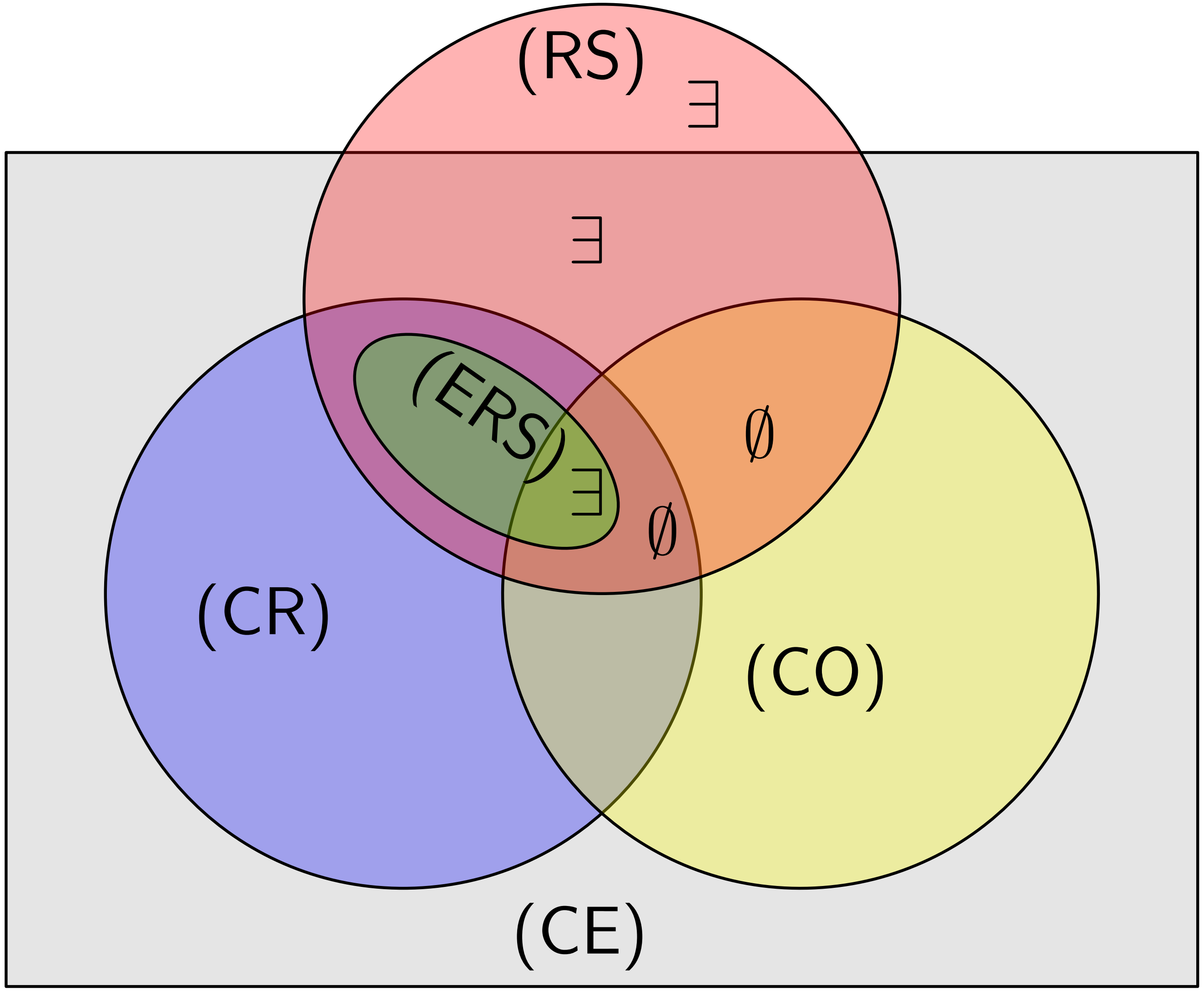


Implications between isomorphisms

For complete multipartite graphs:

RS $+$ CO $=>$ strong iso.

If each partition class has ≥ 3 vertices:
CE $=>$ RS
CR $=>$ ERS
CO $=>$ strong iso.

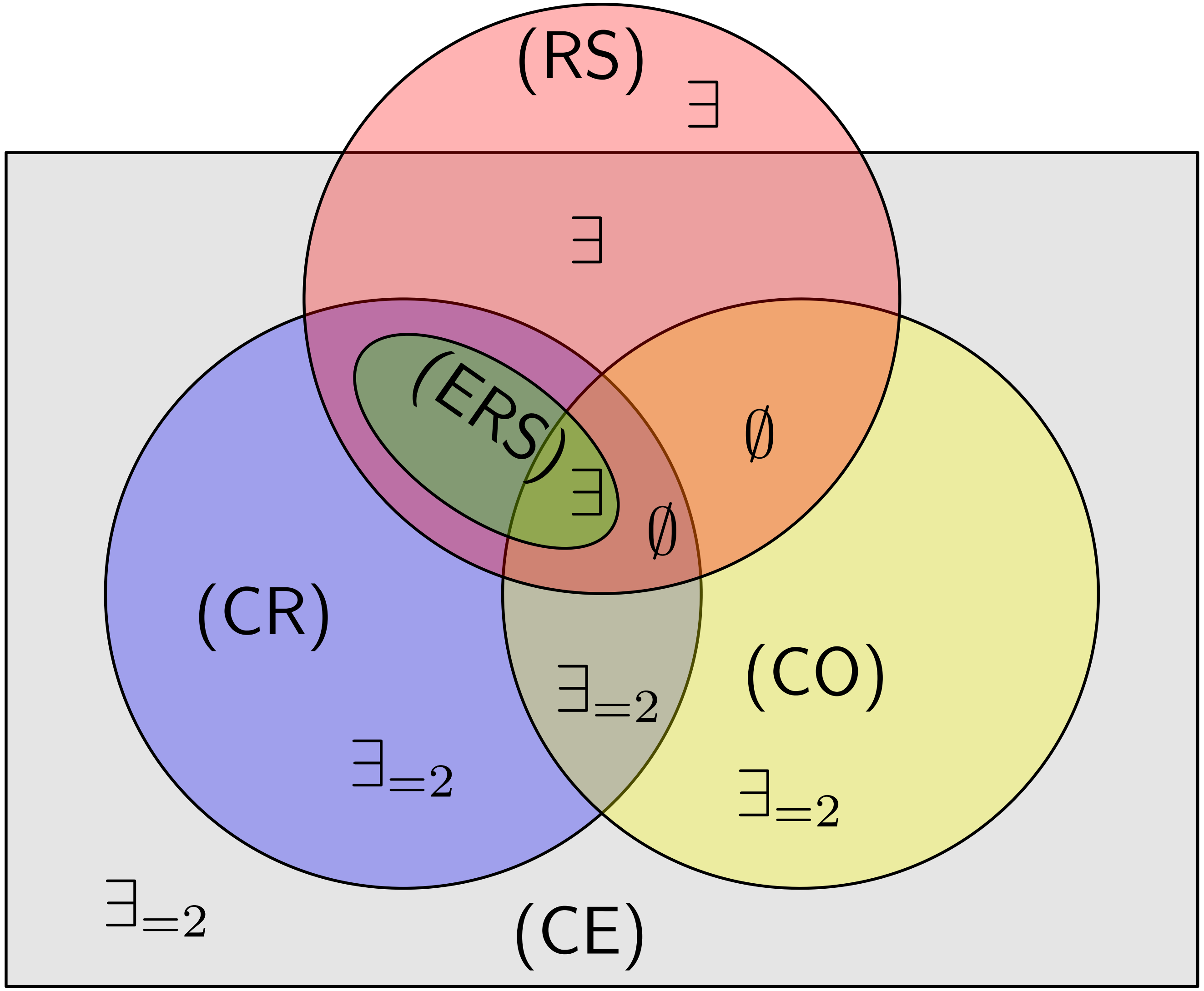


Implications between isomorphisms

For complete multipartite graphs:

RS + CO => strong iso.

If each partition class has ≥ 3 vertices:
CE => RS
CR => ERS
CO => strong iso.

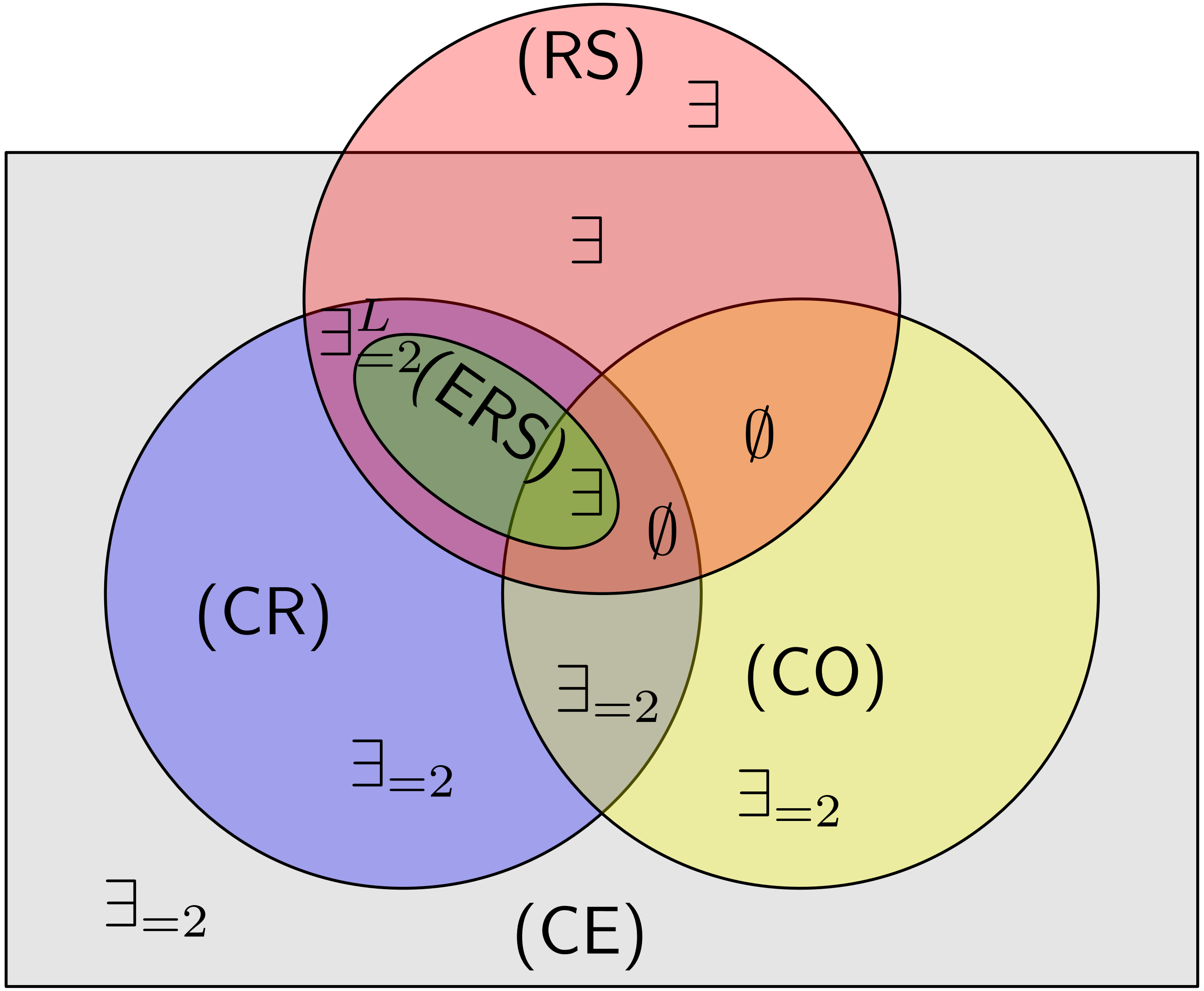


Implications between isomorphisms

For complete multipartite graphs:

RS + CO => strong iso.

If each partition class has ≥ 3 vertices:
CE => RS
CR => ERS
CO => strong iso.



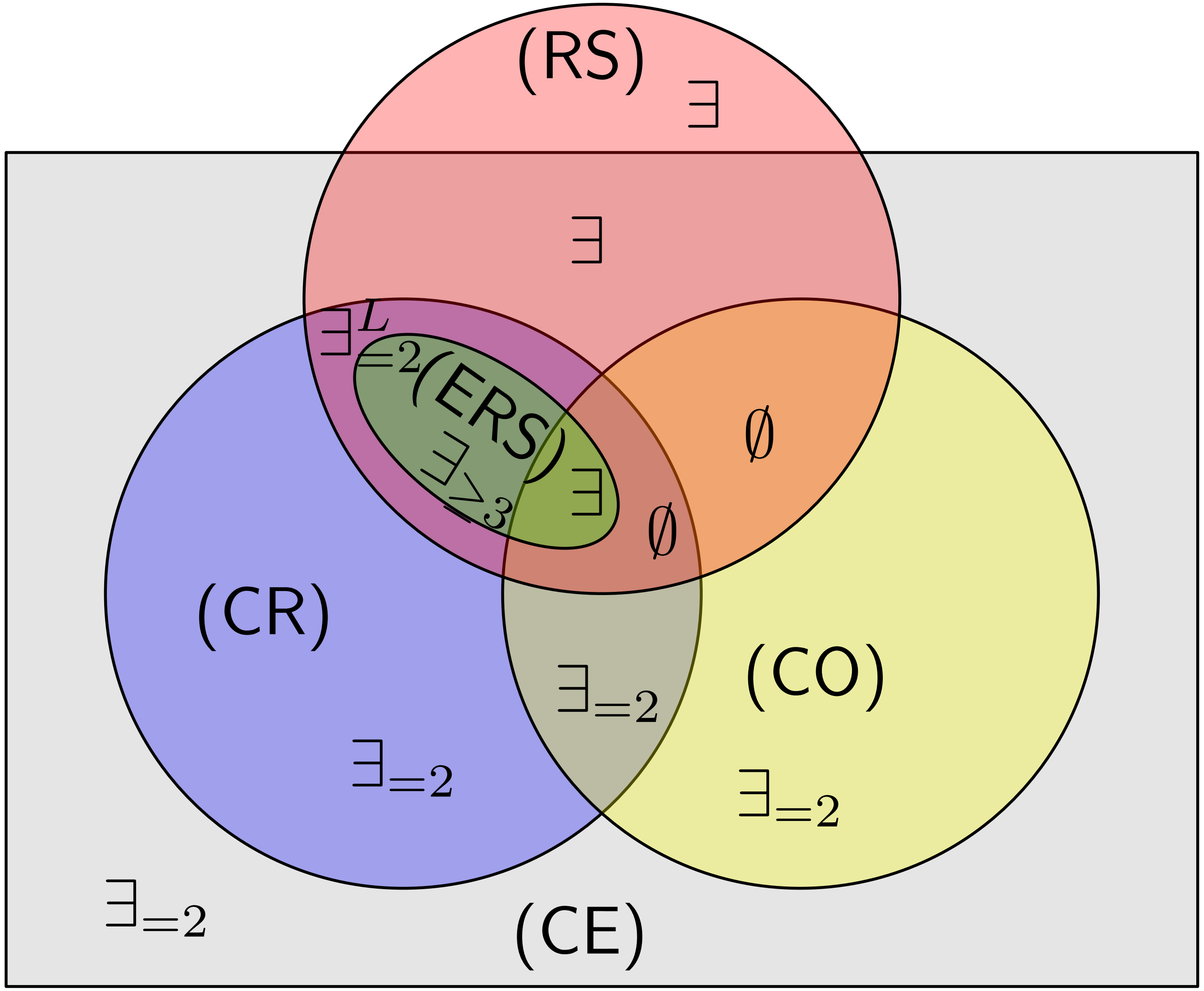
Implications between isomorphisms

For complete multipartite graphs:

RS + CO => strong iso.

If each partition class has ≥ 3 vertices:
CE => RS
CR => ERS
CO => strong iso.

For $K_{2,n}$: ERS => strong iso.[1]



1) [O. Aichholzer, M.K. Chiu, H. Hoang, M. Hoffmann, J. Kynčl, Y. Maus, B. Vogtenhuber, A.W. 2023]

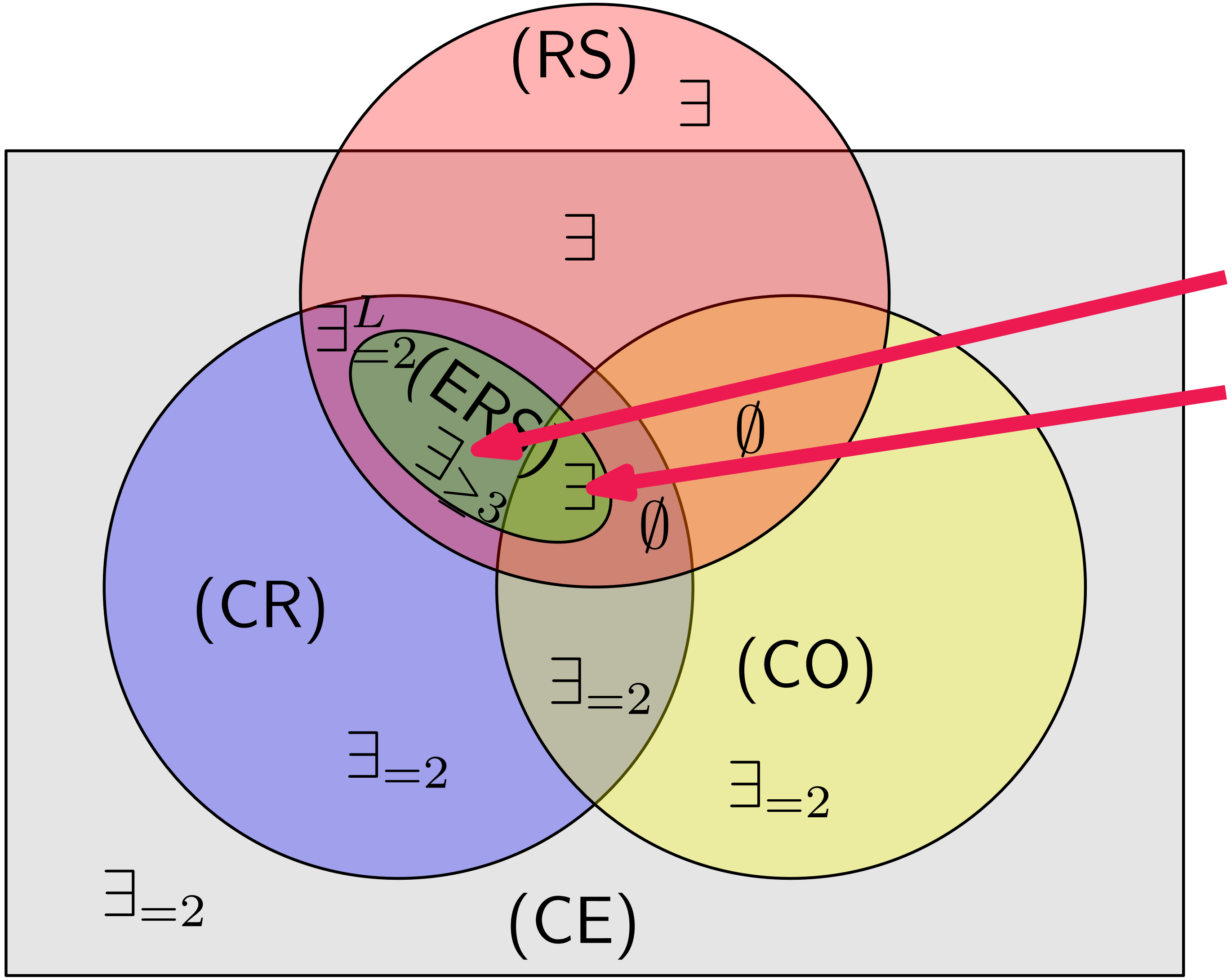
Implications between isomorphisms

For complete multipartite graphs:

RS + CO => strong iso.

If each partition class has ≥ 3 vertices:
CE => RS
CR => ERS
CO => strong iso.

For $K_{2,n}$: ERS => strong iso.[1]



1) [O. Aichholzer, M.K. Chiu, H. Hoang, M. Hoffmann, J. Kynčl, Y. Maus, B. Vogtenhuber, A.W. 2023]

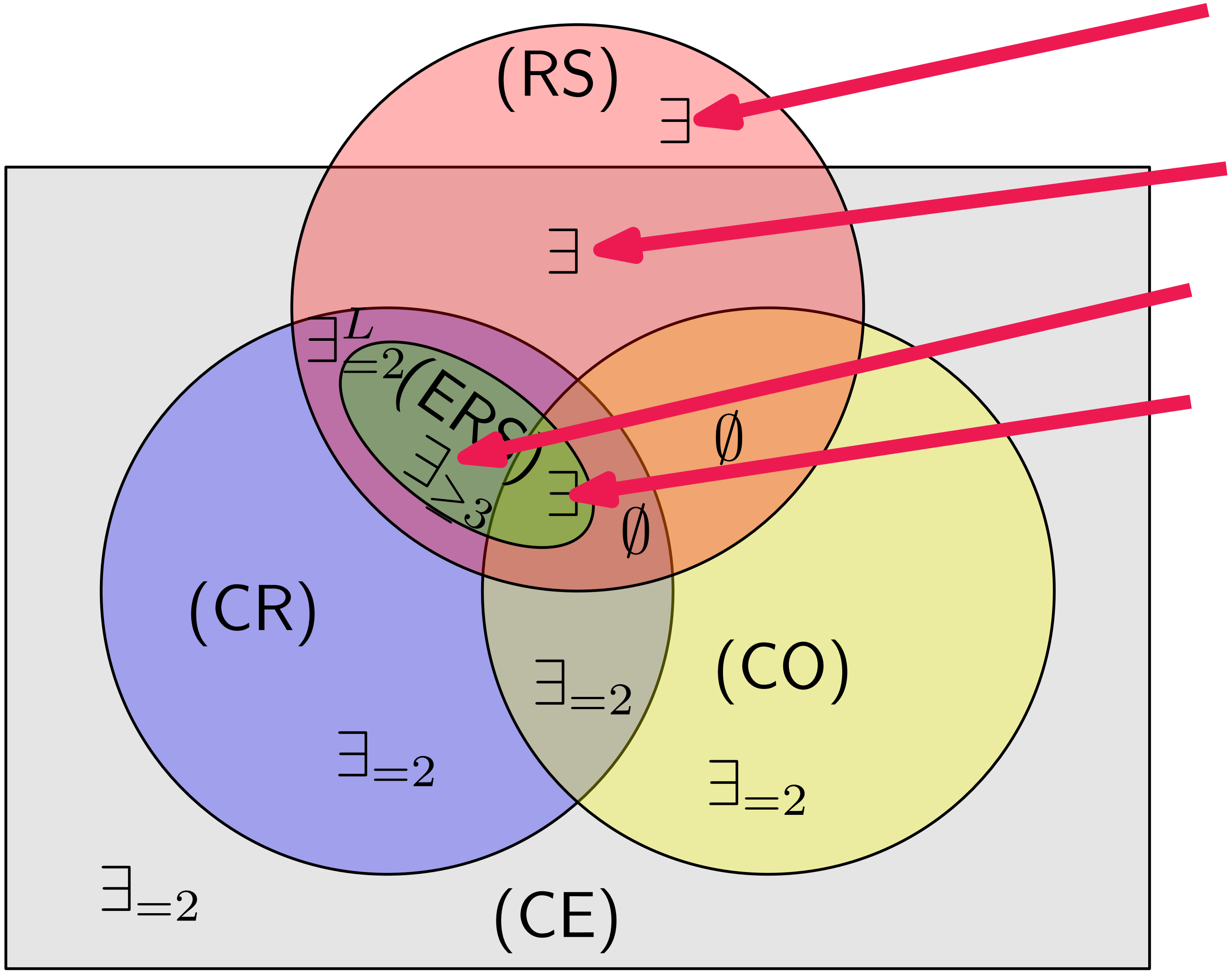
Implications between isomorphisms

For complete multipartite graphs:

RS + CO => strong iso.

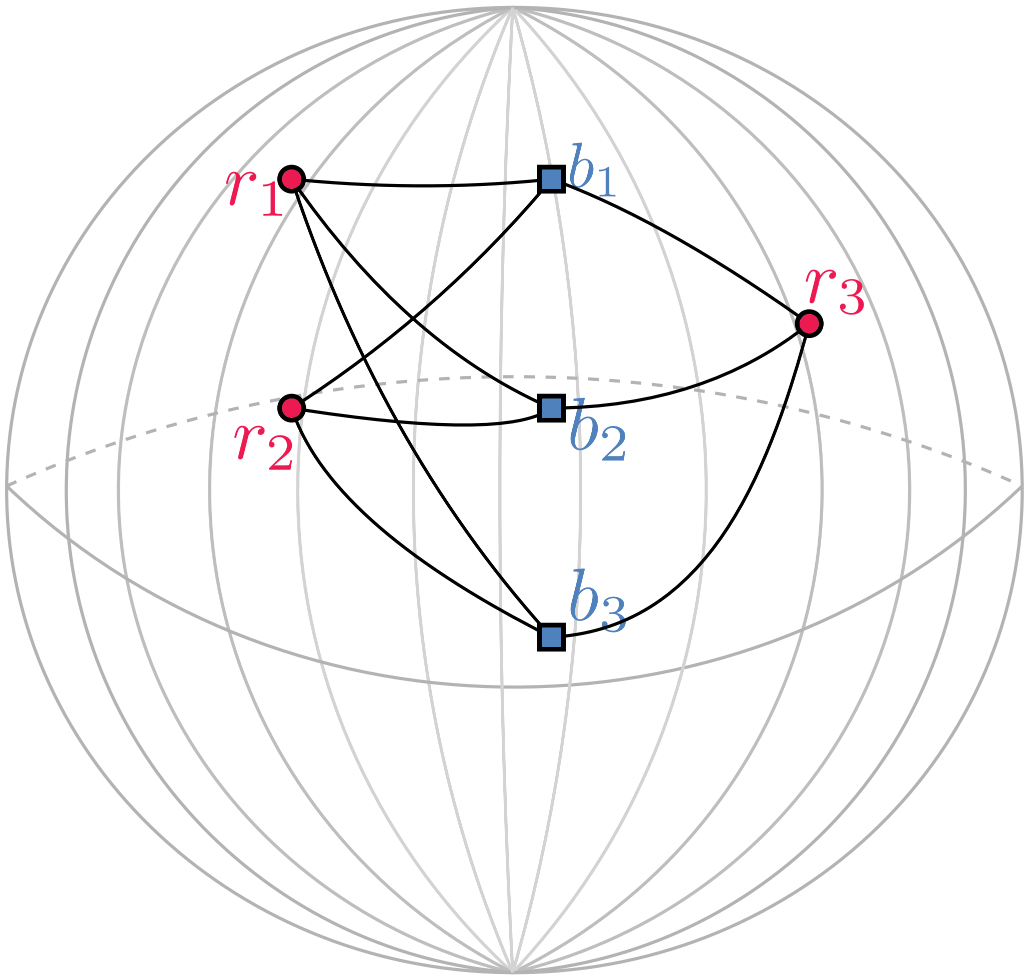
If each partition class has ≥ 3 vertices:
CE => RS
CR => ERS
CO => strong iso.

For $K_{2,n}$: ERS => strong iso.[1]



1) [O. Aichholzer, M.K. Chiu, H. Hoang, M. Hoffmann, J. Kynčl, Y. Maus, B. Vogtenhuber, A.W. 2023]

Simple Drawings



Simple Drawings