A logarithmic bound for simultaneous embeddings

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Simultaneous embeddability

Definition

A collection $\mathcal{G}$ of planar graphs on n vertices is *simultaneously embeddable* if there exists a set $P \subseteq \mathbb{R}^2$ of size n such that every $G \in \mathcal{G}$ admits a crossing-free straight-line embedding on P.
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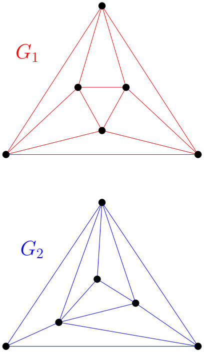

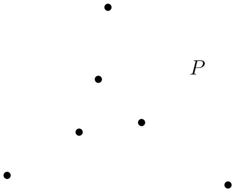

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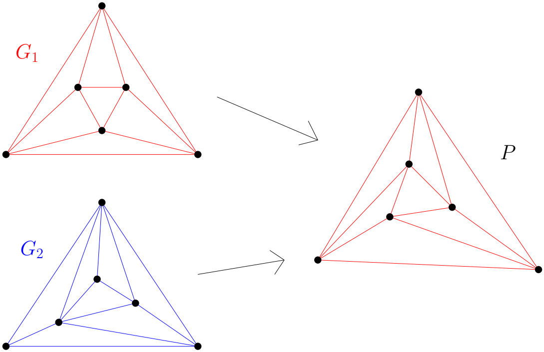

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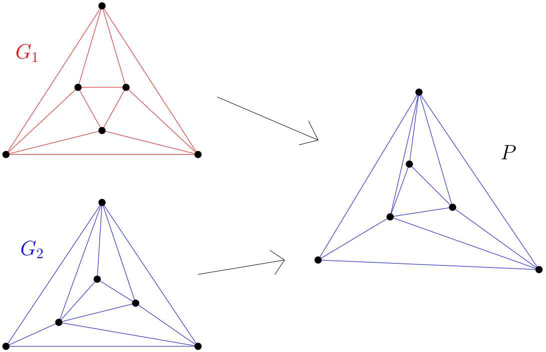

Major open problem in the area

Problem (Brass, Cenek, Duncan, Efrat, Erten, Ismailescu, Kobourov, Lubiw and Mitchell 2007)

Is there a conflict collection $\mathcal{G} = \{G_1, G_2\}$ *of size two?*

Progress so far

$\sigma(n) :=$ minimum size of a conflict collection of n-vertex graphs

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- Scheucher, Schrezenmaier and S. (2019): $\sigma(11) \leq 49 \Rightarrow \sigma \leq 49$.
- Goenka, Semnani and Yip (2023): $\sigma(n) = O(1.135^n)$ via an explicit construction of a conflict collection.

New results

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Theorem (S. 2023)

For $n \geq 5040$ *there exists an explicitly constructed conflict collection consisting of*

$$n^{\underline{6}} + 1 = n(n-1)(n-2)(n-3)(n-4)(n-5) + 1 < n^6$$

planar n-vertex graphs.

Randomly generating stacked triangulations

Random process to generate n-vertex labelled stacked triangulation:

- Start from a K_4 on vertices $\{1, 2, 3, 4\}$.
- For $i = 4, \ldots, n-1$, randomly and uniformly select one of the $2i-4$ faces of the current stacked triangulation and stack a vertex with label $i+1$ into the selected face.

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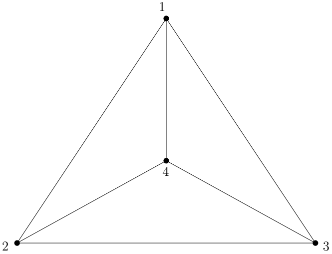

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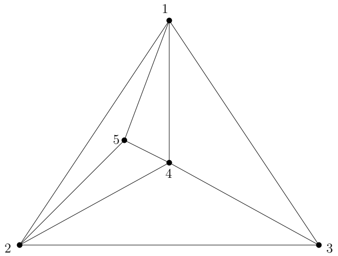

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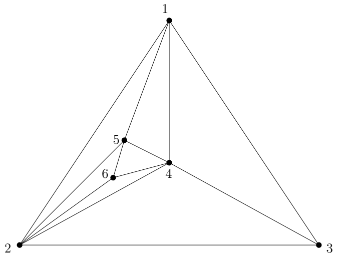

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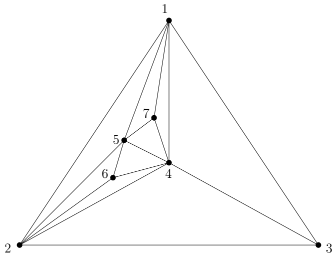

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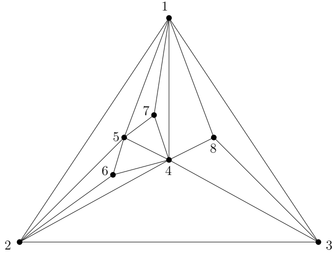

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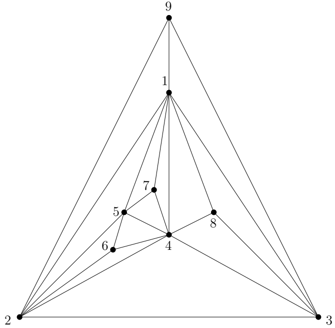

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Important to note:

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Important to note:

- Number of possible outcomes of the process:

$$4 \cdot 6 \cdots (2(n-1)-4) = 2^{n-4}(n-3)!$$

- Each of them is equally likely
- Given a labelling of a point set P, at most one stacked triangulation embeds in a label-preserving way

Main lemma

Lemma

Let **G** *denote the random n-vertex triangulation generated according to the described process. Then for every* $P \subseteq \mathbb{R}^2$ *of size n, we have:*

$$\mathbb{P}(\mathbf{G} \text{ embeds straight-line on } P) \leq \frac{16n(n-1)(n-2)}{2^n} = 2^{-(1-o(1))n}$$

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Proof.

Every straight-line embedding of $\mathbf{G}$ induces a $\{1, \ldots, n\}$-labelling of P. There are $n!$ such labellings. For a fixed labelling of P, at most one triangulation embeds in label-preserving way. Thus, at most $n!$ of the relevant stacked triangulations embed on P. Hence,

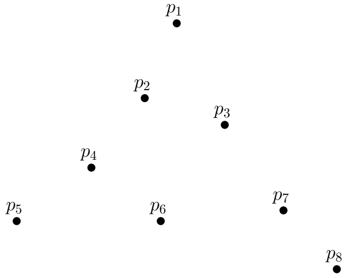
$$\mathbb{P}(\mathbf{G} \text{ embeds on } P) \leq \frac{n!}{2^{n-4}(n-3)!} = \frac{16n(n-1)(n-2)}{2^n}.$$

□

Straight-line embeddability and order types

Definition

Two point sets $P = \{p_1, \ldots, p_n\}, Q = \{q_1, \ldots, q_n\}$have same order-type if $\forall i, j, k$: $p_i p_j p_k$ and $q_i q_j q_k$ have the same orientation.



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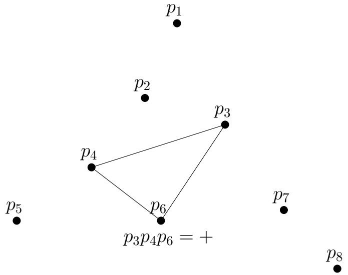

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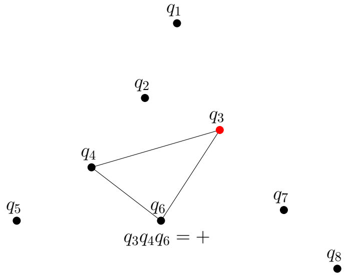

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Corollary

There exists a collection $\mathcal{P}_n$ consisting of n-point sets with $|\mathcal{P}_n| = n^{(4+o(1))n}$ such that the following holds. If planar graphs $G_1, \ldots, G_k$ are simultaneously embeddable, then there is $P \in \mathcal{P}_n$ such that every G_i embeds on P.

Proof of $\sigma(n) \leq (4+o(1))\log_2(n)$.

Theorem

For all $\varepsilon > 0$ *there is* $n_0 = n_0(\varepsilon) \in \mathbb{N}$ *such that* $\sigma(n) \leq (4+\varepsilon)\log_2(n)$ *for all* $n \geq n_0$.

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Proof Sketch.

Let $k = \lfloor (4 + \varepsilon) \log_2(n) \rfloor$. Consider k independently generated random triangulations $\{\mathbf{G}_1, \ldots, \mathbf{G}_k\}$. Then for every $P \in \mathcal{P}_n$:

$$\mathbb{P}\left(\bigwedge_{i=1}^{k} \{\mathbf{G}_i \text{ embeds on } P\}\right) \leq \left(2^{-(1-o(1))n}\right)^k = 2^{-(1-o(1))kn}.$$

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$$= n^{(4+o(1))n} \cdot 2^{-(4+\varepsilon-o(1))n \log_2(n)} = 2^{-(\varepsilon-o(1))n \log_2(n)} \to 0.$$

□

The end

Thank you for your attention!